It follows from a theorem of Titchmarsh ([6], Theorem VII) that the set N of all nilpotent elements of A consists of just those $f \in A$ for which there exists $\varepsilon > 0$ such that $f(t) = 0$ $(0 \leqslant t \leqslant \varepsilon)$. So the closure $\overline{N}$ of N is the set of $f \in A$ such that $f(0) = 0$, and we observe that for all $g \in A$, $g^2 \in \overline{N}$.

Now let $A_1 = A \oplus C.1$ be the algebra obtained by adjoining an identity to A, and let $A' = A_1 \otimes L^\omega[0,1] \otimes \alpha^1(N)$, as above. Since the set of all nilpotent elements of A_1 is N, every $\tilde{z} \in \mathscr{R}(A')$ has $z_1(t) \in N$ a.e. (almost everywhere). So, for every $\tilde{z} \in \overline{\mathscr{R}(A')}$, $z_1(t) \in \overline{N}$ a.e. Thus, if $\tilde{x} = x \otimes \otimes 1 \otimes 1$, with $x \in A \setminus \overline{N}$, then $\tilde{x} \notin \overline{\mathscr{R}(A')}$. However, $(\tilde{x})^2 = x^2 \otimes 1 \otimes 1$ is in the closure of the set of nilpotent elements of A', and hence is in $\overline{\mathscr{R}(A')}$. Thus $A'/\overline{\mathscr{R}(A')}$ has non-zero nilpotent elements. Since it is commutative, it cannot be semisimple.

References

[1] R. Arens, *The space* L^ω *and convex topological rings*, Bull. Amer. Math. Soc. 52 (1946), pp. 931–935.

[2] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955).

[3] П. М. Китайник, *Почти нильпотентные элементы коммутативной* B_0-*алгебры*, Вестник Моск. Унив. Мат. Мех. (1969), No. 6, pp. 69–72.

[4] A. Mallios, *Semisimplicity of tensor products of topological algebras*, Bull. Soc. Math. Grèce (N. S.) 8 (1967), pp. 1–16.

[5] S. Rolewicz, *Some remarks on radicals in commutative* B_0-*algebras*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astron. Phys. 15 (1967), pp. 153–155.

[6] E. C. Titchmarsh, *The zeros of certain integral functions*, Proc. London Math. Soc. (2) 25 (1926), pp. 283–302.

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An application of interpolation theory to Fourier series

by

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Abstract. In this note we introduce a generalization of the weak interpolation theory of Lions and Peetre. With the help of this generalization we present a unified account of some theorems in the theory of Fourier series with positive coefficients.

The generalization consists in considering interpolation not of subspaces of a topological vector space, but of what we call quasi-cones of it (see Definition I.1). We shall in this note present only the minimal amount of interpolation theory of quasi-cones needed for the application to the problem at hand, and hope to return to the general theory in a subsequent paper. We shall assume familiarity with the notion of $L(p,q)$ spaces, as well as with the terminology of the Lions-Peetre interpolation theory.

I. Interpolation of quasi-cones.

Definition 1. Let V be a (real or complex) vector space. A subset Q of V will be called a *quasi-cone* (QC) iff $Q+Q \subset Q$. Q is a *cone* iff we also have $\lambda Q \subset Q$ for all $0 < \lambda$. We shall apply our results to cones, but since no additional work is involved, we shall state the results for quasi-cones. Two cones which will be important in the applications we give are:

$$Q_1 = \{\{x_n\}_1^\infty \mid x_n \downarrow 0\} \quad \text{and} \quad Q_2 = \{\{x_n\}_1^\infty \mid \text{for some } \beta,\ n^{-\beta} x_n \downarrow 0\}.$$

Definition 2. Let B be a vector space over C. A *quasi-norm on* B is a function $\| \ \|: B \to R^+$ satisfying:

(a) $\|b\| = 0$ iff $b = 0$.

(b) For all $\lambda \in C$, $b \in B$: $\|\lambda b\| = |\lambda| \, \|b\|$.

(c) A number $k = k(B)$ exists, so that

$$\|b_1 + b_2\| \leqslant k(\|b_1\| + \|b_2\|), \quad \text{for all} \quad b_1, b_2 \in B.$$

A quasi-normed space is a topological vector space, whose topology is given by a quasi-norm.

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Definition 3. Let $(B, \| \|)$ be a quasi-normed space. Let $\{\mu_n\}_{-\infty}^{\infty}$ be a sequence of positive numbers. Define:

(a) $\|\{b_n\}\|_{l^p_{\mu_n}(B,\| \|)} = \left(\sum_{-\infty}^{\infty} \|\mu_n b_n\|_B^p\right)^{1/p}, \quad 0 < p < \infty,$

(b) $\|\{b_n\}\|_{l^\infty_{\mu_n}(B,\| \|)} = \sup_{-\infty<n<\infty} \{\|\mu_n b_n\|_B\}.$

If $S \subset B$, $l^p_{\mu_n}(S, B)$ will denote the set of all sequences $\{b_n\}_{-\infty}^{\infty}$ of elements of S, so that $\|\{b_n\}\|_{l^p_{\mu_n}(B,\| \|)} < \infty$. $l^p_{\mu_n}(B, B)$ will be denoted by $l^p_{\mu_n}(B)$. Clearly if Q is a QC in B, $l^p_{\mu_n}(Q, B)$ is a QC in $l^p_{\mu_n}(B)$.

