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# Some remarks on a theorem of S. M. Lozinski concerning linear process of approximation of periodic functions

by

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**Abstract.** Let $T_n(x, f)$ be the trigonometric polynominal of order $\leqslant n$ such that $T_n(x_i, f) = f(x_i)$, $i = 0, 1, 2, \ldots, 2n$. Here $x_i$'s are defined by (1.1). Consider the linear process of approximation starting out with interpolation as defined in (1.4). S. M. Lozinski [3] proved under certain conditions on $\lambda$'s that $L_n(f; x)$ converges uniformly to $f(x)$ on the real line provided $f(x) \in C_{2\pi}$. Here, starting with weaker assumptions on $\lambda$'s, we give the estimate of $L_n(f; x) - f(x)$ in terms of modules of continuity of $f(x)$. Our theorem is similar to Korovkin's theorem on linear process of approximation starting with partial sum of fourier series of $f(x)$.

**1.** The simplest and at the same time most natural example of a linear process of approximation of continuous periodic functions by trigonometric sums is the approximation by the sequences of partial sums of their Fourier series expansions or interpolation (trigonometric) by Lagrange formula. However, they do not provide a tool that yields uniform approximation for the whole class of continuous functions. In this connection there arises the question of how it is possible, starting with a Fourier series expansion or interpolation formula of Lagrange, to achieve by means of some variation of the given process uniform convergence of the resulting polynomials for any continuous function.

Linear process of approximation starting out with interpolation was developed by S. N. Bernstein and S. M. Lozinski [3]. Let

$$x_k = x_{kn} = \frac{2k\pi}{2n+1}, \qquad k = 0, 1, \ldots, 2n, \tag{1.1}$$

and

$$T_n(x) = A^{(n)} + \sum_{m=1}^{n} (a_m^{(n)} \cos mx + b_m^{(n)} \sin mx) \tag{1.2}$$

be a trigonometric polynomial which concides at the nodes (1.1).

It is known that

$$(1.3)\quad \begin{cases} A^{(n)} = \dfrac{1}{(2n+1)} \displaystyle\sum_{k=0}^{2n} f(x_k), \\ a_m^{(n)} = \dfrac{2}{(2n+1)} \displaystyle\sum_{k=0}^{2n} f(x_k)\cos m x_k, \\ b_m^{n} = \dfrac{2}{(2n+1)} \displaystyle\sum_{k=0}^{2n} f(x_k)\sin m x_k. \end{cases}$$

Consider

$$(1.4)\qquad L_n(f, x) = \lambda_0^{(n)} A^{(n)} + \sum_{m=1}^{n} \lambda_m^{(n)} (a_m^{(n)} \cos mx + b_m^{(n)} \sin mx)$$

with

$$(1.5)\qquad \lambda_{k}^{(n)} \underset{n\to\infty}{=} 1, \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big|\lambda_0^{(n)} + 2\sum_{k=1}^{n} \lambda_k^{(n)} \cos kt\Big|\, dt < K,$$

then a theorem of S. M. Lozinski [3] states that if $f(x) \in C_{2\pi}$ and $\lambda_k^{(n)}$ satisfy (1.5), then $L_n(f, x)$ converges uniformly to $f(x)$ on the real line. But it may be remarked that the degree of approximation problem is not solved. However, other linear process of approximation starting with the partial sum of Fourier series of $f(x)$ much more is known. Let

$$S_n(f, x) = \tfrac{1}{2} a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx),$$

where $a_k$, $b_k$ are Fourier coefficient of $f(x)$. Denote

$$k_n(t) = \tfrac{1}{2} + \sum_{k=1}^{n} \lambda_k^{(n)} \cos kt, \qquad \lambda_0^{n} = 1$$

and define

$$U_n(f; x, \lambda) = \tfrac{1}{2} a_0 \lambda_0^{(n)} + \sum_{k=1}^{n} \lambda_k^{(n)} (a_k \cos kx + b_k \sin kx)$$

we state the following theorem

THEOREM (Korovkin [2]). *Let $f(x) \in C_{2\pi}$ and $k_n(t) \geqslant 0$; then*

$$|U_n(f, x, \lambda) - f(x)| \leqslant w\left(\frac{1}{m}\right)\left(l + \frac{m\pi}{\sqrt{2}}\sqrt{1-\lambda_1^{(n)}}\right)$$

