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The equicontinuous weak* topology and semi-reflexivity*

by

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Abstract. It is known that a Banach space B is reflexive if and only if B is complete with respect to the relativization of the bounded weak* topology on the second dual space B''. In this paper the equicontinuous weak* (ew*) topology on the dual of a locally convex space E is employed in order to extend this result to more general spaces. A number of properties of the ew* topology on E' are obtained: in particular, that scalar multiplication is always jointly continuous.

1. Introduction. A Banach space B is *reflexive* if the canonical map J of B into its second conjugate space B'' is surjective. It is well known that reflexivity of B is equivalent to weak quasi-completeness: i.e., the requirement that every *bounded* weakly closed subset of B be complete under the relativization of the unique translation-invariant uniformity which generates the weak topology [18, p. 16]. As Day [4] has observed, we can obtain a characterization of reflexivity in terms of *completeness* (rather than *quasi-completeness*) as follows: Let the rbw* topology be the restriction to B (more precisely, to $J(B)$) of the *bounded weak** (*bw**) *topology* on B''. The latter, the finest topology on B'' which agrees with the weak* topology on norm-bounded sets, is known to be a locally convex linear topology; hence the rbw* topology also has that property.

THEOREM 1.1. [4, p. 57]. *A Banach space is reflexive if and only if it is rbw*-complete.*

In this paper we consider the problem of extending the stated result to the setting of a general locally convex space (LCS). We refer the reader to [18] for details of the construction of the second dual space E'' of a LCS E, the concept of semi-reflexivity, and its equivalence to weak quasi-completeness. The natural generalization of the bounded weak* topology is the *equicontinuous weak** (*ew**) *topology* of Collins [2], referred to by Day [4, p. 44] as the *almost-weak** (*aw**) *topology*. Let E^* denote the *strong*

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dual of E [18, p. 141], the space E' of continuous linear functionals endowed with the strong topology $\beta(E', E)$. Then E'', being the dual of E^*, admits a corresponding ew* topology. Referring to the relative topology of E (more precisely, of $J(E)$) as the ew (or aw) topology, we can conjecture a result which generalizes Theorem 1.1.

CONJECTURE 1.2. *A locally convex space is semi-reflexive if and only if it is ew-complete.*

Indeed a proof of this has been proposed [6, p. 74], but it relies on an assertion [4, p. 46, (6)] which we believe to be erroneous: namely, that if E is a LCS, then (E', ew^*) is complete. In Section 3 we present a class of spaces E for which the topological space (E', ew^*) is not completely regular, so that characterizing ew* as complete (or even topologically complete [10, p. 208]) does not seem to be, in general, appropriate.

A number of other results concerning the ew* topology are collected in Section 3. In particular, it is shown that scalar multiplication as a function on $K\times(E', \text{ew}^*)$ to (E', ew^*) is (jointly) continuous (here K is the real or complex field, with the usual topology). Consequently ew* is a linear (vector) topology if and only if addition is continuous. Certain results of Kakutani and Klee [9] on the geometry of linear spaces yield a large class of examples (spaces with weak topology and dual of uncountable dimension) for which addition is not continuous. Such spaces are (hereditarily) paracompact, hence topologically complete, but admit no compatible, translation-invariant uniform structure.

The more pathological features of the ew* topology make it evident that a better-behaved (i.e., locally convex linear) topology is needed to investigate the relationship between semi-reflexivity and completeness. If E is a LCS, the family of all convex neighborhoods of 0 in (E'', ew^*) is a base for a locally convex topology, the *cew* topology* [8]. By restricting cew* to $J(E)$, we obtain a locally convex topology on E. In Section 4 we show that this topology, the *cbw topology*, is the finest locally convex topology which agrees with the weak topology on bounded sets. Semi-reflexivity of E is quickly seen to be equivalent to *cbw-quasi-completeness*. This leads to the following conjecture.

CONJECTURE 1.3. *A locally convex space is semi-reflexive if and only if it is cbw-complete.*

In Section 5 it is shown that this statement is valid for several classes of spaces (e.g., metrizable, (DF)-, and (LF)-spaces) and that cbw-completeness is preserved under locally convex direct sums and products and inherited by closed bornological subspaces.

Finally in Section 6 we present a possible example (depending on the existence of a certain type of topological space) to show that cbw-completeness need not, in general, be implied by semi-reflexivity.

2. Notation, definitions, and preliminary results. We assume the basic results of general topology [5], [10]. If d and d' are two topologies for a set S such that every d-open set is d'-open, we write $d \leqslant d'$ ($d < d'$ if d' is strictly finer than d). The notation for topological vector spaces is taken primarily from [18], but we adopt the language of nets rather than filters, and employ the definition of polarity found in [12, p. 141].

Most results given hold for vector spaces over the real or complex field; in this case the scalar field will be denoted by K and will have the usual topology. The dimension of a vector space E, $\dim E$, is the cardinal of a Hamel basis for E. The functions

$$a(x, y) = x+y \quad \text{on } E\times E \text{ into } E,$$
$$m(t, x) = tx \quad \text{on } K\times E \text{ into } E$$

are the addition and scalar multiplication functions, respectively. A topology d on E is *semi-linear* if a and m are separately continuous. In this case $(E, d)' = E'$ is that subspace of the algebraic dual $E^\#$ which consists of all d-continuous linear functionals.