Definition 4. Let $(B_i, \| \|_i)$ be two quasi-normed spaces. If both are continuously embedded in a topological vector space, we shall say that $(B_0, \| \|_0; B_1, \| \|_1)$ is an *interpolation pair*. We shall omit, when no confusion arises, the quasi-norms, and write (B_0, B_1).

Definition 5. Let (B_0, B_1) be an interpolation pair $Q_i \subset B_i$ quasi-cones. Denote:

$$(Q_0, Q_1)_{\theta,q} = \{b \in Q_0 + Q_1 / \{v_{in}\}_{-\infty}^{\infty} \in l^q_{e^{(i-\theta)n}}(Q_i), \text{ so that } v_{0n} + v_{1n} = b\}.$$

We may well have $Q_i = B_i$, in which case we are back in the theory of interpolation of quasi-normed spaces.

Theorem 6. $\operatorname{Inf}\{\operatorname*{Max}_{i=0,1} \|\{e^{(i-\theta)n} v_{in}\}\|_{l^q(B_i)} / v_{0n} + v_{1n} = b\}$ *is a quasi-norm on* $(B_0, B_1)_{\theta,q}$. *It is equivalent to*

$$\operatorname{Inf}\{\|\{e^{-\theta n} v_{0n}\}\|_{l^q(B_0)}^{1-\theta} \|\{e^{(1-\theta)n} v_{1n}\}\|_{l^q(B_1)}^{\theta} / v_{0n} + v_{1n} = b\}. \tag{1}$$

Proof. See [6], theorem II.12.

We shall denote the expression appearing in (1) by $\| \|_{B,\theta,q}$

Definition 7. Let (A_0, A_1), (B_0, B_1) be two interpolation pairs. Q_i quasi-cones in A_i, R_i in B_i. An operator $T\colon Q_0 + Q_1 \to R_0 + R_1$ will be called a *quasi-linear operator* from (Q_0, Q_1) to (R_0, R_1) iff for every $a_0 \in Q_0$, $a_1 \in Q_1$ we can find $b_i \in R_i$ so that

$$T(a_0 + a_1) = b_0 + b_1 \quad \text{and} \quad \|b_i\|_{B_i} \leqslant K_i \|a_i\|_{A_i}. \tag{2}$$

Theorem 8 (Interpolation theorem). *Let* (A_0, A_1), (B_0, B_1) *be two interpolation pairs.* Q_i *quasi-cones in* A_i, R *in* B_i. *If* T *is a quasi-linear operator from* (Q_0, Q_1) *to* (R_0, R_1), *then for* $0 < \theta < 1$, $0 < q \leqslant \infty$, T *maps* $(Q_0, Q_1)_{\theta,q}$ *to* $(R_0, R_1)_{\theta,q}$ *and*

$$\|Ta\|_{B,\theta,q} \leqslant K_0^{1-\theta} K_1^{\theta} \|a\|_{A,\theta,q}. \tag{3}$$

Proof. Let $a \in (Q_0, Q_1)_{\theta,q}$. $\{a_{in}\} \in l^q_{e^{(i-\theta)n}}(Q_i)$, $a_{0n} + a_{1n} = a$. Let b_{in} be given by (2). Then

$$\|Ta\|_{B,\theta,q} \leqslant \|\{e^{-\theta n} b_{0n}\}\|_{l^q(B_0)}^{1-\theta} \|\{e^{(1-\theta)n} b_{1n}\}\|_{l^q(B_1)}^{\theta}$$
$$\leqslant K_0^{1-\theta} K_1^{\theta} \|\{e^{-\theta n} a_{0n}\}\|_{l^q(A_0)}^{1-\theta} \|\{e^{(1-\theta)} a_{1n}\}\|_{l^q(A_1)}^{\theta}.$$

Taking infimum of last expression over all pairs of sequences, we get (3). That T maps $(Q_0, Q_1)_{\theta,q}$ to $(R_0, R_1)_{\theta,q}$ is clear.

For many purposes the integral definition of the intermediate cones is convenient:

Definition 9. Let (B_0, B_1) be an interpolation pair Q_i quasi-cones in B_i. Define for $b \in Q_0 + Q_1$

$$K(t, b; Q) = \operatorname{Inf}\{\operatorname*{Max}_{i=0,1} t^i \|b_i\|_i / b_i \in Q_i,\ b_0 + b_1 = b\}. \tag{4}$$

We then have

Theorem 10. $b \in (Q_0, Q_1)_{\theta,q}$ *iff* $\int_0^\infty [t^{-\theta} K(t, b; Q)]^q \frac{dt}{t} < \infty$. *Further,* $\| \|_{B,\theta,q}$ *is equivalent to* $\left(\int_0^\infty [t^{-\theta} K(t, b; Q)]^q \frac{dt}{t}\right)^{1/q}$.

Proof. This is well known. See for example [6], Theorems III.2, and IV.11.

The difficulty in applying the interpolation theorems lies of course in identifying the intermediate QC.

Theorem 11. *Let* (B_0, B_1) *be an interpolation pair. Let* Q *be a* QC *in* $B_0 + B_1 \cdot Q_i = Q \cap B_i$. *Then*

$$(Q_0, Q_1)_{\theta,q} \subset Q \cap (B_0, B_1)_{\theta,q}.$$

Proof. Clearly $K(t, b; Q) \geqslant K(t, b; B)$, from which the theorem follows.

