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On Lipschitz mappings between Fréchet spaces

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Abstract. The connection between Lipschitz topological structure and linear topological structure is investigated. It is proved that if a Banach space X is Lipschitz embeddable in a reflexive Banach space Y, then X is isomorphic to a subspace of Y. Some similar results for Fréchet spaces are obtained. Only the real case is considered.

1. Introduction. Let X, Y be locally convex topological vector spaces. A mapping F from a subset A of X into Y is said to satisfy the first order Lipschitz condition iff for every continuous pseudonorm $P(\cdot)$ on Y there exists a continuous pseudonorm $Q(\cdot)$ on X and a positive constant K, such that for every pair $x_1, x_2 \in A$

$$P\big(F(x_1)-F(x_2)\big) \leqslant KQ(x_1-x_2).$$

In the following we shall often use "F is a Lipschitz mapping" instead of "F satisfies the first order Lipschitz condition".

In the present note we intend, roughly speaking, to study the question of the connection between Lipschitz topological structure and linear topological structure of metrizable locally convex spaces over the field of reals. Therefore our main interest will be concentrated on invertible Lipschitz mappings whose inverses satisfy the first order Lipschitz condition.

In Section 2 we prove first the extension of a theorem of Gelfand [8] about the existence of the derivative of a Lipschitz mapping from the real line into a reflexive Banach space. The result is that in many situations the Lipschitz mapping from the Hilbert cube C into a Banach space X possesses the derivative in the direction $a \in C$ for almost all points in C. For example this is the case if X is a reflexive Banach space. The rest of this section is devoted to a study of the analogous property for arbitrary Lipschitz mappings from one metrizable locally convex complete topological vector space into another.

In Section 3 we study Lipschitz embeddings of a separable metrizable locally convex vector space into complete metrizable locally convex vector space. It is a well-known fact that the existence of an isomorphic embedding of a Banach space X into a reflexive Banach space implies that X is also reflexive. Similarly if a Fréchet space may be linearly embedded in a Montel–Fréchet space (Schwartz–Fréchet space, nuclear Fréchet space), then it is a Montel (resp. Schwartz, nuclear) space. We shall prove, that under certain additional assumptions, the above statements remain true provided that there exists a Lipschitz embedding. In fact we shall prove that in those cases the existence of a Lipschitz embedding implies the existence of a linear embedding.

In Section 4 we study Lipschitz homeomorphisms between Fréchet spaces to conclude that if X is Lipschitz homeomorphic with a Montel–Fréchet space Y, then X is isomorphic to Y. Since every Schwartz–Fréchet space and every nuclear Fréchet space is a Montel–Fréchet space we infer that in such spaces the linear topological structure is uniquely induced by the Lipschitz topological structure.

In the last section we are interested in uniformly continuous homeomorphisms between Fréchet spaces and we state that if a Fréchet space X is uniformly homeomorphic with a Montel space, then X is Lipschitz embeddable in it. Combining this result with the statement obtained in Section 3 we conclude that if a Fréchet space X is uniformly homeomorphic with a Montel (Schwartz, nuclear) space, then X is a Montel (Schwartz, nuclear) space too.

The method used in this note is very similar to the method used by Enflo in [6] in order to prove that every Banach space uniformly homeomorphic with a Hilbert space is isomorphic to a Hilbert space.

2. The derivative of Lipschitz mappings. Let F be a Lipschitz mapping from a subset B of locally convex space X into another locally convex space Y, and let $a\in X$, $x\in B$ be such that $x+\lambda a\in B$ for sufficiently small λ, $\lambda\in R$ (R denotes the field of real numbers). We say that there exists a derivative of F at the point x and in the direction a iff

$$\lim_{\lambda\to 0}\frac{F(x+\lambda a)-F(x)}{\lambda}=y$$

(in the topology of Y). In this case we write $F'_a(x)=y$. The classical theorem of Rademacher states that if B is a cube in a finite dimensional Banach space, then for every Lipschitz mapping F from B into another finite dimensional Banach space and every direction $a\in X$ the derivative $F'_a(x)$ exists for almost all x in B. In the following we shall need an extension of this theorem (Theorem 1).

Let C be a Hilbert cube, $C=\prod_{i=1}^{\infty}I_i$, where $I_i=[-2^{-i},2^{-i}]$ with standard metric $\varrho_C(\cdot,\cdot)$ defined for $p_1,p_2\in C$, $p_1=(p_{1,1},p_{2,1},\ldots)$,

$p_2=(p_{1,2},p_{2,2},\ldots)$ by the formula

$$\varrho_C(p_1,p_2)=\sum_{i=1}^{\infty}|p_{i,1}-p_{i,2}|.$$

Therefore we can consider C as a subset of l_1. Let us define the product measure $\bar{\mu}=\mathop{\text{X}}_{i=1}^{\infty}\mu_i$ on C, where for every $i\in N$, μ_i is a normalized Lebesgue measure on I_i. From the definition of μ_i we have $\mu_i(I_i)=1$ for $i\in N$, so $\bar{\mu}(C)=1$. It is easy to see that $\bar{\mu}$ is a Radon measure on C. Let μ be the completion of $\bar{\mu}$.

Theorem 1. *Let X be a reflexive Banach space, F a Lipschitz mapping from C into X, and let $a=(a_1,a_2,\ldots,a_k,0,0,\ldots)\in C$. Then the derivative $F'_a(p)$ of F in the direction a exists for μ-almost all p in C.*

The proof of this theorem is based on the following two lemmas.

Lemma 1. *Let f be a real valued function on C satisfying the first order Lipschitz condition. Then for μ-almost all p in C the derivative $f'_a(p)$ exists.*

Proof of Lemma 1. We put

$$\bar{g}_n(p)=\sup\left\{\frac{f(p+\lambda a)-f(p)}{\lambda}:|\lambda|<\frac{1}{n}\right\},$$

$$g_n^{\sim}(p)=\inf\left\{\frac{f(p+\lambda a)-f(p)}{\lambda}:|\lambda|<\frac{1}{n}\right\},$$

for $n\in N$. The functions $\{\bar{g}_n(p)\}_{n\in N}$ and $\{g_n^{\sim}(p)\}_{n\in N}$ are semicontinuous and therefore measurable. Since

$$\left|\frac{f(p+\lambda a)-f(p)}{\lambda}\right|\leqslant K|\lambda|^{-1}\varrho_C(p+\lambda a,p)=K\sum_{i=1}^{k}|a_i|$$

it follows that these functions are finite for every $n\in N$. Observe that for $p\in C$ the sequence $\{\bar{g}_n(p)\}_{n\in N}$ ($\{g_n(p)\}_{n\in N}$) is decreasing (increasing). Let $\bar{g}(p)=\lim_{n\to\infty}\bar{g}_n(p)$ and $g^{\sim}(p)=\lim_{n\to\infty}g_n^{\sim}(p)$ and put

$$A=\{p\in C:\ \bar{g}(p)=g^{\sim}(p)\}.$$

Obviously A is measurable because the functions $\bar{g}(p)$ and $g^{\sim}(p)$ are measurable. On the other hand, it is easy to see that $p\in A$ if and only if the derivative $f'_a(p)$ exists.