The phrase "locally convex space" means locally convex Hausdorff topological vector space. If E and F are vector spaces in duality, then $\sigma(E, F)$, $\tau(E, F)$, and $\beta(E, F)$ denote, respectively, the weak, Mackey, and strong topologies on E with respect to F.

A uniformity on a vector space E is *translation-invariant* if it has a base of subsets W of $E\times E$ satisfying $W+\Delta \subset W$, where $\Delta = \{(x, x): x\in E\}$. The statement "a topological vector space E is *complete*" means that E, endowed with the unique translation-invariant uniformity which generates the topology, is complete. On the other hand, a topological space (S, d) is *topologically complete* if there is a compatible uniformity on S under which S is complete.

The definition of the ew* topology is due to Collins [2, p. 259].

DEFINITION 2.1. If E is a LCS, the equicontinuous weak* (ew*) topology on E' is the family of all subsets W of E' with the property: corresponding to each equicontinuous subset D of E', there is a $\sigma(E', E)$-open set V with $W \cap D = V \cap D$.

If $\mathscr{U}$ is a neighborhood base at 0 for E, then a subset B of E' is ew*-open (resp., ew*-closed) if and only if for each $U \in \mathscr{U}$, $B \cap U^0$ is relatively $\sigma(E', E)$-open in U^0 (resp., $\sigma(E', E)$-compact). Clearly the ew* topology is the finest on E' which coincides with $\sigma(E', E)$ on equicontinuous sets.

Basic results about ew* may be found in [2], [4], [8]; we list several for future reference.

FACTS 2.2 (i) *If F is a dense linear subspace of a LCS E, with $i\colon F \to E$ the natural inclusion, then the adjoint i^* is a homeomorphism of (E', ew^*) onto (F', ew^*).*

(ii) *The ew* topology is Hausdorff and semi-linear.*

(iii) (Banach–Dieudonné) *If E is a metrizable LCS, then ew* is the topology of uniform convergence on totally bounded subsets of E, hence a locally convex linear topology.*

DEFINITION 2.3 [4, p. 47]. If E is a LCS, the ew topology on E is the *restriction to* E (more precisely, to $J(E)$) of the ew* topology on $E'' = E^{*\prime} = (E', \beta(E', E))'$.

3. Topological and uniform properties of ew*. If (S, d) is a Hausdorff space, and $\mathscr{C}$ is a non-empty collection of compact subsets of S, then the family

$$d(\mathscr{C}) = \{U \subset S: \text{for each } C \in \mathscr{C}, \text{there is a } d\text{-open set } V \text{ with } U \cap C = V \cap C\}$$

is a (Hausdorff) topology for S, evidently the finest which agrees with d on each member of $\mathscr{C}$ [8, p. 59]. We shall refer to $d(\mathscr{C})$ as the $\mathscr{C}$-extension of d. If $d = d(\mathscr{C})$, then S is often said to have the *weak topology with respect to* $\mathscr{C}$ [5, p. 131]. However, to avoid confusion with other uses of "weak", we shall say, in this case, that the topology d is $\mathscr{C}$-saturated.

Remark 3.1. A function on $(S, d(\mathscr{C}))$ to another topological space is continuous if its restriction to each set in $\mathscr{C}$ is continuous. If $\mathscr{C}_1$ and $\mathscr{C}_2$ are two collections of compact subsets of (S, d) such that $\mathscr{C}_1 \subset \mathscr{C}_2$ and for each $D \in \mathscr{C}_2$ there is $D' \in \mathscr{C}_1$ with $D \subset D'$, then $d(\mathscr{C}_1) = d(\mathscr{C}_2)$.

If d is $\mathscr{C}$-saturated, where $\mathscr{C}$ is the family of all compact subsets, then (S, d) is called a *k-space* [5, p. 248]. The following theorem is a generalization of a well-known result for k-spaces due to D. E. Cohen [5, p. 263]. Since a proof can be obtained by simple modifications of an argument due to Bagley and Yang [1, Th. 1], we omit it.

THEOREM 3.2. *Let (X, d) and (Y, t) be Hausdorff spaces with X locally compact. Let $\mathscr{C}$ be the family of all compact subsets of X, and let $\mathscr{D}$ be a family of compact subsets of Y, covering Y, for which t is $\mathscr{D}$-saturated. Then the product topology $d \times t$ on $X \times Y$ is $\mathscr{C} \times \mathscr{D}$-saturated, where $\mathscr{C} \times \mathscr{D} = \{C \times D: C \in \mathscr{C}, D \in \mathscr{D}\}$.*

This result has immediate consequences for the ew* topology.