*and if $m = \sqrt{n}$, $1-\lambda_1^{(n)} = O\left(\frac{1}{n}\right)$, then*

$$|U_n(f, x, \lambda) - f(x)| \leqslant w\left(\frac{1}{\sqrt{n}}\right).$$



The object of this paper is to obtain the estimate of $L_n(f, x, \lambda) - f(x)$ in terms of modulus of continuity of $f(x)$. The result turns out to be similar to Korovkin's theorem although we will not need the condition $k_n(t) \geqslant 0$. We will replace this condition by other general condition (see 1.6 and 1.7). We will give some examples of $L_n(f; x)$ for various choice of $\lambda_k^{(n)}$. Thus our operator need not be a positive one. Further main role is played by Fejer Kernel, a result of O. Shisha and B. Mond [6], and Lemma 2.1.

Let the triangular matrix $\lambda_j^{(n)}$ of positive bounded numbers satisfying the following conditions

$$(1.6)\qquad \lambda_0^{(n)} = 1, \quad \lambda_j^{(n)} = 0, \quad j \geqslant n+1, \quad \lambda_n^{(n)} = O\left(\frac{1}{n}\right),$$

$$(1.7)\quad 1-\lambda_1^{(n)} = O\left(\frac{1}{n}\right), \quad \lambda_{j+1}^{(n)} - 2\lambda_j^{(n)} + \lambda_{j-1}^{(n)} \geqslant 0, \quad j = 1, 2, \ldots, n,$$

or

$$(1.7a)\qquad |\lambda_{j+1}^{(n)} - 2\lambda_j^{(n)} + \lambda_{j-1}^{(n)}| = O\left(\frac{1}{n^2}\right), \quad j = 1, 2, \ldots, n.$$

THEOREM 1. *Let $\lambda_j^{(n)}$ satisfy conditions* (1.6) *and* (1.7) *or* (1.6) *and* (1.7a); *then we have*

$$(1.8)\qquad |L_n(f; x, \lambda) - f(x)| \leqslant c w_f\left(\frac{1}{\sqrt{n}}\right)$$

*and*

$$(1.9)\qquad |L_n(f; x, \lambda) - f(x)| \leqslant c_1 \log n\, w\left(\frac{1}{n}\right),$$

*where $c$ and $c_1$ is a positive constant independent of $n$ and $x$, $w(\delta)$ being the modulus of continuity of $f(x)$.*

Remark. If (1.7) is satisfied, then it is clear from (2.9) that $L_n(f; x) \geqslant 0$ if $f \geqslant 0$ for $[-\pi, \pi]$. But this may not be true if (1.7a) is satisfied. To this end we will give examples at the end of the paper. Further, if (1.7) is satisfied, then $\lambda_n^{(n)} = O\left(\frac{1}{n}\right)$ is automatically satisfied. Since,

$$\sum_{j=1}^{n} \lambda_{j+1}^{(n)} - 2\lambda_j^{(n)} + \lambda_{j-1}^{(n)} = 1 - \lambda_1^{(n)} - \lambda_n^{(n)} \geqslant 0.$$

Therefore

$$\lambda_n^{(n)} \leqslant 1 - \lambda_1^{(n)},$$

but $\lambda_1^{(n)} < \lambda_0^{(n)} = 1$ and $1-\lambda_1^{(n)} = O\left(\frac{1}{n}\right)$ hence

$$\lambda_n^{(n)} = O\left(\frac{1}{n}\right).$$

**2.** First we will express $L_n(f;x,\lambda)$ in a suitable form. Substituting (1.3) into (1.4) and assuming $\lambda_0^{(n)}=1$ we get

$$L_n(f;x,\lambda)=L_n(f;x)=\sum_{k=0}^{2n}f(x_{kn})A_n(x_k-x), \tag{2.1}$$

where

$$A_n(t)=\frac{1}{2n+1}\Big[1+2\sum_{m=1}^{n}\lambda_m^{(n)}\cos mt.\Big]. \tag{2.2}$$

From (1.1), (2.1) and (2.2) it follows that

$$\sum_{k=0}^{2n}A_n(x-x_k)=1. \tag{2.3}$$

Now we will prove the following lemma concerning $A_n(t)$.