Now we consider C as the Cartesian product of a k-dimensional cube $C_k=\prod_{i=1}^{k}I_i$ and a Hilbert cube $C_{\underline{k}}=\prod_{i=k+1}^{\infty}I_i$. In this interpretation the measure

$\bar{\mu}$ becomes the product of measures $\bar{\mu}_k = \bigtimes_{i=1}^{k} \mu_i$ and $\bar{\mu}_{\underline{k}} = \bigtimes_{i=k+1}^{\infty} \mu_i$. In addition we have that $\bar{\mu}_k$ is absolutely continuous with respect to the k-dimensional Lebesgue measure. Therefore we can consider f as a function on the product $C_k \times C_{\underline{k}}$ putting $p = (p_k, p_{\underline{k}})$, where $p_k \in C_k$ and $p_{\underline{k}} \in C_{\underline{k}}$. Since according to the assumption $a = (a_k, 0)$, where $a_k \in C_k$ and $\mathbf{0} = (0, 0, \ldots) \in C_{\underline{k}}$, we infer that the derivative $f'_a(p)$ exists if and only if the function $f_{p_{\underline{k}}}(p_k) = f(p_k, p_{\underline{k}})$ possesses a derivative in the direction a_k at the point p_k. It is easy to see that for every fixed $p_{\underline{k}}$ the function $f_{p_{\underline{k}}}$ defined above is a real valued function satisfying the first order Lipschitz condition. According to the Rademacher theorem we have that for every fixed $p_{\underline{k}}$

$$\bar{\mu}_k(A_{p_{\underline{k}}}) = \bar{\mu}_k(\{p_k \in C_k \colon (p_k, p_{\underline{k}}) \in A\}) = 1.$$

Finally, applying the Fubini theorem we obtain

$$\int_C \chi_A(p)\, d\mu = \int_{C_{\underline{k}}} \Big(\int_{C_k} \chi_{A_{p_{\underline{k}}}}(p_k)\, d\bar{\mu}_k \Big) d\bar{\mu}_{\underline{k}} = \int_{C_{\underline{k}}} 1\, d\bar{\mu}_{\underline{k}} = 1$$

which means that the derivative $f'_a(p)$ exists for μ-almost all $p \in C$ and therefore the lemma is proved.

LEMMA 2. *Let Y be a separable Banach space under the norm $P(\cdot)$. Then there exists an equivalent norm $\|\cdot\|$ on Y such that the sequence $\{y_n\}_{n \in N} \subset Y$ tends to $y_0 \in Y$ if and only if $y_n \to y_0$ in the weak topology and $\|y_n\| \to \|y_0\|$.*

Proof of Lemma 2. Such a norm exists in $C([0, 1])$ (it follows immediately from Kadec's renorming theorem [10]). Since every separable Banach space is isomorphic to a subspace of $C([0, 1])$ we infer that such a norm exists in Y which concludes the proof of the lemma.

Proof of Theorem 1. Let $Y = \overline{\operatorname{span} F(C)}$. Obviously Y is a separable reflexive space. Let $\|\cdot\|$ be a norm on Y satisfying the conditions of Lemma 2.

Y^* is separable. Let $\{z_n\}_{n \in N}$ be a dense sequence in the unit sphere of Y^*. For every $n \in N$, $z_n(F(p))$ is a real valued function satisfying the first order Lipschitz condition. Put

$$M = \{p \in C \colon (z_n \circ F)'_a(p) \text{ exists for every } n \in N\}.$$

It follows from Lemma 1 that $\mu(M) = 1$. Since

$$\left\| \frac{F(p + \lambda a) - F(p)}{\lambda} \right\| \leqslant K \sum_{i=1}^{k} |a_i|$$

and the ball $\{x \in X \colon \|x\| \leqslant K\}$ is weakly compact we have that for $p \in M$ there exists a weak derivative $F^*_a(p)$. Indeed, let $p \in M$; then for $n \in N$ there exists

$$\lim_{\lambda \to 0} z_n \left(\frac{F(p + \lambda a) - F(p)}{\lambda} \right).$$

Hence it is enough to observe that since $\{z_n\}_{n \in N}$ is a fundamental subset of Y^*, $\{\lambda^{-1}(F(p + \lambda a) - F(p))\}$ is a weak Cauchy sequence $(\lambda \to 0)$.

Take an $\varepsilon > 0$. The Lusin theorem implies that there exists a compact subset $M_\varepsilon \subset M$ such that for every $n \in N$, $(z_n \circ F)'_a(p)$ restricted to M_ε is continuous and $\mu(M_\varepsilon) \geqslant \mu(M) - \varepsilon = 1 - \varepsilon$. Since for $p \in M_\varepsilon$

$$\|F^*_a(p)\| = \sup \{ |(z_n \circ F)'_a(p)| \colon n \in N \}$$

and for every $n \in N$ the function $(z_n \circ F)'_a(p)$ is continuous in M_ε it follows that $\|F^*_a(p)\|$ restricted to M_ε is measurable. Putting $\varepsilon_n = 1/n$ we obtain that $\|F^*_a(p)\|$ restricted to $M_\infty = \bigcup_{n=1}^{\infty} M_{1/n}$ is measurable and $1 \geqslant \mu(M_\infty) \geqslant \mu(M_{1/n}) \geqslant 1 - 1/n$ for $n \in N$. Therefore the measurability of $\|F^*_a(p)\|$ is established.