THEOREM 3.3. *Let E and F be locally convex spaces with E finite-dimensional. If $G = E \times F$, then the natural map $T: E' \times F' \to G'$ is a homeomorphism of $(E', ew^*) \times (F', ew^*)$ onto (G', ew^*).*

Proof. Choose 0-neighborhood bases $\mathscr{U}$ and $\mathscr{V}$ for E and F, respectively. Since $\{U^0 \times V^0: U \in \mathscr{U}, V \in \mathscr{V}\}$ is a fundamental family of equicontinuous sets in G' (using the map T), (G', ew^*) is the $\mathscr{C} \times \mathscr{D}$-extension of $\sigma(G', G)$, where $\mathscr{C} = \{U^0: U \in \mathscr{U}\}$ and $\mathscr{D} = \{V^0: V \in \mathscr{V}\}$. But $\sigma(G', G) = \sigma(E', E) \times \sigma(F', F)$, so that the product of the ew* topologies on E' and F' agrees with $\sigma(G', G)$ on each set in $\mathscr{C} \times \mathscr{D}$. Thus T is an open map.

Conversely let $\mathscr{C}' = \{C \subset E': C \text{ is } \sigma(E', E)\text{-compact}\}$. Since E is finite-dimensional, the topologies $\sigma(E', E)$ and ew* coincide on E' and are locally compact, while (F', ew^*) is $\mathscr{D}$-saturated. Now (3.2) shows that $(E', \text{ew}^*) \times (F', \text{ew}^*)$ is $\mathscr{C}' \times \mathscr{D}$-saturated; hence by (3.1) it is $\mathscr{C} \times \mathscr{D}$-saturated. This completes the proof.

COROLLARY 3.4. *If F is a LCS, then scalar multiplication on $K \times (F', ew^*)$ to (F', ew^*) is continuous.*

Proof. Let E be a locally convex space of dimension one over K. Then $K \times (F', \text{ew}^*)$ can be identified with $(E', \text{ew}^*) \times (F', \text{ew}^*)$. By virtue of (3.3) and (3.1), m is continuous if its restriction to each set of the form $\{a \in K: |a| \leqslant n\} \times V^0$ (n a positive integer, V a neighborhood of 0 in F) is continuous. Since the image of each such set is equicontinuous in F', the assertion follows from the continuity of m for the vector topology $\sigma(F', F)$.

Arguments similar to those given in (3.4) show that if E is a LCS, and V a fixed neighborhood of 0 in E, then addition on $(E', \text{ew}^*) \times (E', \text{ew}^*)$ to (E', ew^*) is continuous when restricted to $V^0 \times E'$.

THEOREM 3.5. *Let E be a LCS. If the natural map $T: (E', ew^*) \times (E', ew^*) \to ((E \times E)', ew^*)$ is a homeomorphism, then the ew* topology on E' is a vector topology. The space (E', ew^*) is a topological vector space if and only if there is a compatible, translation-invariant uniformity on E'.*

Proof. If T is a homeomorphism, then $(E', \text{ew}^*) \times (E', \text{ew}^*)$ is $\mathscr{C}$-saturated, where $\mathscr{C} = \{U^0 \times V^0: U, V \in \mathscr{U}\}$, $\mathscr{U}$ a neighborhood base at 0 for E. Applying (3.1), the continuity of addition is a consequence of its continuity on each set in $\mathscr{C}$. Thus (3.4) shows that ew* is a vector topology on E'.

Only the sufficiency of the second criterion requires proof. Suppose there is a uniformity on E' with a base $\mathscr{V}$ of subsets V of $E' \times E'$ satisfying

$$(*) \qquad (x, y) \in V, \quad z \in E' \Rightarrow (x+z, y+z) \in V$$

and suppose that ew* coincides with the uniform topology. If U is a neighborhood of 0 in E', there is $V \in \mathscr{V}$ with $V[0] \subset U$, and there is $V' \in \mathscr{V}$ with $V' \circ V' \subset V$ (notation as in [10]). Let $W = V'[0]$; then W is an ew*-neighborhood of 0, and $(*)$ yields $W + W \subset U$. Thus addition is continuous.

Komura [13] gave an example showing that ew* is not, in general, a vector topology. We prove the following generalization of this result.

THEOREM 3.6. *If (E, d) is a LCS with $d = \sigma(E, E')$ and $\dim E' > \aleph_0$, then the ew* topology is not a vector topology.*

A finite-dimensional vector space admits a unique Hausdorff vector topology [18, p. 21]. Following [9], this topology is called the *natural topology* of the space. Then the *finite topology* on a vector space E is the family of all sets S with the property: If F is a finite-dimensional subspace of E, then $S \cap F$ is open in the natural topology of F. In view of a result of Kakutani and Klee [9], (3.6) is an immediate consequence of the following proposition.

PROPOSITION 3.7. *If (E, d) is a LCS with $d = \sigma(E, E')$, then the ew* topology and the finite topology on E' coincide.*

Proof. Since equicontinuous sets in E' have finite-dimensional linear span, it is immediate that sets open in the finite topology are ew*-open.

Conversely let U be ew*-open and fix a finite-dimensional subspace H of F. There is a family $\{A_n: n \in Z\}$ of naturally open subsets of H which are equicontinuous in E' and satisfy $H = \bigcup_{n=1}^{\infty} A_n$. For each n there is a $\sigma(E', E)$-open set V_n with $U \cap A_n = V_n \cap A_n$. Thus $U \cap H = \bigcup_{n=1}^{\infty} U \cap A_n = \bigcup_{n=1}^{\infty} (V_n \cap H) \cap A_n$ is naturally open.