Motivated by this we make the following definition:

Definition 12. Let (B_0, B_1) be an interpolation pair Q a QC in $B_0 + B_1$. Q will be called a *Marcinkiewicz quasi-cone* (MQC) iff, for $Q_i = Q \cap B_i$, we have

$$(Q_0, Q_1)_{\theta,q} = Q \cap (B_0, B_1)_{\theta,q} \quad \text{for all} \quad 0 < \theta < 1,\ 0 < q \leqslant \infty. \tag{5}$$

Theorem 13 (Reiteration theorem). *Let* (B_0, B_1) *be an interpolation pair.* Q_i *a* QC *in* B_i. $0 < \theta_i < 1$, $\theta_0 \neq \theta_1$, $0 < q_i \leqslant \infty$. $R_i = (Q_0, Q_1)_{\theta_i,q_i}$. *Then for* $0 < \lambda < 1$, $0 < q \leqslant \infty$ *we have*

$$(R_0, R_1)_{\lambda,q} = (Q_0, Q_1)_{\theta,q},$$

where $\theta = (1-\lambda)\theta_0 + \lambda\theta_1$.

Proof. The beautiful proof of Holmstedt ([4], Theorems 2.1, 3.1), although stated for spaces rather than QC, actually proves the theorem above.

Theorem 14. *Let* (B_0, B_1) *be an interpolation pair.* $E_i = (B_0, B_1)_{\theta_i,q_i}$, *where* $0 < \theta_i < 1$, $\theta_0 \neq \theta_1$, $0 < q_i \leqslant \infty$. *Then if* Q *is a* MQC *in* $B_0 + B_1$, $Q \cap (E_0 + E_1)$ *is a* MQC *in* $E_0 + E_1$.

Proof. Let $\overline{Q} = Q \cap (E_0+E_1)$, $\overline{Q}_i = Q \cap E_i$. Since Q is a MQC in B_0+B_1, $Q_i = (Q_0, Q_1)_{\theta_i,q_i}$, where $Q_i = Q \cap B_i$. Using the reiteration theorem we have:

$$\overline{Q} \cap (E_0, E_1)_{\lambda,q} = Q \cap (E_0, E_1)_{\lambda,q} = Q \cap (B_0, B_1)_{\theta,q} = (Q_0, Q_1)_{\theta,q} = (\overline{Q}_0, \overline{Q}_1)_{\lambda,q},$$

and the theorem is proved.

THEOREM 15. *Let Q be a QC in $L(p_0, q_0)+L(p_1, q_1)$, where $p_0 \neq p_1$, $0 < p_i < \infty$, $0 < q_i \leqslant \infty$. Then if for every $f \in Q$ and every $0 < y$ the functions*

$$(6) \qquad f^y = \begin{cases} f & \text{if } y < |f|, \\ 0 & \text{if } |f| \leqslant y, \end{cases} \quad \text{and} \quad f_y = \begin{cases} 0 & \text{if } y < |f|, \\ f & \text{if } |f| \leqslant y \end{cases}$$

belong to Q, then Q is a Marcinkiewicz quasi-cone.

Proof. Since $(L^p, L^\infty)_{\theta,q} = L\left(\dfrac{p}{1-\theta}, q\right)$, suffices if we prove the theorem for Q a QC in L^p+L^∞.

An inspection of the proof of Theorem IV.12 in [6], shows that for Q as above we have for $f \in Q$,

$$K(t, f, Q) \sim K(t, f, L)$$

from which follows that Q is a MQC.

II. Applications to Fourier series. For the sake of simplicity, we shall denote the $L(p, q)$ spaces over the positive integers with measure 1 carried by each integer, by $l(p, q)$. It can easily be shown that

$$\|\{a_k\}\|^*_{p,q} \sim \Big(\sum_1^\infty a_k^{*q} k^{q/p-1}\Big)^{1/q},$$

where $\{a_k^*\}$ is the non-decreasing rearrangement of $\{|a_k|\}$. (Similarly for $q = \infty$.) As a first application, we give the theorem of Hardy and Littelwood:

THEOREM 1. (a) *If $a_k \downarrow 0$, a necessary and sufficient condition that $C(x) = \sum_1^\infty a_k \cos kx$ should belong to $L(p', q)$ is that $\{a_k\} \in l(p, q)$. The obvious norm inequality holds.*

(b) *The same result holds for $S(x) = \sum_1^\infty a_k \sin kx$.*

Proof. The cone $Q_1 = \{\{x_k\}_1^\infty / x_k \downarrow 0\}$ is a MQC in $l^r + l^\infty$ by Theorem I.15. Therefore:

$$(Q_1 \cap l^1, Q_1 \cap l^\infty)_{\theta,q} = Q_1 \cap (l^1, l^\infty)_{\theta,q} = Q_1 \cap l\left(\frac{1}{1-\theta}, q\right).$$

The operator $T(\{a_k\}) = \sum_1^\infty a_k \cos kx$, is well defined, since the series converges for $x \neq 0$. Summation by parts also yields: $|T(\{a_k\})| \leqslant \dfrac{a_1}{x}$, and so $T\colon Q_1 \cap l^\infty \to L(1, \infty)$. Clearly also $T\colon l^1 \to L^\infty$, and so for $1 < p < \infty$, $0 < q \leqslant \infty$ we have $T\colon Q_1 \cap l(p, q) \to L(p', q)$, with the obvious norm inequality.