Now we shall prove that the set A of $p \in C$ such that $F'_a(p)$ exists is measurable. Obviously the existence of $F'_a(p)$ implies the existence of $F^*_a(p)$ and we have $F'_a(p) = F^*_a(p)$. Take an $\varepsilon > 0$. We can find a compact subset $M_\varepsilon \subset M_\infty$ such that $\|F^*_a(p)\|$ and $(z_n \circ F)'_a(p)$ restricted to M_ε are continuous and $\mu(M_\varepsilon) \geqslant \mu(M_\infty) - \varepsilon = 1 - \varepsilon$. It follows from the properties of the norm $\|\cdot\|$ that $F^*_a(p)$ restricted to M_ε is continuous.

Define for $n \in N$ and $p \in M_\varepsilon$

$$g_n(p) = \sup \left\{ \left\| \frac{F(p + \lambda a) - F(p)}{\lambda} - F^*_a(p) \right\| \colon |\lambda| < \frac{1}{n} \right\}.$$

It is easy to see that for every $n \in N$, $g_n(p)$ is a semicontinuous measurable bounded function and that the sequence $\{g_n(p)\}_{n \in N}$ decreases for $p \in M_\varepsilon$. Therefore $\lim_{n \to \infty} g_n(p) = g(p)$ for every $p \in M_\varepsilon$ and $g(p)$ is measurable. It follows from the definition of $g(p)$ that for $p \in M_\varepsilon$, $F'_a(p)$ exists if and only if $g(p) = 0$. Hence the set $A \cap M_\varepsilon$ is measurable for every $\varepsilon > 0$, and in an analogous way as before we can obtain that A is measurable.

The rest of this proof is similar to the final part of the proof of Lemma 1, so we omit it. The only difference is that instead of the Rademacher theorem we use the following extension of Gelfand's theorem:

(G) *For every Lipschitz mapping F of an n-dimensional cube C_n in R^n into a reflexive Banach space X and for every direction $a \in R^n$ the derivative $F'_a(p)$ exists for almost all $p \in C_n$.*

In fact Gelfand has proved this theorem for $n = 1$. However, the n-dimensional case is an easy consequence of this result and the methods used above.

THEOREM 2. *Let X be a Fréchet space and let X_n be a reflexive Banach space for every $n \in N$. Then for every countable set of directions $A = \{a_n\}_{n\in N}$ in X and for every sequence $\{F_n\}_{n\in N}$ of Lipschitz mappings, $F_n\colon X \to X_n$, the set D of $x \in X$ such that the derivatives $F'_{n,a_k}(x)$ exist for $n, k = 1, 2, \ldots$ is dense in X.*

Proof. It is sufficient to prove that the closure of D contains the origin. The case when $x_0 \in X$ is an arbitrary point may easily be reduced to the previous one by the substitution

$$F_n^{\sim}(x) = F_n(x - x_0).$$

Since for every mapping F the existence of $F'_a(x)$ is equivalent to the existence of $F'_{\lambda a}(x)$ for every λ different from 0 it follows that without loss of generality we can assume that the set A is bounded (otherwise we multiply each element a_k of A by a sufficiently small constant). Let $B = \{b_i\}_{i\in N}$ be a maximal linearly independent subset of A. The mapping $F\colon C \to X$, defined for $p = (a_1, a_2, \ldots) \in C$ by the formula

$$F(p) = F\big((a_1, a_2, \ldots)\big) = \sum_{i=1}^{\infty} a_i b_i$$

satisfies the first order Lipschitz condition because for every $p, q \in C$, $p = (a_1, a_2, \ldots)$, $q = (\beta_1, \beta_2, \ldots)$ and every continuous pseudonorm $Q(\cdot)$ on X

$$Q\big(F(p) - F(q)\big) = Q\Big(\sum_{i=1}^{\infty}(a_i - \beta_i) b_i\Big) \leqslant \sum_{i=1}^{\infty} |a_i - \beta_i| Q(b_i)$$
$$\leqslant \sup\{Q(b_i)\colon i \in N\} \sum_{i=1}^{\infty} |a_i - \beta_i| = K\varrho_C(p, q).$$

Hence for every $n \in N$ the mapping $\bar{F}_n = F_n \circ F$, $\bar{F}_n\colon C \to X_n$ satisfies the first order Lipschitz condition (the composition of two Lipschitz mappings is a Lipschitz mapping too). It follows from the definition of B that every element of A may be uniquely written in the form

$$a_k = a_{k,1} b_1 + a_{k,2} b_2 + \ldots + a_{k,m_k} b_{m_k}.$$

Dividing each a_k by $\lambda_k = \max\{|a_{k,i}| 2^i\colon i \leqslant m_k\}$ we have that $\bar{a}_k = \lambda_k^{-1} a_k \in F(C)$ for every $k \in N$. It is clear that $p_k = (\lambda_k^{-1} a_{k,1}, \lambda_k^{-1} a_{k,2}, \ldots \ldots, \lambda_k^{-1} a_{k,m_k}, 0, 0, \ldots)$ belong to C and $F(p_k) = \bar{a}_k$ for every $k \in N$. Applying Theorem 1 to the Lipschitz mapping $\bar{F}_n$ and the direction p_k, we obtain

that the set $M_{n,k}$ of those $p \in C$ such that the derivative $\bar{F}'_{n,p_k}(p)$ exists has full measure for $n, k = 1, 2, \ldots$ Hence the set

$$M = \bigcap_{n,k=1}^{\infty} M_{n,k} = \{p \in C\colon \bar{F}'_{n,p_k}(p) \text{ exists for } n, k \in N\}$$

has measure equal to 1. It is easy to verify that for every $n, k \in N$ and for every p in the radial interior of C the existence of the derivative $\bar{F}'_{n,p_k}(p) = (F_n \circ F)'_{p_k}(p)$ implies the existence of the derivative $F'_{n,F(p_k)}(F(p)) = F'_{n,a_k}(F(p))$. Since every open subset of C has positive measure we infer that there exists a sequence $\{q_m\}_{m\in N}$, $q_m \in M$, $q_m \to (0, 0, \ldots)$. It follows from the continuity of F that $F(q_m) \to \mathbf{0}$ and the derivatives $F'_{n,a_k}(F(q_m))$ exist for every $m, n, k \in N$, so the closure of D contains the origin. This concludes the proof of the theorem.

DEFINITION 1. A Fréchet space X is said to be a *super reflexive Fréchet space* iff the topology on X can be defined by a system of pseudonorms $\{Q_m\}_{m\in N}$ such that for every $m \in N$ the completion $\tilde{X}_m$ of the quotient space $X_m = X/Q_m$ is a reflexive Banach space.