If E is metrizable, (E', ew^*) is σ-compact regular, hence paracompact [10, p. 172]. Spaces with weak topologies also have this property.

THEOREM 3.8. *If (E, d) is a LCS with $d = \sigma(E, E')$, then the ew* topology on E' is hereditarily paracompact and is perfectly normal (normal, and every closed subset is a G_δ).*

Proof. That a real vector space with finite topology is paracompact and perfectly normal is proved in [16, p. 87]. Moreover, since every open subset is an F_σ, the hereditary paracompactness is a consequence of [5, p. 165].

The result is now immediate from (3.7) and the following observation, whose proof is routine and therefore omitted: If F is a complex vector space, then the finite topologies determined by F and the associated real vector space F_R coincide on the point-set F.

We now give the promised example of a situation in which ew* is not completely regular. Let ω denote the finest locally convex topology on a given vector space [18, p. 56].

THEOREM 3.9. *If the LCS (E, ω) has uncountable dimension, then (E', ew^*) is not completely regular.*

Proof. If $\{x_a: a \in A\}$ is a Hamel basis for E, then $(E', \sigma(E', E))$ is topologically isomorphic to K^A, endowed with the product topology d_A. Since (E, ω) is barrelled, the family of all compact subsets of (K^A, d_A) corresponds to a fundamental family of equicontinuous subsets of E'. Thus (E', ew^*) is homeomorphic to the set K^A endowed with the $\mathscr{C}$-extension $d_A(\mathscr{C})$ of the product topology, where $\mathscr{C}$ is the family of all d_A-compact sets. It is known [10, p. 240] that $d_A(\mathscr{C}) > d_A$ when $\operatorname{card} A > \aleph_0$.

We now reduce the argument to the case: $\operatorname{card} A = \aleph_1$. Indeed if $B \subset A$ with $\operatorname{card} B = \aleph_1$ and if F is the linear span of $\{x_a: a \in B\}$, then (F', ew^*) is homeomorphic to $(K^B, d_B(\mathscr{C}))$. A routine verification shows that the natural embedding of (K^B, d_B) in (K^A, d_A) remains a homeomorphism for the topologies $d_B(\mathscr{C})$ and $d_A(\mathscr{C})$. Thus if $(K^B, d_B(\mathscr{C}))$ is not completely regular, neither is $(K^A, d_A(\mathscr{C}))$.

Suppose, then, that $\operatorname{card} A = \aleph_1$, and let f be any real-valued $d_A(\mathscr{C})$-continuous function on K^A. Then f is d_A-sequentially continuous; a result of Mazur [17, p. 237] shows that f is d_A-continuous.

Since $d_A(\mathscr{C})$ is strictly finer than d_A, yet determines the same continuous real-valued functions on K^A, it cannot be completely regular.

COROLLARY 3.10. *If $\dim E > \aleph_0$, the ew* topology on $(E, \omega)'$ is not locally convex.*

Proof. If it were, it would be a vector topology, hence completely regular.

This extends a result of Collins [2, p. 272].

EXAMPLE 3.11. *If $\operatorname{card} A > \aleph_0$, then K^A is a reflexive LCS which is not ew-topologically complete.*

Proof. The space K^A is reflexive [18, p. 146]. Also since K^A is the entire algebraic dual of $(K^A)'$, it is easy to see that the strong topology on $(K^A)'$ coincides with ω. But $\dim((K^A)') = \operatorname{card} A$, hence by (3.9) the ew topology on K^A (= ew* topology on the dual of $(K^A)'$) is not completely regular. Thus there is no uniformity (complete or not) compatible with (K^A, ew).

4. Locally convex topologies associated with ew*. As the results of Section 3 illustrate, the ew* topology frequently exhibits pathological properties which make it difficult to apply the usual techniques of duality theory. However, the situation improves considerably if we restrict attention to convex neighborhoods of 0 [2, p. 266].

DEFINITION 4.1. The convex ew* (cew*) topology on the dual of a LCS E is the unique locally convex topology with a base at 0 consisting of all convex neighborhoods of 0 in the ew* topology.

The term "cew*" is used by Husain [8]. In view of (2.2i), any dense subspace F of E determines the same cew* topology when E' and F' are identified. We shall also need the following observation due to Kelley [11, p. 246].

FACT 4.2. *If E is a LCS, then cew* on E' is the topology of uniform convergence on compact subsets of the completion of E.*

The restriction to $J(E)$ of the cew* topology on E'' (considered as the dual of E^*) yields a corresponding *rcew* topology* on E.

DEFINITION 4.3. [4, p. 41]. If E is a LCS, the bounded weak (bw) topology on E is the collection of all subsets U of E satisfying: for each bounded subset B of E, there is a $\sigma(E, E')$-open set V with $U \cap B = V \cap B$.

Thus the bw topology is the finest which agrees with the weak topology on bounded sets. In (4.8) we will show that the bw topology on a Banach space need not coincide with the rbw* topology mentioned in Section 1, and, indeed, need not be locally convex.