The proof of the converse theorem follows along the lines of the original proof (see [9]).

Take $G(x) = \int_0^x C(t)\,dt = \sum_1^\infty \dfrac{a_n}{n} \sin nx.$

$$G\left(\frac{\pi}{n}\right) = \sum_{m=1}^{n-1}\left(\frac{a_m}{m} - \frac{a_{m+n}}{m+n} + \ldots\right)\sin m\frac{\pi}{n} \geqslant \sum_{m=1}^{n-1}\left(\frac{a_m}{m} - \frac{a_{m+n}}{m+n}\right)\sin m\frac{\pi}{n}$$
$$\geqslant B \sum_{[\frac{n}{3}]+1}^{[\frac{2n}{3}]}\left(\frac{a_m}{m} - \frac{a_{m+n}}{m+n}\right) \geqslant B \sum_{[\frac{n}{3}]+1}^{[\frac{2n}{3}]} \frac{a_m}{m} \geqslant B a_n \sum_{[\frac{n}{3}]+1}^{[\frac{2n}{3}]} \frac{1}{m} \geqslant B a_n.$$

Denote $H(x) = \int_0^x |C(t)|\,dt$. $C \in L(p', q)$ if and only if

$$\int_0^\pi [(C^*)^{**}(t)]^q t^{q/p'} \frac{dt}{t} < \infty.$$

Where:

$$(C^*)^{**}(t) = \frac{1}{t}\int_0^t C^*(u)\,du \geqslant \frac{1}{t}\int_0^t |C(u)|\,du = \frac{1}{t} H(t).$$

Therefore:

$$\sum_2^\infty a_n^q n^{q/p-1} \leqslant B \sum_2^\infty G\left(\frac{\pi}{n}\right)^q n^{q/p-1} \leqslant B \sum_2^\infty H\left(\frac{\pi}{n}\right)^q n^{q/p-1}$$
$$\leqslant B \sum_2^\infty \int_{\frac{\pi}{n}}^{\frac{\pi}{n-1}} H(t)^q t^{-q/p} \frac{dt}{t} = B \sum_2^\infty \int_{\frac{\pi}{n}}^{\frac{\pi}{n-1}} \left(\frac{1}{t} H(t)\right)^q t^{q/p'} \frac{dt}{t}$$
$$= B \int_0^\pi \left(\frac{1}{t} H(t)\right)^q t^{q/p'} \frac{dt}{t} \leqslant B \|C\|^q_{p',q}.$$

Case (b) follows from case (a), since the Hilbert transform maps $L(p, q)$ to $L(p, q)$ for $1 < p < \infty$, $0 < q \leqslant \infty$. Actually we have shown more than claimed in the theorem:

$$(1) \qquad \int_0^\pi \left(\frac{1}{t}\int_0^t |C(u)|\,du\right)^q t^{q/p'} \frac{dt}{t} < \infty \Leftrightarrow \{a_k\} \in l(p, q) \Leftrightarrow C \in L(p', q).$$

THEOREM 2. (a) *Let* $a_k \downarrow 0$, $1 \leqslant q \leqslant \infty$, $C(x) = \sum_1^\infty a_k \cos kx$. *Then* $\{a_k\} \in l(p,q)$ *iff* $\int_0^\pi |C(t)|^q t^{q/p'} \frac{dt}{t} < \infty$, *with obvious inequalities.*

(b) *Some result holds for* $S(x) = \sum_{k=1}^\infty a_k \sin kx$.

Proof. We shall denote $|||C|||_{p,q} = \left(\int_0^\pi |C(t)|^q t^{q/p} \frac{dt}{t}\right)^{1/q}$. The space of functions such that $|||C|||_{p,q} < \infty$ will be denoted by $L^q_{t^{1/p}}$. This is of course the space L^q with weight function $t^{1/p}$.

When $q = p'$, the results of Theorem 2 coincide of course with those of Theorem 1.

In one direction, the theorem is a consequence of Hardy's inequality and remark (1); since

$$\left(\int_0^\pi \left(\frac{1}{t}\int_0^t |C(u)|\,du\right)^q t^{q/p'} \frac{dt}{t}\right)^{1/q} \leqslant p\left(\int_0^\pi |C(t)|^q t^{q/p} \frac{dt}{t}\right)^{1/q}.$$

For the proof of the other half, we make use of the result of Stein and Weiss on interpolation with change of measure. In terms of interpolation spaces it can be stated in the following form:

$$(2) \qquad (L^{p_0}_{w_0}, L^{p_1}_{w_1})_{\theta,p} = L_{w_0^{1-\theta} w_1^\theta},$$

where $L^p_w = \{f \mid \|f\|^p_w = (\int |wf|^p d\mu)^{1/p} < \infty\}$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

We have seen $\{a_k\} \in Q_1 \cap l^\infty \Rightarrow |C(u)| \leqslant \frac{a_1}{u}$. From this we have deduced $T\colon Q_1 \cap l^\infty \to L(1,\infty)$. However, we can also express the result by $T\colon Q_1 \cap l^\infty \to L^\infty_u$. Again interpolating between this and $T\colon l^1 \to L^\infty$, we get

$$T\colon Q_1 \cap (l^1, l^\infty)_{\theta,q} \to (L^\infty, L^\infty_u)_{\theta,q}.$$