The following theorem is a nearly immediate consequence of Definition 1 and Theorem 2.

THEOREM 3. *Let X be a Fréchet space and let for every $n \in N$, X_n be a super reflexive Fréchet space. Then for every countable set of directions $A = \{a_k\}_{k\in N}$ in X and for every sequence $\{F_n\}_{n\in N}$ of Lipschitz mappings, $F_n\colon X \to X_n$, the set D of $x \in X$ such that the derivatives $F'_{n,a_k}(x)$ exist for $n, k = 1, 2, \ldots$ is dense in X.*

Proof. For every $n \in N$, let $\{Q_{n,m}\}_{m\in N}$ be a sequence of pseudonorms defining the topology on X_n such that for every $m \in N$ the completion of $X_{n,m} = X_n/Q_{n,m}$ is a reflexive Banach space. Consider a double sequence $\{\bar{F}_{n,m}\}_{n,m\in N}$, $\bar{F}_{n,m} = I_{n,m} \circ F_n$ of mappings from X into $\tilde{X}_{n,m}$, where $I_{n,m}$ denotes the canonical mapping from X_n into $\tilde{X}_{n,m}$, $n, m = 1, 2, \ldots$ It is clear that for every $n, m \in N$ the mapping $\bar{F}_{n,m}$ satisfies the first order Lipschitz condition. Applying the above Theorem to the family of mappings $\{\bar{F}_{n,m}\}_{n,m\in N}$ and the set of directions A we obtain that the set of $x \in X$ such that the derivatives $\bar{F}'_{n,m,a_k}(x)$ exist for $n, m, k = 1, 2, \ldots$ is dense in X. Obviously it is equal to D and this concludes the proof of the theorem.

Theorem 3 implies the following corollary.

COROLLARY 1. *For every Lipschitz mapping F from a separable Fréchet space X into a super reflexive Fréchet space the set of $x \in X$ such that the derivative $F'_a(x)$ exists for a dense subset of directions is dense in X.*

3. Lipschitz embeddings into super reflexive Fréchet spaces.

DEFINITION 2. A mapping F from X into Y is said to be a *Lipschitz embedding* of X into Y iff it is one-to-one and both F and F^{-1} satisfy the first order Lipschitz condition.

Let F be a Lipschitz embedding of a Fréchet space X into a super reflexive Fréchet space Y. Without loss of generality we may assume that the topologies on X and Y are respectively given by increasing sequences of pseudonorms $\{Q_n\}_{n\in N}$, $\{P_n\}_{n\in N}$ and that there exist sequences of constants $\{K_n\}_{n\in N}$ and $\{M_n\}_{n\in N}$ such that for every $n\in N$ and every pair of points $x_1, x_2\in X$ the following inequalities hold:

$$Q_n(x_1-x_2)\leqslant K_nP_{n+1}\big(F(x_1)-F(x_2)\big), \tag{1}$$

$$P_n\big(F(x_1)-F(x_2)\big)\leqslant M_nQ_n(x_1-x_2), \tag{2}$$

and in addition the completion of $Y_n = Y/P_n$ for every positive integer n is a reflexive Banach space.

Let $\bar F$ be another Lipschitz embedding of $X_0\subset X$ into Y. We say that $\bar F$ satisfies the first order Lipschitz condition with the same set of constants (or $\bar F$ is a Lipschitz embedding with the same set of constants) as F iff $\bar F$ satisfies the inequalities

$$Q_n(x_1-x_2)\leqslant K_nP_{n+1}\big(\bar F(x_1)-\bar F(x_2)\big), \tag{1'}$$

$$P_n\big(\bar F(x_1)-\bar F(x_2)\big)\leqslant M_nQ(x_1-x_2), \tag{2'}$$

for $x_1, x_2\in X_0$.

Let F be a Lipschitz embedding of X into Y. We define for every $x_0\in X$

$$A(x_0) = \{a\in X\colon\ F'_a(x_0)\ \text{exists}\}.$$

Suppose that $A(x_0)$ is dense in X. Let X_0 be the set of $x\in X$ of the form $x = x_0+va$, where $v\in R$, $a\in A(x_0)$. Define the mapping $F^{\sim}_{x_0}\colon X_0\to Y$ by the formula

$$F^{\sim}_{x_0}(x_0+va) = F(x_0)+vF'_a(x_0).$$

It is clear that the mapping $F^{\sim}_{x_0}$ is the Lipschitz embedding of X_0 into Y with the same set of constants as F. This last remark follows immediately from the inequality

$$\begin{aligned}Q_{n-1}(v_1a_1-v_2a_2) &= |\lambda|^{-1}Q_{n-1}(\lambda v_1a_1-\lambda v_2a_2)\\ &\leqslant K_{n-1}P_n\left(\frac{F(x_0+\lambda v_1a_1)-F(x_0+\lambda v_2a_2)}{\lambda}\right)\\ &\leqslant K_{n-1}M_nQ_n(v_1a_1-v_2a_2)\end{aligned}$$

for $v_1, v_2, \lambda\in R$, $|\lambda|\neq 0$ and $a_1, a_2\in A(x_0)$, and from the identity

$$\begin{aligned}F^{\sim}_{x_0}(x_0+v_1a_1)-F^{\sim}_{x_0}(x_0+v_2a_2) &= v_1F'_{a_1}(x_0)-v_2F'_{a_2}(x_0)\\ &= \lim_{\lambda\to 0}\frac{\big(F(x_0+\lambda v_1a_1)-F(x_0)\big)-\big(F(x_0+\lambda v_2a_2)-F(x_0)\big)}{\lambda}\\ &= \lim_{\lambda\to 0}\frac{F(x_0+\lambda v_1a_1)-F(x_0+\lambda v_2a_2)}{\lambda}\end{aligned}$$

for $v_1, v_2\in R$, $a_1, a_2\in A(x_0)$.

Since $A(x_0)$ is dense in X, the set X_0 is dense in X and the mapping $F^{\sim}_{x_0}$ can be extended by continuity to the mapping F_{x_0} defined on X. Obviously F_{x_0} is the Lipschitz embedding of X into Y with the same set of constants as F.

We shall need the following result (Enflo [6]).