LEMMA 2. *Let $\lambda_m^{(n)}$ satisfy conditions* (1.6) *and* (1.7) *or* (1.6) *and* (1.7a). *Then there exists positive constants $c_1$ and $c_2$ such that*

$$\sum_{k=0}^{2n}|A_n(x-x_{kn})|\leqslant c_1, \tag{2.4}$$

$$\sum_{k=0}^{2n}\sin^2\tfrac{1}{2}(x-x_{kn})|A_n(x-x_{kn})|\leqslant\frac{c_2}{n}. \tag{2.5}$$

Proof. We set [9]

$$t_{2n+1}(x)=1+2\sum_{i=1}^{2n}\Big(1-\frac{i}{2n+1}\Big)\cos ix \tag{2.6}$$

which is the Fejer kernel. It is well known that

$$t_{2n+1}(x)=\frac{1}{(2n+1)}\left(\frac{\sin(n+\frac{1}{2})x}{\sin\frac{1}{2}x}\right)^2, \tag{2.7}$$

$$\sum_{k=1}^{2n}t_{2n+1}(x-x_{kn})=2n+1. \tag{2.8}$$

From (2.6) it follows that

$$(j+1)t_{j+1}(x)-2jt_j(x)+(j-1)t_{j-1}(x)=2\cos jx.$$

Using this formula we find that

$$A_n(x)=\frac{1}{(2n+1)}\sum_{j=1}^{n}a_j^{(n)}jt_j(x)+\frac{\lambda_n^{(n)}(n+1)t_{n+1}(x)}{2n+1}, \tag{2.9}$$



where we have set

$$a_j^{(n)}=\lambda_{j+1}^{(n)}-2\lambda_j^{(n)}+\lambda_{j-1}^{(n)}.$$

In case $\lambda_j^{(n)}$ is a convex sequence of positive numbers i.e. (1.7) is satisfied then it follows from (2.9) that $A_n(x)\geqslant 0$. But then on using (2.3) we obtain

$$\sum_{k=0}^{2n}|A_n(x-x_{kn})|=\sum_{k=0}^{2n}A_n(x-x_{kn})=1.$$

Again, from (2.9) we have

$$\sin^2\frac{x}{2}A_n(x)=\frac{1}{(2n+1)}\sum_{j=1}^{n}a_j^{(n)}\sin^2\frac{jx}{2}+\frac{\lambda_n^{(n)}}{(2n+1)}\sin^2\left(\frac{n+1}{2}\right)x.$$

Therefore on using (1.6) and (1.7) we obtain

$$\sin^2\frac{x}{2}|A_n(x)|\leqslant O\left(\frac{1}{n^2}\right)+\frac{1}{(2n+1)}\sum_{j=1}^{n}a_j^{(n)}.$$

But

$$\sum_{j=1}^{n}a_j^{(n)}=1-\lambda_1^{(n)}-\lambda_n^{(n)}\leqslant|1-\lambda_1^n|+|\lambda_n^n|=O\left(\frac{1}{n}\right).$$

Replacing $x$ by $x-x_{kn}$ we obtain

$$\sum_{k=0}^{2n}\sin^2\frac{x-x_{kn}}{2}|A_n(x-x_{kn})|\leqslant O\left(\frac{1}{n}\right).$$

This proves (2.5) on the assumption that (1.6) and (1.7) are satisfied.