Taking $q = \infty$ and using (2) we get:

$$(3) \qquad T\colon Q_1 \cap l(p,\infty) \to L^\infty_{u^{1/p'}}.$$

From Theorem 1, however:

$$(4) \qquad T\colon Q_1 \cap l(r,r') \to L^{r'}.$$

Interpolating again, we get:

$$T\colon Q_1 \cap \big(l(p,\infty), l(r,r')\big)_{\theta,q} \to (L^\infty_{u^{1/p'}}, L^{r'})_{\theta,q}.$$

Further, $(L^\infty_{u^{1/p'}}, L^{r'})_{\theta,q} = L^q_{u^{(1-\theta)/p'}}$, where $\frac{1}{q} = \frac{\theta}{r'}$. Therefore:

$$T\colon Q_1 \cap l(s,q) \to L^q_{u^{1/s'-1/q}}.$$

Finally, $C(u) \in L^q_{u^{1/s'-1/q}}$ means $\int_0^\pi |C(u)|^q u^{q/s'} \frac{du}{u} < \infty$, and the proof of (a) is complete. The proof of (b) is analogous.

We now take up $Q_2 = \{\{x_n\}^\infty \mid x_n n^{-\beta} \downarrow 0 \text{ some } \beta > 0\}$. Sequences as above are called *quasi-monotone*. See [2] where some of the results below are proved by different methods.

THEOREM 3. *Let* $\{b_n\}$ *be a quasi-monotone sequence. Denote*:

$$(5) \qquad |||\{b_n\}|||_{p,q} = \begin{cases} \left(\sum_1^\infty b_n^q n^{q/p-1}\right)^{1/q}, & q < \infty, \\ \operatorname{Sup}_{0<n} n^{1/p} b_n, & q = \infty. \end{cases}$$

We then have for $1 \leqslant q$, $\|\{b_n\}\|^*_{p,q} < \infty \Leftrightarrow |||\{b_n\}|||_{p,q} < \infty$.

Proof. Let β be such that $b_n n^{-\beta} \downarrow 0$.

$$|||\{b_n\}|||_{p,q} = \left(\sum_1^\infty b_n^q n^{q/p-1}\right)^{1/q} = \left(\sum_1^\infty (b_n n^{-\beta})^q n^{q\beta+q/p-1}\right)^{1/q} = \|\{b_n n^{-\beta}\}\|^*_{p/(p\beta+1),q}.$$

Using the multiplication theorem for $L(p,q)$ spaces (see [5], Theorem 4.5);

$$\|\{b_n n^{-\beta}\}\|^*_{p/(p\beta+1),q} \leqslant B\|\{b_n\}\|^*_{p,q}\|\{n^{-\beta}\}\|^*_{1/\beta,\infty} = B\|\{b_n\}\|^*_{p,q}.$$

For the other half of the theorem, we can easily check that if $\{b_n\}$ is quasi-monotone, there exist $0 < n_0$, a, so that for all $n > n_0$, $b_{n+1} \leqslant b_n\left(1 + \frac{a}{n}\right)$. A finite number of terms will not affect the result, and so we assume $b_{n+1} \leqslant b_n\left(1 + \frac{a}{n}\right)$ for all n. We then define:

$$(6) \qquad c_n = \sum_{j=n}^\infty \frac{b_j}{j}, \qquad a_n = b_n + ac_n.$$

We have $a_{n+1} - a_n = b_{n+1} - b_n - a\frac{b_n}{n} \leqslant 0$. We have $b_n \leqslant a_n$, $a_n \downarrow$, and so $b_n^* \leqslant a_n$. Therefore:

$$\|\{b_n\}\|^*_{p,q} \leqslant \|\{a_n\}\|^*_{p,q} \leqslant |||\{b_n\}|||_{p,q} + a|||\{c_n\}|||_{p,q}.$$

By Hardy's inequality, however:

$$|||\{c_n\}|||_{p,q} \leqslant B|||\{b_n\}|||_{p,q},$$

and the proof is complete.

THEOREM 4. *Let* $\{b_n\}$ *be a quasi-monotone sequence. Denote by* $C(u) = \sum_1^\infty b_n \cos nu$, $S(u) = \sum_1^\infty b_n \sin nu$. *We then have for* $1<p<\infty$, $1 \leqslant q \leqslant \infty$

(a) $\|\{b_n\}\|_{p,q}^* < \infty \Leftrightarrow \|C(u)\|_{p',q} < \infty$,

(b) $\|\{b_n\}\|_{p,q}^* < \infty \Leftrightarrow \|S(u)\|_{p',q} < \infty$,

with the obvious norm inequalities.

Proof. We again define $\{a_n\}$, $\{c_n\}$ as in (6).