LEMMA 3. (On linearization of Lipschitz embeddings "step by step"). *Let F be a Lipschitz embedding of a locally convex space X into a locally convex space Y. If the following two statements hold,*

(i) *there exists a linear subspace $X_0\subset X$ with the property that for every $x\in X$ the mapping F restricted to span $\{X_0, x\}$ is linear, and*

(ii) *there exists $x_0\in X\setminus X_0$ such that $A(x_0)$ is a dense subset of directions, then the above defined mapping F_{x_0} satisfies conditions*

(iii) *F_{x_0} is a Lipschitz embedding of X into Y with the same set of constants as F,*

(iv) *for every $x\in X$, F_{x_0} restricted to* span$\{X_0, x_0, x\}$ *is linear, and*

(v) *F_{x_0} restricted to* span$\{X_0, x_0\}$ *is equal to F restricted to* span$\{X_0, x_0\}$.

Implication (ii) ⇒ (iii) was proved, and the other properties of F_{x_0} can be easily verified.

THEOREM 4. *If a separable Fréchet space X is Lipschitz embeddable in a super reflexive Fréchet space Y, then X is isomorphically embeddable in Y with the same set of constants.*

Proof. Let F be a Lipschitz embedding of X into Y and let $\{a_n\}_{n\in N}$ be a dense sequence in X. Let $[n, m](k) = [n(k), m(k)]$ be one-to-one function from the set N of positive integers onto $N\times N$. By Corollary 1 there exists a point $x_0\in X$ such that $A(x_0)$ is a dense set of directions. Without loss of generality we can assume that $x_0 = \mathbf{0}$ and $F(\mathbf{0}) = \mathbf{0}$. It was shown that $F_\mathbf{0}$ is the Lipschitz embedding of X into Y with the same set of constants as F. It is easy to see that $F_\mathbf{0}$ is a linear mapping on each one-dimensional subspace of X, so it satisfies the assumption of the previous lemma. We shall define by induction the sequence $\{F_k\}_{k\in N}$ of Lipschitz embeddings. Suppose that we have defined the embeddings

$F_0, F_1, F_2, \ldots, F_{k-1}$. Let ϱ_X be an arbitrary fixed metric on X. It follows from Corollary 1 that there exists a point $x_k \in X$ such that

1° the set $A(x_k)$ of directions for which the derivatives of F_{k-1} exist is dense in X,

2° $\varrho_X(a_{n(k)}, x_k) < \dfrac{1}{m(k)}$.

Define $F_k = F_{k-1,x_k}$. By Lemma 3 we have that F_k is a Lipschitz embedding of X into Y with the same set of constants as F. In this way we can assume that we have defined the whole sequence $\{F_k\}_{k\in N}$. Put

$$X_k = \operatorname{span}\{x_i\colon\ i \leqslant k+1\}.$$

It follows from the lemma on linearization of Lipschitz embeddings that for every $k, l \in N$

3° F_k is a Lipschitz embedding with the same set of constants as F.

4° F_k restricted to X_k is a linear mapping.

5° $F_k(x) = F_l(x)$ for $x \in X_k \cap X_l$.

Put for $x \in X_\infty = \bigcup_{k=1}^{\infty} X_k$

$$F_\infty^{\sim}(x) = \lim_{k\to\infty} F_k(x).$$

According to 5° the limit on the right-hand side is unessential because for $x \in X_n$ we have $F_k(x) = F_n(x)$ for $k \geqslant n$. It is clear that the mapping $F_\infty^{\sim}$ from X_∞ into Y is a Lipschitz embedding of X_∞ into Y with the same set of constants as F. Obviously $F_\infty^{\sim}$ is a linear mapping. Since according to 2°

$$\overline{\{x_k\}_{k\in N}} \supset \overline{\{a_n\}_{n\in N}} \supset X,$$

the closure of X_∞ contains $\{a_n\}_{n\in N}$. Hence X_∞ is a dense subspace of X. Therefore there exists a unique extension F_∞ of $F_\infty^{\sim}$ to the Lipschitz embedding of $\overline{X}_\infty = X$ into Y with the same set of constants as F. Since $F_\infty^{\sim}$ is linear on X_∞, we infer that F_∞ is linear on X. Hence F_∞ is an isomorphic embedding of X into Y with the same set of constants as F so the theorem is proved.

COROLLARY 2. *If a separable Fréchet space X is Lipschitz embeddable in a super reflexive Fréchet space, then X is a super reflexive Fréchet space.*

This corollary is an easy consequence of Theorem 4.

Since every Montel–Fréchet space is separable, we have the following

COROLLARY 3. (i) *If a Fréchet space X is Lipschitz embeddable in a super reflexive Montel–Fréchet space, then X is a Montel space.*

(ii) *If a Fréchet space X is Lipschitz embeddable in a super reflexive Schwartz–Fréchet space, then X is a Schwartz space.*

(iii) *If a Fréchet space X is Lipschitz embeddable in a nuclear space, then X is a nuclear space.*

(iii) follows from the fact that the topology on a nuclear space can be given by a system of prehilbertian pseudonorms (Pietsch [12]). Hence every nuclear space is a separable super reflexive Fréchet space.

Similarly we have the following

THEOREM 5. *If a Banach space X is Lipschitz embeddable in a reflexive Banach space Y, then X is a reflexive Banach space.*

Proof. It is an immediate consequence of Theorem 4 and the following well-known fact: a Banach space X is reflexive if and only if every separable closed subspace of X is reflexive.

Remark. Observe that since for $1 < p < \infty$, $p \neq 2$, L_p is not isomorphic to any subspace of l_p it follows from Theorem 4 that L_p is not Lipschitz homeomorphic with l_p for $1 < p < \infty$, $p \neq 2$.