The ew and bw topologies on E are semi-linear, as follows from (2.2ii) and [2, p. 265]. A general result of Collins [2, p. 266], which can be extended to the complex case, makes the following definitions valid.

DEFINITIONS 4.4. The convex ew (cew) topology (resp., the convex bw (cbw) topology) on a LCS E is the *unique locally convex topology* with a base of all convex neighborhoods of 0 in the ew (resp., bw) topology.

Having now defined locally convex topologies cbw, cew, and rcew* on a LCS E, we show that this apparent variety is illusory.

LEMMA 4.5. *If E is a LCS, let $(E, cbw) = F$. Then F^* is the completion of E^*.*

Proof. A straightforward application of [2, p. 266, Th. 8 (2)] reveals that

$$F' = \{f \in E^{\#}: f \text{ is } \sigma(E, E')\text{-continuous on each bounded subset of } E\}.$$

Then F', endowed with the topology of uniform convergence on $\sigma(E, E')$-bounded subsets of E, is the completion of E^* [18, p. 149]. A subset of E is $\sigma(E, E')$-bounded if and only if it is $\sigma(F, F')$-bounded. The conclusion follows.

THEOREM 4.6. *If E is a LCS, the topologies cbw, cew, and rcew* coincide on E.*

Proof. Since cew* $\leqslant$ ew* on E'', rcew* $\leqslant$ ew on E. But cew is the finest locally convex topology on E which is coarser than ew; hence rcew* $\leqslant$ cew. If B is bounded in E, then ew* agrees with $\sigma(E'', E')$ on $J(B)$. Thus ew agrees with $\sigma(E, E')$ on B, so that ew $\leqslant$ bw on E. In view of (4.4), cew $\leqslant$ cbw. Hence it remains to show that cbw $\leqslant$ rcew*. By virtue of (4.2) and (4.5), it suffices to prove that any $\sigma(F', F)$-closed equicontinuous subset S of F' is $\beta(F', F)$-compact.

Let B be bounded in F. Since S^0 (here polarity is with respect to the duality $\langle F, F'\rangle$) is a cbw-neighborhood of 0, there is a *finite* subset D of $E' \subset F'$ with $S^0 \supset (2B_1) \cap D^0$, where $B_1 = B^{00}$. Then $S^{00} \subset (2B_1 \cap D^0)^0 \subset (1/2)B_1^0 + D^{00}$. There is a finite subset D_1 of F' with $D^{00} \subset D_1 + (2B_1)^0$. Thus $S \subset S^{00} \subset D_1 + B_1^0 = D_1 + B^0$, so that S is $\beta(F', F)$-totally bounded. Since F^* is complete (4.5), the result follows.

COROLLARY 4.7. *If E is a LCS, the finest locally convex topology which agrees with $\sigma(E, E')$ on bounded sets is that of uniform convergence on compact subsets of the completion of E^*.*

An application of (4.6) leads to a counter-example to a remark of Day [4, p. 42].

EXAMPLE 4.8. *The bw topology on the Banach space c_0 is strictly finer than the rbw* topology (= ew topology) and is not locally convex.*

Proof. Since ew* = cew* on c_0'' (2.2iii), ew = rcew* = cbw follows from (4.6). Thus we need only find a bw-closed set which is not ew-closed.

Let $\{e_n: n \in Z\}$ be the usual unit vectors in c_0, and define

$$A = \{x_{n,k}: n, k \in Z\},$$

where $x_{n,k} = \sum_{i=1}^{n} (1/k) e_i + k e_{n+1}$. For fixed k, the sequence $\{x_{n,k}: n \in Z\}$ is $\sigma(l^\infty, l^1)$-convergent to $(1/k)x_0$, where $x_0 = (1, 1, 1, \ldots) \in l^\infty$. It follows that A is bw-closed. Now the ew-closure of A in c_0 is $\bar{A} \cap c_0$, where $\bar{A}$ is the ew*-closure of A in l^∞. Since $0 \in \bar{A}$, A is not ew-closed. Thus bw is strictly finer than cbw; hence by (4.4) it cannot be locally convex.

THEOREM 4.9. *Let E be a LCS, and let $F = (E, cbw)$. Then the following are equivalent.*

(i) *E is semi-reflexive.*

(ii) *F is semi-reflexive.*

(iii) *F is quasi-complete.*

Proof. The equivalence of (i) and (ii) follows from (4.5), since E^* and F^* have the same dual, and it is well known [18, p. 144] that (ii) $\Rightarrow$ (iii). Finally let F be quasi-complete and let $\{x_a: a \in A\}$ be a Cauchy net in $(E, \sigma(E, E'))$, with range in a $\sigma(E, E')$-closed, convex, circled, bounded set B. If W is a cbw-neighborhood of 0, there is a finite subset D of E' with $((1/2)W) \cap B \supset (2D)^0 \cap B$. For some $a_0 \in A$, $a, \beta \geqslant a_0 \Rightarrow x_a - x_\beta \in D^0$. In this case $(1/2)(x_a - x_\beta) \in (2D)^0 \cap B$, so that $x_a - x_\beta \in W$. Thus $\{x_a: a \in A\}$ is a cbw-Cauchy net in B (cbw-bounded and closed), hence there is $x_0 \in B$ with cbw-$\lim x_a = x_0$. Since this holds for the coarser topology $\sigma(E, E')$, E is weakly quasi-complete and therefore semi-reflexive.