If $\|\{b_n\}\|_{p,q}^* < \infty$, $\|\{c_n\}\|_{p,q}^* < \infty$, and so $\|\{a_n\}\|_{p,q}^* < \infty$. Therefore, by Theorem 1, since both $\{a_n\}$ and $\{c_n\}$ are monotone, $\sum_1^\infty a_n \cos nu$, $\sum_1^\infty c_n \cos nu$ belong to $L(p', q)$. Therefore also

$$C(u) = \sum_1^\infty a_n \cos nu - a\sum_1^\infty c_n \cos nu \in L(p', q).$$

Conversely, let $C(u) \in L(p', q)$. Define in this case $C_1(u) = \sum_{n=1}^\infty c_n \cos nu$. We have, by summation by parts:

$$C_1(u) = \sum_{n=1}^\infty \frac{b_n}{n} D_n(u), \quad \text{where} \quad D_n(u) = \frac{\sin(n+\frac{1}{2})\ u}{2\sin u/2}.$$

Therefore:

$$C_1(u) = \tfrac{1}{2}\sum_{n=1}^\infty \frac{b_n}{n}\cos nu + \frac{\cos u/2}{2\sin u/2}\sum_{n=1}^\infty \frac{b_n}{n}\sin nu.$$

The first summand is in L^∞, with L^∞ norm bounded by a multiple of the $L(p',q)$ norm of $C(u)$. For the second summand, note that $0 < \frac{1}{2\sin u/2} \leqslant \frac{1}{u}\cdot\frac{\pi}{2}$, and so:

$$\left|\frac{\cos u/2}{2\sin u/2}\sum_{n=1}^\infty \frac{b_n}{n}\sin nu\right| \leqslant B\cdot\frac{1}{u}\int_0^u |C(t)|\,dt \leqslant BC^{**}(u). \tag{7}$$

Therefore

$$\left\|\frac{\cos u/2}{2\sin u/2}\sum_{n=1}^\infty \frac{b_n}{n}\sin nu\right\|_{p',q} \leqslant B\|C(u)\|_{p',q}.$$

Noting that the coefficients of $C_1(u)$ are monotone, we get $\|\{c_n\}\|_{p,q}^* \leqslant B\|C(u)\|_{p',q}^*$.

$$\sum_1^\infty a_n \cos nu = \sum_1^\infty b_n \cos nu + a\sum_1^\infty c_n \cos nu = C(u) + aC_1(u) \in L(p', q)$$

and so $\|\{a_n\}\|_{p,q}^* \leqslant B\|C(u)\|_{p',q}$, since a_n are monotone. Finally, since $b_n = a_n - ac_n$, $\|\{b_n\}\|_{p,q}^* \leqslant B\|C(u)\|_{p',q}^*$. The proof is complete.

The proof for the case of sine series again is a consequence of the properties of the Hilbert transform, or else can be done directly.

THEOREM 5. *Let* $1<p<\infty$, $1 \leqslant q \leqslant \infty$. *Let* $\{b_n\}$ *be a quasi-monotone sequence. We have*

(a) $\|\{b_n\}\|_{p,q}^* < \infty \Leftrightarrow |||\{b_n\}|||_{p,q} < \infty \Leftrightarrow \|C(u)\|_{p',q} < \infty \Leftrightarrow |||C(u)|||_{p',q} < \infty$.

(b) *Same result for* $S(u)$.

Proof. Only the last equivalence remains to be shown. If $\|C(u)\|_{p',q} < \infty$. We again construct $C_1(u)$ as in the proof of Theorem 4. Since we have

$$\left\|\frac{\cos u/2}{2\sin u/2}\sum_{n=1}^\infty \frac{b_n}{n}\sin nu\right\|_{p',q} \leqslant B\left\|\frac{1}{u}\int_0^u |C(t)|\,dt\right\|_{p',q} \leqslant B|||C(u)|||_{p',q}$$

(the last inequality from Hardy's inequality). The proof then proceeds along same lines as those of Theorem 4, using Theorem 2 rather than Theorem 1.

For the converse implication: If $\|\{b_n\}\|_{p,q}^* < \infty$, then both $\|\{c_n\}\|_{p,q}^* < \infty$ and $\|\{a_n\}\|_{p,q}^* < \infty$. From this we have by Theorem 2,

$$\left\|\sum_1^\infty c_n \cos nu\right\|_{p',q} < \infty, \qquad \left\|\sum_1^\infty a_n \cos nu\right\|_{p',q} < \infty.$$

From this, however, the result follows. The result for the sine series follows similarly. From these theorems follows that if $\{b_k\}$ is a quasi-monotone sequence $\{b_k^*\}$ its non-increasing rearrangement, then

$$\sum_{k=1}^\infty b_k \cos kx \in L(p, q) \Leftrightarrow \sum_{k=1}^\infty b_k^* \cos kx \in L(p, q)$$

for $1<p<\infty$, $1 \leqslant q \leqslant \infty$.

We shall deal now with the connection between the $l(p, q)$ norms of quasi-monotone sequences, and the differentiability properties of the corresponding series. It is in this context that quasi-monotone sequences are natural: The class of trigonometrical series with coefficients in Q_2 is closed under term by term differentiation. The results generalize results of Askey [1]. We shall need some results about the interpolation between Sobolev spaces. We refer the reader to [3], for the details. We include statement of the results:

DEFINITION 6. $f \in L(p, q)$ will be said to belong to the Sobolev space $W(r, p, q)$, $r = 1, 2, 3, \ldots$, iff $f^{(j)} \in L(p, q)$ for $j = 1, 2, \ldots, r$. On $W(r, p, q)$

we define the norm

$$\|f\|_{W(r,p,q)} = \sum_{j=0}^{r} \|f^{(j)}\|_{p,q}.$$

We will denote $W(r,p,p) = W(r,p)$.

We will deal only with the case $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$. With respect to the norm defined above, $W(r,p,q)$ is then a Banach space.

DEFINITION 7. (a) $\Delta_h^r f(x) = \sum_{l=0}^{r} (-1)^{r-l}\binom{r}{l} f(x+lh)$.