Let (1.6) and (1.7a) be satisfied. On using, (2.9), (2.8) we obtain

$$\sum_{k=0}^{2n}|A_n(x-x_{kn})|\leqslant\sum_{j=1}^{n}|a_j^{(n)}|j+(n+1)|\lambda_n^{(n)}|$$

$$=O\left(\frac{1}{n^2}\right)\sum_{j=1}^{n}j+(n+1)O\left(\frac{1}{n}\right)=O(1).$$

Again, from (2.9) we obtain

$$\sin^2\frac{x}{2}|A_n(x)|\leqslant\frac{1}{(2n+1)}\sum_{j=1}^{n}|a_j^n|+\frac{\lambda_n^{(n)}}{(2n+1)}$$

on using (1.7a) and $\lambda_n^{(n)}=O\left(\frac{1}{n}\right)$, replacing $x$ by $x-x_{kn}$ and adding for $k=0,1,\ldots,2n$ we get

$$\sum_{k=0}^{2n}\sin^2\frac{x-x_{kn}}{2}|A_n(x-x_{kn})|\leqslant O\left(\frac{1}{n}\right).$$

This proves the lemma completely.

**3.**

Proof of Theorem 1. In view of (2.1) and (2.3) we have $L_n(f;x)-$ $-f(x) = \sum_{k=0}^{2n}[f(x_{kn})-f(x)]A_n(x_k-x)$. But from a known result [5] that for any $2\pi$ periodic continuous function $f(x)$ having modulus of continuity $w(\delta)$ we have

$$|f(x)-f(t)| \leqslant \left(1+\frac{\pi^2}{\delta^2}\sin^2\frac{x_k-x}{2}\right)w(\delta)$$

for all $t$ and $x$. Replacing $t$ by $x_{kn}$ in this inequality we obtain on using Lemma 2

$$|L_n(f;x)-f(x)| \leqslant w(\delta)\sum_{k=0}^{2n}\left(1+\frac{\pi^2}{\delta^2}\sin^2\frac{t-x}{2}\right)|A_n(x_k-x)|$$

$$\leqslant w(\delta)\left(c_1A+\frac{\pi^2c_2}{\delta^2 n}\right).$$

Let $\delta = \frac{1}{\sqrt{n}}$ and (1.8) is proved.

It remains to prove (1.9). Let $x$ be arbitrary but fixed. Let $x_{in}$ is the nearest to $x$; then obviously

$$(4.1)\qquad |x-x_{in}| \leqslant \frac{\pi}{2n+1}, \qquad |x-x_{kn}| \geqslant \frac{2|k-i|-1}{(2n+1)}\pi.$$

Since $f(x)$ and $A_n(x)$ are both periodic functions we have

$$(4.2)\qquad L_n(f;x)-f(x) = \sum_{k=0}^{2n}[f(x_{kn})-f(x)]A_n(x_{kn}-x)$$

$$= \sum_{k=i-n}^{i+n}[f(x_{kn})-f(x)]A_n(x_{kn}-x).$$

But $w(\lambda\delta) \leqslant (\lambda+1)w(\delta)$ we have

$$(4.3)\qquad |f(x)-f(x_{kn})| \leqslant (n|x-x_{kn}|+1)w\left(\frac{1}{n}\right).$$

On using (4.3), (4.1) and (4.2) we obtain

$$|L_n(f;x)-f(x)| \leqslant w\left(\frac{1}{n}\right)\Big[|A_n(x_{in}-x)|(1+n|x-x_{in}|)+O(1)+$$

$$+n\sum_{\substack{k=i-n\\k\neq i}}^{i+n}|A_n(x_{kn}-x)||x_{kn}-x|\Big],$$



on using (2.4) and (4.1) we obtain

$$|A_n(x_{in}-x)|(1+n|x-x_{in}|) \leqslant c_1\left[1+\frac{n\pi}{2n+1}\right] \leqslant c_3,$$

on using (2.9), (2.7) and later part of (4.1) we obtain

$$\sum_{\substack{k=i-n\\k\neq i}}^{i+n}|A_n(x_{kn}-x)||x_{kn}-x| = O\left(\frac{1}{n}\right)\log n.$$

Thus we have

$$|L_n(f;x)-f(x)| \leqslant c_2\log n\, w\left(\frac{1}{n}\right).$$

This proves the theorem.