5. Strong semi-reflexivity. Theorem (4.9) reassures us that the cbw topology does indeed bear a relation to semi-reflexivity. In particular, it shows that a cbw-complete space is semi-reflexive. However, it is the converse of this implication which is of interest; we introduce it as a definition.

DEFINITION 5.1. A LCS E is *strongly semi-reflexive* if the space (E, cbw) is complete.

Thus a strongly semi-reflexive LCS is semi-reflexive.

The remainder of this section is devoted to establishing certain permanence properties of strongly semi-reflexive spaces, and to demonstrating that in several well-known classes of spaces, the concepts of semi-reflexivity and strong semi-reflexivity are equivalent.

LEMMA 5.2. *If E is a complete bornological LCS, then (E', cew^*) is complete.*

Proof. Let $f \in E^{\#}$, and suppose that the restriction of f to each compact subset of E is continuous. Then f is sequentially continuous, and so $f \in E'$ [18, p. 62]. In view of (4.2) and [18, p. 149], (E', cew^*) is complete.

THEOREM 5.3. *A semi-reflexive space E of any of the following types is strongly semi-reflexive.*

(i) *Metrizable locally convex spaces* [14, p. 392].

(ii) *(DF)-spaces.*

(iii) *(LF)-spaces.*

(iv) *Spaces of countable dimension.*

Proof. (i) E is both bornological and reflexive; thus E^* is both complete and bornological [18, p. 153]. Lemma (5.2) proves that $(E, \text{cbw}) = (E'', \text{cew}^*)$ is complete.

(ii) The space E^* is metrizable, so that $(E, \text{cbw}) = (E'', \text{cew}^*) = (E'', \text{ew}^*)$. The result now follows from (2.2iii) and [18, p. 149].

(iii) If E is a semi-reflexive (LF)-space [18, p. 58], then E is the union of an increasing sequence $\{E_n: n \in Z\}$ of closed subspaces, each of which, with the induced topology, is a reflexive Fréchet space. We recall that

(1) E^* is complete [18, p. 62, p. 148].

(2) If B is a bounded subset of E, then, for some $n \in Z$, $B \subset E_n$ [18, p. 59].

To prove that (E, cbw) is complete, we need to show that, given $L \in E'^{\#}$ and supposing the restriction of L to each compact subset of E^* to be continuous, there is $x \in E$ with $L(f) = f(x)$ for each $f \in E'$.

Let π_n denote the map $f \to f|E_n$ of E' onto E'_n. We claim there is $n \in Z$ such that whenever $\pi_n(f) = 0$, $L(f) = 0$. Suppose not. Then for each n there is $g_n \in E'$ with $\pi_n(g_n) = 0$ but $L(g_n) = 1$. If $s_n = \sum_{k=1}^{n} g_k$, an application of (1) and (2) shows that $\{s_n: n \in Z\}$ converges to some $s_0 \in E^*$. This is a contradiction, since $L(s_0) = \lim L(s_n)$. Thus there must be $m \in Z$ such that $\pi_m(f) = 0$ implies $L(f) = 0$.

Define φ on E'_m by $\varphi(f) = L(g)$, where g is a member of E' such that $\pi_m(g) = f$. Then φ is a well-defined member of $E'^{\#}_m$, and $L = \varphi \circ \pi_m$.

Since E^*_m is bornological (see (5.3i)), it now suffices to show that if D is $\beta(E'_m, E_m)$-bounded, then $\varphi(D)$ is bounded. There is a $\beta(E', E)$-bounded subset C of E' with $\pi_m(C) = D$ [18, p. 133]. Now L, being bounded on compact subsets of E^*, is easily seen, by a sequential argument, to be bounded on bounded sets. Thus $\varphi(D) = L(C)$ is bounded, so that $\varphi \in E''_m = E_m$. This completes the proof.

(iv) If $\dim E = \aleph_0$, it follows from [18, p. 63] that bounded sets have finite-dimensional linear span, so that $\beta(E', E) = \sigma(E', E)$. Thus the completion of E^* is $(E^{\#}, \sigma(E^{\#}, E))$, and $(E, \text{cbw}) = (E, \tau(E, E^{\#})) = (E, \omega)$ is complete.

We remark that a large class of such spaces has been introduced by Köthe [15, p. 378].

EXAMPLE 5.4. The positive results of the preceding theorem hold for certain spaces of continuous functions. If $E = (C(S), \text{co-op})$ denotes the real-valued continuous functions on a completely regular Hausdorff space S, endowed with the *compact-open topology*, then semi-reflexivity of E is equivalent to discreteness of S [19, p. 274]. In this case E is isomorphic to R^S; the cbw topology of R^S coincides with the product topology, using (4.7), and therefore is complete.