4. The Lipschitz homeomorphisms of super reflexive Montel–Fréchet spaces. Let F be a Lipschitz homeomorphism([1]) between two Fréchet spaces X, Y. Suppose that one of them, say X, is a Montel space. The definition of Lipschitz mappings implies that the image of a bounded set is bounded. Hence Y is a Montel space. Assume that Y is a super reflexive Fréchet space. Let $x_0 \in X$ be such that $A(x_0)$ is dense in X. Consider F_{x_0} (for the definition of $A(x_0)$ and F_{x_0} see above). Then we have the following

LEMMA 4. *F_{x_0} is a Lipschitz homeomorphism between X and Y with the same set of constants as F.*

Proof. Without loss of generality we can assume that $x_0 = \mathbf{0}$ and $F(\mathbf{0}) = \mathbf{0}$. Since X and Y are complete and F_0 is a Lipschitz embedding, it is enough to show that $F_0(A(\mathbf{0}))$ is a dense subset of Y. Assume the contrary. Let $y_0 \notin \overline{F_0(A(\mathbf{0}))}$. Since $F_0(\mathbf{0}) = \mathbf{0}$ we infer that $y_0 \neq \mathbf{0}$. Hence there exists $n \in N$ and $\varepsilon > 0$ such that $P_{n-1}(y_0) > 0$ and

$$(3)\qquad \inf\{P_n(y_0 - F_0(x))\colon\ x \in A(\mathbf{0})\} > \varepsilon.$$

Let us define

$$(4)\qquad V = \bigcup_{\gamma\in R} \{y \in Y\colon\ P_n(\gamma y_0 - y) \leqslant \varepsilon\gamma\}.$$

It is easy to see that V is radial. Moreover, $V = \overline{V}$. Indeed, suppose that $y_n \in V$, $y_n \to \bar{y}$. Hence there exists a bounded sequence $\{\gamma_m\}_{m\in N}$ of scalars such that

$$(5)\qquad P_n(\gamma_m y_0 - y_m) \leqslant \varepsilon\gamma_m.$$

([1]) Which means that F is a Lipschitz embedding of X onto Y.

If $\{\gamma_m\}_{m\in N}$ tends to zero, then we have $P_n(\bar{y}) = 0$ and $\bar{y}\in V$. Otherwise we can choose an increasing subsequence $\{m_k\}_{k\in N}$ of positive integers such that the sequence $\{\gamma_{m_k}^{-1}\}_{k\in N}$ is convergent to some γ_0^{-1}. Therefore it follows from (5) that

$$P_n(y_0 - \gamma_0^{-1}\bar{y}) = \lim_{k\to\infty} P_n(y_0 - \gamma_{m_k}^{-1} y_{m_k}) \leqslant \varepsilon.$$

This implies that $P_n(\gamma_0 y_0 - \bar{y}) \leqslant \varepsilon\gamma_0$. Hence $\bar{y}\in V$.

Put

$$A^{\sim}(\mathbf{0}) = A(\mathbf{0}) - \{a\in A(\mathbf{0}) \colon Q_{n-1}(a) = 0\}.$$

Since $A(\mathbf{0})$ is dense in X and the set $\{x\in X\colon Q_{n-1}(x) = 0\}$ is nowhere dense in X we infer that $A^{\sim}(\mathbf{0})$ is dense in X. Since $F_0(A(\mathbf{0}))$ is radial we obtain, using (1), (2) and (3), (4) that $V\cap F_0(A^{\sim}(\mathbf{0})) = \emptyset$. Since V is closed and

$$F_0(a) = F'_a(\mathbf{0}) = \lim_{\lambda\to 0}\frac{F(\lambda a)}{\lambda} \quad \text{for } a\in A^{\sim}(\mathbf{0})$$

we obtain that for every $a\in A^{\sim}(\mathbf{0})$ there exists $\delta(a) > 0$ such that $\lambda^{-1}F(\lambda a)\notin V$ for $|\lambda| < \delta(a)$, $\lambda\neq 0$. Hence $F(\lambda a)\notin V$ and

$$\inf\left\{\frac{P_n(\gamma y_0 - F(\lambda a))}{|\gamma|}\colon \gamma\neq 0\right\} > \varepsilon. \tag{6}$$

Observe that according to (2) we have for $\gamma\neq 0$

$$0 < P_{n-1}(\gamma y_0) \leqslant M_{n-1}Q_{n-1}(F^{-1}(\gamma y_0)).$$

Define $x(\gamma) = F^{-1}(\gamma y_0)$ for $\gamma\neq 0$. Applying (1), (2) and (6) we obtain the following inequality for $0\neq\gamma$, $a\in A^{\sim}(0)$ and $|\lambda| < \delta(a)$,

$$\frac{Q_n(x(\gamma) - \lambda a)}{Q_{n-1}(x(\gamma))} \geqslant \frac{P_n(\gamma y_0 - F(\lambda a))}{M_n K_{n-1} P_n(\gamma y_0)} = \frac{P_n(\gamma y_0 - F(\lambda a))}{M_n K_{n-1} P_n(y_0)|\gamma|} > \frac{\varepsilon}{M_n K_{n-1} P_n(y_0)}. \tag{7}$$

On the other hand it is easy to verify that the set B

$$B = \left\{\frac{x(\gamma)}{Q_{n-1}(x(\gamma))}\right\}_{\gamma\neq 0}$$

is bounded. Indeed, for every pseudonorm Q_m on X we have

$$Q_m\left(\frac{x(\gamma)}{Q_{n-1}(x(\gamma))}\right) \leqslant \frac{K_m M_{n-1} P_{m+1}(\gamma y_0)}{P_{n-1}(\gamma y_0)} = \frac{K_m M_{n-1} P_{m+1}(y_0)}{P_{n-1}(y_0)}$$

for $\gamma\in R$, $\gamma\neq 0$. Since X is a Montel space it follows that the set B is precompact. Let $\{\gamma_k\}_{k\in N}$ be a sequence of scalars such that the sequence

$$\left\{\frac{x(\gamma_k)}{Q_{n-1}(x(\gamma_k))}\right\}_{k\in N}$$

tends to certain point $x_0\in X$, and $\gamma_k\to 0$. Observe that $Q_{n-1}(x_0) = 1$. Since $A^{\sim}(0)$ is dense in X we infer that there exists $a\in A^{\sim}(\mathbf{0})$ such that

$$\frac{\varepsilon}{M_n K_{n-1} P_n(y_0)} > Q_n(x_0 - a) = \lim_{k\to\infty} Q_n\left(\frac{x(\gamma_k)}{Q_{n-1}(x(\gamma_k))} - a\right) = \lim_{k\to\infty}\frac{Q_n\big(x(\gamma_k) - Q_{n-1}(x(\gamma_k))a\big)}{Q_{n-1}(x(\gamma_k))}. \tag{8}$$

Since $\gamma_k y_0 = F(x(\gamma_k))$ tends to the origin (as $k\to\infty$), we have tha $x(\gamma_k)\to 0$. Thus $Q_{n-1}(x(\gamma_k))\to 0$ $(k\to\infty)$. Therefore there exists k_0 such that for $k > k_0$, $Q_{n-1}(x(\gamma_k)) < \delta(a)$. From (8) we have that there exists $k_1 > k_0$ such that

$$\frac{Q_n\big(x(\gamma_k) - Q_{n-1}(x(\gamma_k))a\big)}{Q_{n-1}(x(\gamma_k))} < \frac{\varepsilon}{M_n K_{n-1} P_n(y_0)}$$

for $k > k_1$, $a\in A^{\sim}(\mathbf{0})$ and also $Q_{n-1}(x(\gamma_k)) < \delta(a)$, so we obtain the contradiction with (7) which concludes the proof of the lemma.

