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# An embedding theorem for commutative $B_0$-algebras

by

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**Abstract.** If $A$ is a commutative Banach algebra with identity, then it may be embedded in a commutative $B_0$-algebra $A'$ such that: (i) the Jacobson radical of $A'$ intersects $A$ in precisely the set of nilpotent elements of $A$; and (ii) if $A$ has no non-zero nilpotent elements, then $A'$ is semisimple. An application of this result gives an example of a $B_0$-algebra whose quotient by the closure of its radical is not semisimple.

**1.** Throughout this paper, the algebras considered will be *commutative* algebras over the complex field, and will each possess an *identity*, denoted by 1. The results proved will also hold for real algebras, by obvious modifications to the proofs. The term *radical* will mean the Jacobson radical, and an algebra will be called semisimple if it has zero radical. A *$B_0$-algebra* is a complete, metrizable, locally convex algebra.

In [5] Rolewicz proved that if $A$ is a Banach algebra with a non-nilpotent element $x$, then $A$ can be embedded in a $B_0$-algebra $A'$ whose radical $\mathscr{R}(A')$ does not contain $x$. He then asked (Problem 1) whether this would still be true if $A$ were just a $B_0$-algebra. This question was answered in the negative by Kitainik [3], who proved that such an embedding is possible if and only if $x$ is not 'almost nilpotent'. Rolewicz also asked (Problem 2) whether, if the Banach algebra $A$ has no non-zero nilpotent elements, it can be embedded in a semisimple $B_0$-algebra. In the present paper we shall show that this is so. In fact, we shall prove that every $B_0$-algebra $A$ can be embedded in a $B_0$-algebra $A'$ such that $\mathscr{R}(A') \cap A$ is precisely the set of almost nilpotent elements of $A$, and $\mathscr{R}(A') = 0$ if $A$ has no non-zero almost nilpotent elements. Using his embedding theorem, Rolewicz found that not every $B_0$-algebra has a closed radical. His Problem 3 asked whether $\overline{A/\mathscr{R}(A)}$ is semisimple for every $B_0$-algebra $A$. We shall use our stronger embedding theorem to prove that this is not true.

**2.** In what follows we shall make essential use of the algebra $L^\omega[0,1]$ introduced by Arens [1]. The space $L^\omega[0,1]$ is the space of those measurable, complex-valued functions $f$ on the unit interval such that

$$\|f\|_p = \Big[\int_0^1 |f(t)|^p\,dt\Big]^{1/p} < \infty \qquad (p = 1, 2, 3, \ldots),$$

with functions equal almost everywhere identified. Under pointwise multiplication and the topology given by the seminorms $\|\cdot\|_p$ $(p = 1, 2, 3, \ldots)$, it becomes a $B_0$-algebra. We prove one technical lemma about $L^\omega[0, 1]$.

LEMMA. *Let $S$ be a subset of $[0, 1]$ of positive measure, and let $A_n$, $B_n$ $(n = 1, 2, 3, \ldots)$ be sequences of positive numbers. Then there exists a function $f \in L^\omega[0, 1]$ such that $f(t) = 0$ $(t \notin S)$, $f(t) \geqslant 1$ $(t \in S)$, and*

$$\int_0^1 f(t)^n dt \geqslant A_n + B_n \int_0^1 f(t)^{n-1} dt \quad (n = 1, 2, 3, \ldots).$$

Proof. It is clearly sufficient to prove the result for $S = [0, 1]$. For this, we shall choose sequences of positive numbers $a_n$, $\delta_n$ $(n \geqslant 1)$, with $\delta_1 = 1$, $\delta_n \downarrow 0$, and we shall then define

$$f_n(t) = \begin{cases} a_i & (\delta_{i+1} < t \leqslant \delta_i, \quad i \leqslant n-1), \\ a_n & (0 < t \leqslant \delta_n), \\ 1 & (t = 0); \end{cases}$$

$$f(t) = \lim_{n\to\infty} f_n(t) \quad (0 \leqslant t \leqslant 1).$$

Put $a_1 = A_1 + 2B_1 + 1$. Suppose that, for $1 \leqslant m \leqslant n-1$, $a_m$, $\delta_m$ have been chosen so that:

(i)$_m$ $a_m \geqslant 1$;

(ii)$_m$ $I(m, r) \leqslant I(r, r) + 1 - 2^{r-m}$ $(1 \leqslant r \leqslant m-1)$;

(iii)$_m$ $I(m, m) \geqslant A_m + B_m\big(I(m-1, m-1) + 1\big)$;

where

$$I(m, r) = \int_0^1 f_m(t)^r dt = \sum_{i=1}^{m-1} a_i^r(\delta_i - \delta_{i+1}) + a_m^r \delta_m \quad (r, m \geqslant 1),$$

and $I(0, 0) = 0$. We put $\delta_n = (a_n - a_{n-1})^{-(n-\frac{1}{2})}$, and choose $a_n$ large enough so that:

(i) $a_n - a_{n-1} \geqslant [2a_{n-1}]^{2n}$;

(ii) $a_n - a_{n-1} \geqslant \big[A_n + B_n\big(I(n-1, n-1) + 1\big)\big]^2$.