Collins [3] has shown that the space $F = (C^*(S), \beta)$, the bounded continuous complex-valued functions on a locally compact space S with the *strict topology*, is semi-reflexive if and only if S is discrete. In this case F^* is the Banach space $l^1(S)$, and $F = F'' = l^\infty(S)$ (algebraically). Then (F, cbw) is $(l^1(S))'$ with the topology of compact convergence, hence is complete.

If X is a Banach space and G is a norm-closed total subspace of X', then the strong dual of $(X, \tau(X, G))$ is G with its norm topology [6, (2.6)]. This leads to a proof of the following variant of [6, (3.2)]: A Banach space X is quasi-reflexive of order n if and only if there is a subspace G of X', norm-closed, total, and of co-dimension n, for which $(X, \tau(X, G))$ is strongly semi-reflexive. In this case the characterization of $\text{aw}(X, G)$ mentioned on page 76 of [6] is valid.

We now obtain several permanence properties of strong semi-reflexivity.

LEMMA 5.5. *Let $\{E_a: a \in A\}$ be a collection of locally convex spaces with locally convex direct sum F and product G. Then the natural maps*

$$S: (F', cew^*) \to \Pi_a(E'_a, cew^*),$$

and

$$T: (G', cew^*) \to \oplus_a(E'_a, cew^*)$$

are topological isomorphisms.

Proof. Let the completion of each E_a be denoted by H_a; then $\oplus_a H_a$ and $\Pi_a H_a$ are the completions of F and G, respectively. In view of the remark following (4.1), for the purpose of verifying the claims concerning S and T we may assume that each E_a is complete. Then the cew* topology on E'_a is the topology of compact convergence, with the same holding for F' and G'. The assertions concerning S and T are now a consequence of [18, p. 136].

THEOREM 5.6. *If $\{E_a: a \in A\}$ is a family of strongly semi-reflexive spaces, then $\Pi_a E_a$ and $\oplus_a E_a$ are strongly semi-reflexive.*

Proof. We prove only that the product is strongly semi-reflexive, the argument for the locally convex direct sum being similar. Let $F_a = (E'_a, \beta(E'_a, E_a))$; then $(E_a, \text{cbw}) = (F'_a, \text{cew}^*)$ is complete for each a, by hypothesis. Let $G = \Pi_a E_a$. Then G is semi-reflexive [18, p. 146], so that $(G, \text{cbw}) = (G'', \text{cew}^*)$. Also $(G', \beta(G', G)) = \oplus_a (F_a, \beta(F_a, E_a))$ [18, p. 192, Problem 8]. The preceding lemma now shows that (G'', cew^*) is isomorphic to $\Pi_a(F^*_a, \text{cew}^*)$; thus (G, cbw) is isomorphic to $\Pi_a(E_a, \text{cbw})$, hence is complete.

THEOREM 5.7. *If E is strongly semi-reflexive and M is a closed subspace, which, with the relative topology of E, is bornological, then M is strongly semi-reflexive.*

Proof. If $i: M \to E$ is the natural embedding, then the adjoint i^* is a continuous map of E^* onto M^*. Since M^* is complete, i^* has a continuous extension T to the completion of E^*. Hence (4.5) shows that (with $F = (E, \text{cbw})$) T is a continuous map of F^* onto M^*; moreover, if i is interpreted as the natural (algebraic) embedding of M in F, then T coincides with the restriction of its (algebraic) adjoint to F'.

Since $\sigma(E, E') \leqslant \text{cbw}$ on E, M is closed, hence complete in the relative topology of F. Denote this topology on M by γ_1, in order to distinguish it from γ_2, the cbw topology which M has in its own right. Since γ_1 agrees with $\sigma(M, M')$ on bounded subsets of M, $\gamma_1 \leqslant \gamma_2$.

We have $(M, \gamma_2)' = M'$ from (4.5). Thus it suffices to show that $(M, \gamma_1)' = M'$, since an application of [18, p. 18] will then complete the proof.

Now (M, γ_1) has the topology of uniform convergence on $\{T(C)$: C a compact, convex, circled subset of $F^*\}$, as follows from (4.7). In view of the strong continuity of T and [18, p. 131], $(M, \gamma_1)' = M'$.

COROLLARY 5.8. *A complete bornological nuclear space is strongly semi-reflexive.*

Proof. Such a space is isomorphic to a closed subspace of a product of reflexive Banach spaces [18, p. 101]. Now apply (5.3i), (5.6), and (5.7).

Remark 5.9. According to (5.3i) and [18, p. 195], a Hausdorff quotient of a strongly semi-reflexive space need not even be semi-reflexive.

THEOREM 5.10. *Let (E, d) be a bornological LCS.*

(i) *If E is strongly semi-reflexive, then E is complete.*

(ii) *If E is complete and a Montel space, then E is strongly semi-reflexive.*

Proof. (i) The topology d is the Mackey topology for the pairing $\langle E, E'\rangle$, while $(E, \text{cbw})' = E'$ since E^* is complete.

(ii) In this case E^* is also a complete Montel space [18, p. 147]. Since closed bounded subsets of E^* are compact, $\text{cbw} = \beta(E, E') = d$.