THEOREM 6. *Let a Fréchet space X be Lipschitz homeomorphic with a super reflexive Montel–Fréchet space Y. Then X is isomorphic to Y (with the same set of constants).*

Proof. Observe that if X is Lipschitz homeomorphic with a Montel space, then it is a Montel space. Hence by Corollary 3 we have that X is a super reflexive Montel–Fréchet space.

The idea of this proof is similar to the idea of the proof of Theorem 4. The only difference is that having Lemma 4 we can modify the proof to derive the linear embedding F_∞ of X into Y which is onto. Let F be a Lipschitz homeomorphism of X onto Y such that $F(\mathbf{0}) = \mathbf{0}$ and $A(\mathbf{0})$ is dense in X. Let $\{a_n\}_{n\in N}$ be a countable subset dense in X and $\{b_n\}_{n\in N}$ be a countable subset dense in Y. Let F_0 and $[n, m](k)$ be the same as in the proof of Theorem 4 and ϱ_Y be an arbitrary fixed metric on Y. We shall define the sequence $\{F_k\}_{k\in N}$ of Lipschitz homeomorphisms between X and Y by induction. Suppose that we have defined the Lipschitz homeomorphisms $F_0, F_1, F_2, \ldots, F_{k-1}$ (with the same set of constants as F). Consider $[n, m](k)$. If $m(k)$ is even, then we define F_k as in the proof of Theorem 4, and by Lemma 4 we obtain that F_k is a Lipschitz homeomorphism with the same set of constants as F. Otherwise using Corollary 1 we can find x_k such that

1° the set $A(x_k)$ of directions for which the derivatives of F_{k-1} exist is dense in X,

2° $\varrho_Y(b_{n(k)}, F_{k-1}(x_k)) < 1/m(k)$.

In this case we define $F_k = F_{k-1,x_k}$. By Lemma 4, F_k is a Lipschitz homeomorphism with the same set of constants as F. In this way we can assume that the whole sequence $\{F_k\}_{k\in N}$ of Lipschitz homeomorphisms is defined. Put

$$X_k = \operatorname{span}\{x_i:\ i \leqslant k+1\}.$$

It follows from the lemma on linearization of Lipschitz embeddings that for every $k, l \in N$

3° F_k is a Lipschitz homeomorphism with the same set of constants as F,

4° F_k restricted to X_k is linear,

5° $F_k(x) = F_l(x)$ for $x \in X_k \cap X_l$.

Put for $x \in X_\infty = \bigcup_{k=1}^{\infty} X_k$

$$F_\infty^{\sim}(x) = \lim_{k\to\infty} F_k(x).$$

By similar argument as in the proof of Theorem 4 we can obtain that X_∞ is dense in X and $F_\infty^{\sim}$ is a well defined Lipschitz embedding of X_∞ into Y with the same set of constants as F. In addition since according to 2° $\overline{\{b_n\}_{n\in N}} \subset \overline{\{F_\infty^{\sim}(x_k)\}_{k\in N}}$, it follows that $\overline{\{F_\infty^{\sim}(x_k)\}_{k\in N}} = \overline{\{b_n\}_{n\in N}} = Y$. This and 5° yields that $F_\infty^{\sim}(X_\infty)$ is dense in Y. Since $F_\infty^{\sim}$ is a Lipschitz embedding of a dense subset of a complete space X onto a dense subset of a complete space Y it follows that $F_\infty^{\sim}$ can be uniquely extended to a Lipschitz homeomorphism F_∞ of X onto Y with the same set of constants as F. Observe that $F_\infty^{\sim}$ is a linear mapping on X_∞. Hence F_∞ is linear mapping and so the theorem is proved.

Since every nuclear Fréchet space is a super reflexive Montel–Fréchet space we have the following

Corollary 4. *If a Fréchet space X is Lipschitz homeomorphic with a nuclear space, then X is isomorphic to this space.*

Remark. The results of the Sections 2 and 3 remain true if we consider the Lipschitz mappings defined on an arbitrary non-empty open subset of X. For the results of the present section to remain true in this case is sufficient to assume that there exists a Lipschitz embedding of an open non-empty subset of X onto an open non-empty subset of Y.

5. Uniformly continuous homeomorphisms between super reflexive Montel–Fréchet spaces.

Definition 3. An embedding F of a Fréchet space X into a Fréchet space Y is said *to satisfy the first order Lipschitz condition on large distance* iff the topologies on X and Y can be respectively defined by two increasing sequences of pseudonorms $\{Q_n\}_{n\geqslant 0}$ and $\{P_n\}_{n\in N}$ and there exist sequences of constants $\{K_n\}_{n\in N}$ and $\{M_n\}_{n\in N}$ such that for $x_1, x_2 \in X$ the inequality $Q_0(x_1-x_2) \geqslant 1$ implies that for every $n \in N$

$$Q_n(x_1-x_2) \leqslant K_n P_{n+1}\big(F(x_1)-F(x_2)\big) \tag{9}$$

and

$$P_n\big(F(x_1)-F(x_2)\big) \leqslant M_n Q_n(x_1-x_2). \tag{10}$$

The following lemma states the well-known property of uniformly continuous homeomorphisms between Fréchet spaces.