**4. Examples.** Let us consider

$$A_n(t) = \frac{1}{(2n+1)}\left[1+\frac{2}{(n+1)^m}\sum_{j=1}^{n}(n+1-j)^m\cos jt\right],$$

where $m \geqslant 1$. For $m=1$ this corresponds to Fejer kernel. It turns out as it easy to see that $A_n(t) \geqslant 0$ for each fixed integer $m$. Thus corresponding to this choice of $\lambda_j^{(n)}$ we obtain $L_n(f;x,\lambda)$ as positive operator. Here $\lambda_j^{(n)}$ satisfy conditions (1.6) and (1.7). Next we consider

$$A_n(t) = \frac{1}{(2n+1)}\left[1+2\sum_{j=1}^{n}\frac{(n-j+1)^m}{(n-j+1)^m+j^m}\cos jt\right].$$

Again for $m=1$ this also reduces to Fejer kernel, but here $A_n(t) \geqslant 0$ is not satisfied for $m>1$. Here it can be verified that $\lambda_j^{(n)}$ satisfy (1.6) and (1.7a).

We consider

$$A_n(t) = \frac{1}{(2n+1)}\left[1+2\sum_{j=1}^{n}\lambda_j^{(n)}\cos jt\right],$$

where

$$\lambda_j^{(n)} = \frac{(2n+2-j)^m-2(n+1-j)^m-j^m}{(2n+2-j)^m+(n+1+j)^m-3\{(n+1-j)^m+j^m\}}.$$

Here, again for $m=3$ this reduces to Fejer kernel. Again for $m>3$ the property of positivity breaks down. But $\lambda_j^{(n)}$ satisfy (1.6) and (1.7a). The examples second and third occurred to the author in another interpolation problem.

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# Fractional powers of operators and Bessel potentials on Hilbert space

by

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**Abstract.** Two candidates for the title "the Bessel potential" over a real separable Hilbert space are studied with the theory of fractional powers of operators and shown to define equivalent Sobolev spaces $L_p^a(H)$. $L_p^a(H)$ is shown to be equivalent to $D(T^a)$ when $(-T)$ is the infinitesmal generator of the Poisson integral and when $D(T^a)$ is equipped with the graph norm. The Bessel potentials of purely imaginary order are shown to be bounded on the reflexive $L_p(H)$ and to form a strongly continuous boundary value group for the Bessel potentials $J^a$ with $\mathrm{Re}(a) > 0$.

**Introduction.** In [3] we defined the Bessel potential over a real separable Hilbert space, $H$, and studied the family of singular integral operators $G^a$: $L_p^a(H) \to L_p(H)$, where $L_p^a(H)$ is the image of $L_p(H)$ under the Bessel potential $J^a$. $J^a(f) = \Gamma(a)^{-1}\int_0^\infty P_t(f)t^{a-1}e^{-t}dt$, where $P_t(f)$ is the Poisson integral of $f$; [2]. The norm in $L_p^a(H)$ is $\|g\|_{a,p} = \|f\|_p$ when $g = J^a(f)$. The purpose of this paper is to examine the Bessel potential operators more closely than they were studied in [3]. Specifically, we shall examine two prominent candidates for the designation of "the Bessel potential" over an infinite dimensional Hilbert space and show that the spaces $L_p^a(H)$ defined using these operators are equivalent to the domain of a certain closed densely defined operator when this domain is equipped with the graph norm. Secondly, we shall examine the semigroup $J^a$ in $\mathrm{Re}(a) \geqslant 0$ and show that the boundary values, $J^{i\gamma}$, form a strongly continuous group of bounded operators on $L_p(H)$ if $1 < p < \infty$. The paper closes with a discussion of the infinitesmal generators of $J^\beta$, $\beta > 0$, and $J^{i\gamma}$.

Throughout this paper $K$, $K(a)$, $K(p, a)$ etc. ($M$, $M(a)$, $M(p, a)$, etc.) denote positive (complex) constants which depend only on the parameters shown. If $T$ is a linear operator on a Banach space $X$, $D(T)$ denotes the domain of $T$ and $R(T)$ denotes the range of $T$.

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