The preceding result need not hold if E is not bornological.

EXAMPLE 5.11. Komura [13] has obtained an example of a reflexive LCS which is not complete. The space can be interpreted as follows: For each positive integer n, let ω_n be a certain product of copies of R. A subspace E_n of $\omega_n \times \omega_{n+1}$ is selected (the subtlety of the example lies of course in this choice). Then $E = \Pi_n E_n$ is not given the product topology but rather $\sigma(E, F)$, where F is a certain total subspace of $\oplus_n E'_n$. It is shown that $\{\Pi_n B_n$: B_n bounded with finite-dimensional linear span in $E_n\}$ is a fundamental family of bounded subsets of E. Thus $E^* = (F, \beta(F, E))$ is topologically embedded as a dense subspace of $\oplus_n(E^*_n, \sigma(E'_n, E_n))$. Consequently the completion of E^* is $\oplus_n(E^{\#}_n, \sigma(E^{\#}_n, E_n))$, so that $(E, \text{cbw}) = \Pi_n(E_n, \tau(E_n, E^{\#}_n)) = \Pi_n(E_n, \omega)$. Thus E is strongly semi-reflexive but not complete.

6. A possible counter-example.

We are unable to establish with certainty whether there is a semi-reflexive LCS which is not strongly semi-reflexive. One possibility is the following: Suppose S is a completely regular Hausdorff space such that (1) no closed discrete subspace has *measurable cardinal* [7]; (2) S is a *k-space*; (3) S is not *realcompact* and (4) if T is a closed non-compact subspace of S, there is a real-valued continuous function on S whose restriction to T is *unbounded*. (To date we have not succeeded in constructing such a space.) Then, using results from [19] and [7, p. 229], it can be shown that the dual of $(C(S), \text{co-op})$ (notation as in (4.5)), endowed with the weak* topology, is a semi-reflexive space which is not strongly semi-reflexive.

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An extension of Brudno–Mazur–Orlicz theorem

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Abstract. In this paper the Brudno–Mazur–Orlicz consistency theorem is extended to certain BK-spaces. Let E be a closed subspace of the BK-space D, where $\{e^j\}_{j\geqslant 0}$, $e^j = (0, \ldots, 0, 1, 0, \ldots)$, where 1 is in the j-th place, is a Schauder-basis for E, and $\{e^j\}_{j\geqslant 0}$ together with $e \equiv (1, 1, 1, \ldots)$ is a Schauder-basis for D. With such a BK-space E two dual spaces $(\lambda^* E)$ and (λ^*, D), where associated in [4]. Define a matrix A to be regular with respect to D if its convergence field c_A includes D, $\lim_A e = 1$ and $\lim_A x = 0$ for each $x \in E$. The extension mentioned above is the following: Suppose E is solid and $\{e^j\}_{j\geqslant 0}$ is a Schauder-basis for (λ^*, E). If A and B are regular matrices with respect to D and $c_A \supset c_B \cap \mu^*$, then A and B are consistent on (μ^*, E). For $E = c_0$, and $D = c$ we get the Brudno–Mazur–Orlicz theorem. Examples of spaces E, D satisfying all the assumptions are given.

§ 1. Introduction. In this paper Brudno–Mazur–Orlicz famous consistency theorem stating that *if A and B are regular matrices such that the bounded convergence field of A includes that of B, then A and B are consistent for bounded sequences* (See [2, p. 198], also Mazur and Orlicz [5, Theorem 6] and [6] and Zeller [7, Theorem 6.4]) is extended to pairs of matrices which are "regular" in a certain extended sense. We obtain this extension of Brudno–Mazur–Orlicz theorem by increasing the scope of the principle of "aping sequences". The basic idea of the principle of aping sequences was introduced independently by Agnew [1, Theorem 4.1] and Brudno [2]. A general form of this principle was given by Erdös and Piranian [3, Theorem 1]. This principle states, roughly, that a regular matrix transforms each pair of similar bounded sequences into similar bounded sequences.

Terminology and Notation. We denote by E a closed subspace of co-dimension one of a fixed BK-space $(D, \|\cdot\|_D)$. We assume that $\{e^j\}_{j\geqslant 0}$, $e^j = (0, \ldots, 0, 1, 0, \ldots)$, where 1 is in the j-th place, is a Schauder-basis for E and $\{e^j\}_{j\geqslant 0}$ together with $e = (1, 1, 1, \ldots)$ is a Schauder-basis for D. The space $\lambda^* = (\lambda^*, E)$ is defined (see [4, § 5]) as the set of all infinite sequences y satisfying $\|y\|_{\lambda^*} \equiv \sup_{n\geqslant 0} \sup_{\|x\|_E\leqslant 1} |\sum_{k=0}^{n} x_k y_k| < \infty$. It is known (see [4, § 5]) that $(\lambda^*, \|\cdot\|_{\lambda^*})$ is a BK-space and $e^j \in \lambda^*, j \geqslant 0$. We assume that $\{e^j\}_{j\geqslant 0}$ in a Schauder-basis for λ^*, too. Similarly the space μ^* and