Lemma 5. *Let F be an uniformly continuous homeomorphism of a Fréchet space X onto a Fréchet space Y. Then F satisfies the first order Lipschitz condition on large distance.*

Proof. Let $\{U_n\}_{n\geqslant 0}$ and $\{V_n\}_{n\in N}$ denote respectively bases of symmetric convex neighbourhoods of the origin in X and Y. Let Q_0 be the Minkowski functional of U_0. Since F is a uniformly continuous homeomorphism, there exists a symmetric convex neighbourhood $V_1^{\sim} \subset Y$ of the origin such that for every $x \in X$, $F^{-1}\big(V_1^{\sim}+F(x)\big) \subset U_0+x$. Without loss of generality we can assume that $V_1^{\sim} \subset V_1$. Let P_1 be the Minkowski functional of $V_1^{\sim}$. It is easy to see that if $Q_0(x_1-x_2) \geqslant 1$, then $P_1\big(F(x_1)-F(x_2)\big) \geqslant 1$ and

$$Q_0(x_1-x_2) \leqslant 2P_1\big(F(x_1)-F(x_2)\big).$$

Using a similar argument we can find a symmetric convex neighbourhood $U_1^{\sim} \subset U_1$ such that for every $x \in X$

$$F(U_1^{\sim}+x) \subset V_1^{\sim}+F(x)$$

and if $Q_0(x_1-x_2) \geqslant 1$, then

$$Q_0(x_1-x_2) \leqslant 2P_1\big(F(x_1)-F(x_2)\big) \leqslant 2^2 Q_1(x_1-x_2),$$

where Q_1 is the Minkowski functional of $U_1^{\sim}$. Continuing this process we can define systems of pseudonorms $\{Q_n\}_{n\geqslant 0}$ and $\{P_n\}_{n\in N}$ such that if $Q_0(x_1-x_2) \geqslant 1$, then inequalities (9) and (10) hold for $n = 1, 2, \ldots$ (with a certain set of constants). Hence the lemma is proved.

Lemma 6. *A subset Z of a separable Fréchet space X is Lipschitz embeddable in a Montel–Fréchet space Y with the given set of constants if and only if every finite subset of Z is Lipschitz embeddable in Y with this set of constants.*

Proof. The necessity is trivial. Conversely suppose that every finite subset of Z is Lipschitz embeddable in Y with the given set of constants. Let $\{x_n\}_{n\geqslant 0}$ be a sequence of points dense in Z and let F_n be a Lipschitz embedding of the set $A_n = \{x_0, x_1, \ldots, x_n\}$ into Y for $n = 1, 2, \ldots$ with the given set of constants such that $F_n(x_0) = 0$ for $n \in N$. Since for every fixed $n_0 \in N$ the set $\{F_m(x_{n_0})\}_{m\geqslant n_0}$ is bounded it follows that $\{F_m(x_{n_0})\}_{m\geqslant n_0}$ is precompact for $n_0 = 1, 2, \ldots$ Hence we can choose an increasing subsequence $\{m_{k,1}\}_{k\in N}$ of positive integers such that the sequence $\{F_{m_{k,1}}(x_1)\}_{k\in N}$ converges to some $y_1 \in Y$. Further we can choose a subsequence $\{m_{k,2}\}_{k\in N}$ of the sequence $\{m_{k,1}\}_{k\in N}$ such that the sequence $\{F_{m_{k,2}}(x_2)\}_{k\in N}$ converges to some $y_2 \in Y$. Continuing this process and then using "the diagonal procedure" we can define a sequence of embeddings $\{F_{m_{k,k}}\}_{k\in N}$ such that for $n = 0, 1, 2, \ldots$ and $k \geqslant n$

$$F_{m_{k,k}}(x_n) \underset{k\to\infty}{\to} y_n,$$

where additionally we put $y_0 = 0$. It is easy to see that the mapping $F^{\sim}_{\infty}$ defined by formula

$$F^{\sim}_{\infty}(x_n) = y_n$$

is a Lipschitz embedding of $\{x_n\}_{n\geqslant 0}$ into Y with the same set of constants. Since the sequence $\{x_n\}_{n\geqslant 0}$ is dense in Z then by the same argument as before we obtain that the extension F_∞ of $F^{\sim}_{\infty}$ is a Lipschitz embedding of Z into Y with the given set of constants. This concludes the proof of the lemma.

THEOREM 7. *If a Fréchet space X is uniformly homeomorphic with a super reflexive Montel–Fréchet space Y, then X is isomorphic to a subspace of Y.*

Proof. Let F be a uniform homeomorphism from X onto Y. It follows from Lemma 5 that the topologies on X and Y can be given respectively by sequences of pseudonorms $\{Q_n\}_{n\geqslant 0}$ and $\{P_n\}_{n\in N}$ such that F satisfies the first order Lipschitz condition with some set of constants if $Q_0(x_1 - x_2) \geqslant 1$. Obviously X is a separable space. Let $\{x_n\}_{n\in N}$ be a sequence of points dense in X with the property that $Q_0(x_i - x_j) > 0$ for $i \neq j$, $i, j \in N$. The existence of such a sequence follows from the fact that the set $\{x \in X\colon Q_0(x) = 0\}$ is nowhere dense in X. Put

$$\lambda_n = \inf\{Q_0(x_i - x_j)\colon i < j \leqslant n\}$$

for $n = 1, 2, \ldots$. It follows from the definition of the numbers λ_n that $Q_0(\lambda_n^{-1}x_i - \lambda_n^{-1}x_j) \geqslant 1$ for $i < j \leqslant n$. Since F satisfies the first order Lipschitz condition on large distance, the embedding F_n defined by the formula

$$F_n(x_i) = \lambda_n F(\lambda_n^{-1}x_i)$$

for $i \leqslant n$ of the set $\{x_1, x_2, \ldots, x_n\} \subset X$ into Y satisfies the first order Lipschitz condition with the given set of constants for $n \in N$. Observe that for every positive integer n the embedding F_n is Lipschitz with the same set of constants. Hence by Lemma 6 there exists a Lipschitz embedding $F^{\sim}$ of $\{x_n\}_{n\in N}$ into Y with the same set of constants, and this implies the existence of the Lipschitz embedding F of X into Y. Applying Theorem 4 to the Lipschitz embedding F we obtain that there exists an isomorphic embedding of X into Y and so the theorem is proved.

As an easy consequence of this theorem we obtain

COROLLARY 5. *If a Fréchet space X is uniformly homeomorphic*

(i) *with a super reflexive Montel–Fréchet space, then X is a super reflexive Montel–Fréchet space;*

(ii) *with a super reflexive Schwartz–Fréchet space, then X is a super reflexive Schwartz–Fréchet space,*

(iii) *with a nuclear space, then X is a nuclear space.*

Remark. In [2] it is proved that every infinite dimensional subspace of the space of all sequences s is isomorphic to s. This result and Theorem 7 imply that every Fréchet space uniformly homeomorphic to s is isomorphic to s.

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