(b) $\omega_r(t,f;p) = \sup\{\|\Delta_h^r f\|_p / |h| \leqslant t\}$.

DEFINITION 8. $\mathrm{Lip}(a,r,q;p)$ is the space of all functions in L^p so that $(0<a<r)$

$$\Big(\int_0^{\pi} (t^{-a}\omega_r(t,f;p))^q \frac{dt}{t}\Big)^{1/q} < \infty \quad \text{if} \quad q<\infty,$$

$$\sup\{t^{-a}\omega_r(t,f;p)| \ 0<t<\pi\} \quad \text{if} \quad q = \infty.$$

$\mathrm{Lip}(a,r,p;q)$ is a Banach space with the norm

$$\|f\|_p + \Big(\int_0^{\pi} (t^{-a}\omega_r(t,f;p))^q \frac{dt}{t}\Big)^{1/q}.$$

These are the Besov spaces. The basic theorem we shall use is the following:

THEOREM 9. $(L^p, W(r,p))_{a/r,q} = \mathrm{Lip}(a,r,q;p)$.

Proof. See [3], Theorem 4.3.6.

THEOREM 10. *Let* $0<a<r$, $1\leqslant q\leqslant\infty$, $a=k+\beta$, *where* $0<\beta\leqslant 1$. $f\in\mathrm{Lip}(a,r,q;p)$ *if and only if* $f\in W(k,p)$, *and* $f^{(k)}\in\mathrm{Lip}(\beta,1,q;p)$ *when* $0<\beta<1$, *and* $f^{(k)}\in\mathrm{Lip}(1,2,q;p)$ *when* $\beta=1$.

Proof. See [3], Theorem 4.3.8.

THEOREM 11. *Let* $\{b_n\}$ *be a quasi-monotone sequence.*

$$C(u) = \sum_1^{\infty} b_n \cos nu, \quad S(u) = \sum_1^{\infty} b_n \sin nu, \quad 1<p<\infty, \ 1\leqslant q\leqslant\infty.$$

Then

(a) $C(u)\in W(r,p,q)$ *iff* $\{b_n\}\in l\Big(\frac{p'}{p'r+1}, q\Big)$.

(b) *Same result for* $S(u)$.

Proof. Let $C(u)\in W(r,p,q)$. Then $\sum_1^{\infty} b_n n^r \begin{Bmatrix}\cos nu\\ \sin nu\end{Bmatrix} \in L(p,q)$. If now $\{b_n\}$ is quasi-monotone so is $\{b_n n^r\}$. Therefore $\{b_n n^r\}\in l(p',q)$. By Theorem 3, $|||\{b_n n^r\}|||_{p',q} < \infty$. But $|||\{b_n n^r\}|||_{p',q} = |||\{b_n\}|||_{\frac{p'}{p'r+1},q}$.

Again using Theorem 3, $\{b_n\}\in l\Big(\frac{p'}{p'r+1}, q\Big)$.

The converse is proved similarly: If $\{b_n\}\in l\Big(\frac{p'}{p'r+1}, q\Big)$ using Theorem 3 twice, we get $\{n^r b_n\}\in l(p',q)$ from which we conclude $C^{(r)}(u)\in L(p,q)$, and so $C(u)\in W(r,p,q)$. The Hilbert transform preserves $W(r,p,q)$ for $1<p<\infty$, from which the result for the sine series follows.

THEOREM 12. $T: f \to \frac{1}{2\sin u/2}\int_0^u f$ *preserves* $\mathrm{Lip}(a,r,q;p)$ *for* $1<p<\infty$.

Proof. The transformation clearly preserves L^p and $W(r,p)$. By interpolation the result follows. We shall need the theorem in the following form:

$$(8) \qquad \begin{aligned} T_1&: \sum b_n \sin nu \to \frac{1}{2\sin u/2}\sum \frac{b_n}{n}(1-\cos nu),\\ T_2&: \sum b_n \cos nu \to \frac{1}{2\sin u/2}\sum \frac{b_n}{n}\sin nu, \end{aligned}$$

preserve $\mathrm{Lip}(a,r,q;p)$.

THEOREM 13. *Let* $\{b_n\}$ *be a quasi-monotone sequence.*

$$C(u) = \sum b_n \cos nu, \quad S(u) = \sum b_n \sin nu.$$

Then: for $1<p<\infty$, $1\leqslant q\leqslant\infty$, $C(u)\in\mathrm{Lip}(a,r,q;p)$ *iff* $b_n\in l\Big(\frac{p'}{ap'+1}, q\Big)$.

Same result for $S(u)$.

Proof. Let $\{b_n\}\in l\Big(\frac{p'}{ap'+1}, q\Big) = \Big(l(p',p), l\Big(\frac{p'}{rp'+1}, p\Big)\Big)_{a/r,q}$. Q_2 is a MQC, and $T\{b_n\} = \sum b_n \cos nu$ is linear and maps $l(p',p)$ to L^p, and $l\Big(\frac{p'}{rp'+1}, p\Big)$ to $W(r,p)$. It therefore maps $l\Big(\frac{p'}{ap'+1}, q\Big)$ to $\mathrm{Lip}(a,r,q;p)$.

Conversely, let $C(u)\in\mathrm{Lip}(a,r,q;p)$. $a=k+\beta$, with $0<\beta\leqslant 1$. Then $C^{(k)}(u)\in\mathrm{Lip}(\beta,1,q;p)$ if $0<\beta<1$, $C^{(k)}(u)\in\mathrm{Lip}(1,2,q;p)$. If the coefficients of $C^{(k)}(u)$, i.e. $\{b_n n^k\}$ are monotone decreasing, from Askey's results [1], we have $\{b_n n^k\}\in l\Big(\frac{p'}{p'\beta+1}, q\Big)$ from which follows $\{b_n\}\in l\Big(\frac{p'}{p'a+1}, q\Big)$. Using the decomposition (6) the problem is therefore reduced to showing that if $\{b_n\}$ is quasi-monotone,

$$C(u)\in\mathrm{Lip}(a,r,q;p)$$

then

$$\sum_{n=1}^{\infty}\Big(\sum_{j=n}^{\infty}\frac{b_j}{j}\Big)\cos nu \in\mathrm{Lip}(a,r,q;p).$$

Using summation by parts

$$\left(\sum_{j=n}^{\infty}\frac{b_j}{j}<\infty \text{ follows from } C(u)\in L^p, 1<p\right),$$

$$\sum_{n=1}^{\infty}\left(\sum_{j=n}^{\infty}\frac{b_j}{j}\right)\cos nu = \sum_{1}^{\infty}\frac{b_n}{n}D_n(u) = \frac{\cos u/2}{2\sin u/2}\sum\frac{b_n}{n}\sin nu +$$

$$+\tfrac{1}{2}\sum\frac{b_n}{n}\cos nu.$$

Using Theorem 12, the proof for the cosine series is complete. The proof for the sine series is similar. Let us remark that, comparing Theorems 13 and 11, we get for Fourier series with quasi-monotone coefficients: $f\in W(k,p,q)$ iff $f\in \mathrm{Lip}(k,r,q;p)$ $(k<r)$. The assumption of quasi-monotonicity is crucial: The Weierstrass function $\sum_{k=1}^{\infty}2^{-k}\cos 2^k x$ belongs to the Zygmund class $\mathrm{Lip}(1,2,\infty;p)$ but does not have a finite derivative at any point. In particular it does not belong to $W(1,p,\infty)$.

III. A remark on quasi-monotonicity. We have seen that the class of Fourier series with quasi-monotone coefficients, being closed under differentiation, is in some problems more natural than the class of Fourier series with monotone coefficients.

The decomposition II.6 we employed for quasi-monotone sequences seems suitable for generalizing results on Fourier series with monotone coefficients, to those with quasi-monotone ones. Since this is outside the scope of this note, we limit ourselves to one example.

We shall present a simple proof of a theorem of Shah [7]. This theorem generalizes a theorem of Chaundy and Jaulliffe who proved it for series with monotone coefficients. See [9], Chapter V.

THEOREM 1. *Let $\{b_n\}$ be quasi-monotone. A necessary and sufficient condition that $S(u)=\sum b_n\sin nu$ should converge uniformly is $nb_n=o(1)$.*

Proof. Assume $nb_n\to 0$. Then $n\sum_{j=n}^{\infty}\frac{b_j}{j}\to 0$, and we therefore have $na_n\to 0$. By the original theorem of Chaundy and Jaulliffe, $\sum c_n\sin nu$ and $\sum a_n\sin nu$ converge uniformly, and hence $\sum b_n\sin nu$ does.

Conversely, if $\sum b_n\sin nu$ converges uniformly, $\sum\frac{b_j}{j}<\infty$.

$$\sum c_n\sin nu = \sum\frac{b_n}{n}\tilde{D}_n(u) = \tfrac{1}{2}\cot u/2\sum\frac{b_n}{n}(1-\cos nu)+\tfrac{1}{2}\sum\frac{b_n}{n}\sin nu.$$

The last series converges uniformly, while, denoting $S^N(u)=\sum_{n=N}^{\infty}b_n\sin nu$,

we have

$$\frac{1}{2}\cot u/2\sum_{N}^{\infty}\frac{b_n}{n}(1-\cos nu) = \frac{1}{2}\cot u/2\int_0^u S^N(t)\,dt,$$

and so is small uniformly in u. We therefore have also uniform convergence of $\sum a_n\sin nu$. From these follows by the original theorem $na_n\to 0$, $nc_n\to 0$ and finally $nb_n\to 0$.

References

[1] R. Askey, *Smoothness conditions for Fourier series with monotone coefficients* Acta Sci. Math. (Szeged) 28 (1967), pp. 169–171.
[2] — and S. Wainger, *Integrability theorems for Fourier series*, Duke Math. J. 33 (1966), pp. 223–228.
[3] P. L. Butzer and H. Berens, *Semigroups of operators and approximation*, 1967.
[4] T. Holmstedt, *Interpolation of quasi-normed spaces*, Math. Scand. 26 (1970), pp. 177–199.
[5] R. Hunt, *On $L(p,q)$ spaces*, L'Enseignement Math. 12 (1966), pp. 45–71.
[6] Y. Sagher, *Interpolation of r-Banach spaces*, this volume, pp. 45–70.
[7] S. M. Shah, *Trigonometric series with quasi-monotone coefficients*, Proc. Amer. Math. Soc. 13 (1962), pp. 266–273.
[8] E. Stein and G. Weiss, *Interpolation of operators with change of measures*, Trans. Amer. Math. Soc. 87 (1958), pp. 159–172.
[9] A. Zygmund, *Trigonometric series*, 1968.

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