Then (i)$_{n-1}$ and (i) imply (i)$_n$; (i)$_{n-1}$, (ii)$_{n-1}$ and (i) imply (ii)$_n$; whilst (iii)$_n$ follows directly from (ii). Thus the sequence of functions $f_n$ satisfies (ii)$_n$ and (iii)$_n$ for every $n$. Letting $n \to \infty$ in (ii)$_n$, we obtain that $f \in L^\omega[0, 1]$ and

$$\int_0^1 f(t)^r dt \leqslant I(r, r) + 1,$$



for all $r$. Substituting this in (iii)$_n$ gives:

$$\int_0^1 f_n(t)^n dt \geqslant A_n + B_n \int_0^1 f(t)^{n-1} dt.$$

The required inequality for $f$ follows immediately.

**3.** We shall also use the projective tensor product construction. We recall that if $A$, $B$ are $B_0$-algebras with seminorms $\{p_i\}$, $\{q_j\}$ respectively, then the projective tensor product $A \hat{\otimes} B$ is the completion of $A \otimes B$ in the topology defined by the seminorms

$$(p_i \otimes q_j)(u) = \inf\Big\{\sum_{n=1}^{k} p_i(x_n) q_j(y_n) \colon u = \sum_{n=1}^{k} x_n \otimes y_n\Big\}.$$

The space $A \hat{\otimes} B$ becomes a $B_0$-algebra in the natural way. If $B = L^\omega[0, 1]$, then it follows as in [2], p. 59, that $A \hat{\otimes} B$ can be identified with an algebra $A_A^\omega[0, 1]$ defined as follows. Let $C_A[0, 1]$ be the space of all continuous maps from $[0, 1]$ into $A$. Then $A_A^\omega[0, 1]$ is the completion of $C_A[0, 1]$ in the topology given by the seminorms

$$p_{in}(f) = \Big[\int_0^1 p_i\big(f(t)\big)^n dt\Big]^{1/n}.$$

We shall think of $A_A^\omega[0, 1]$ as an algebra of measurable functions from $[0, 1]$ into $A$ (modulo equality almost everywhere).

Let $L^1(N)$ denote the semigroup algebra of the semigroup $N$ of all non-negative integers (i.e. $L^1(N)$ is the Banach algebra of all absolutely summable sequences of complex numbers, with convolution multiplication and the $l^1$ norm). Then $A \hat{\otimes} L^1(N)$ can be identified with the algebra of sequences $\tilde{x} = (x_1, x_2, x_3, \ldots)$ $(x_i \in A)$ such that

$$\tilde{p}_i(\tilde{x}) = \sum_{n=1}^{\infty} p_i(x_n) < \infty,$$

for all $i$; the topology being given by the seminorms $\tilde{p}_i$.

**4.** DEFINITION (Kitainik [3]). An element $x$ of a $B_0$-algebra $A$ is said to be *almost nilpotent* if, for every continuous seminorm $p$ on $A$, there exists a number $n$ such that $p(x^n) = 0$.

THEOREM. *Let $A$ be a $B_0$-algebra. Then there exists a $B_0$-algebra $A'$ containing $A$, such that $R(A') \cap A$ is precisely the set of almost nilpotent elements of $A$, and $R(A') = 0$ if $A$ has no non-zero almost nilpotent elements.*

Proof. Let $A' = A \otimes L^{\omega}[0,1] \otimes L^1(N)$, and identify $A$ with its image in $A'$ under the mapping $x \mapsto x \otimes 1 \otimes 1$. As explained above, we shall view $A'$ as an algebra of sequences of elements of $A_A^{\omega}[0,1]$. Suppose $\tilde{z} = (z_1, z_2, \ldots) \in A'$ is such a sequence, and $z_1 \in A_A^{\omega}[0,1]$ is such that the set of $t \in [0,1]$ for which $z_1(t)$ is not almost nilpotent is of non-zero measure. We suppose $\tilde{z} \in \mathscr{R}(A')$, and work to obtain a contradiction. Then, for every $\tilde{y} \in A'$, $\tilde{z}\tilde{y}$ is quasi-regular. Take $\tilde{y} = (0, y, 0, 0, \ldots)$, $(y \in A_A^{\omega}[0,1])$, and let $\tilde{u} = (u_0, u_1, u_2, \ldots)$ be the quasi-inverse of $\tilde{z}\tilde{y}$. Then $\tilde{u} + \tilde{z}\tilde{y} + \tilde{u}\tilde{z}\tilde{y} = 0$, and so

$$\begin{aligned} u_0 &= 0, \\ u_1 &= -z_1 y, \\ u_2 &= z_1^2 y^2 - z_2 y, \\ u_3 &= -z_1^3 y^3 + 2 z_1 z_2 y^2 - z_3 y, \\ u_4 &= z_1^4 y^4 - 3 z_1^2 z_2 y^3 + (2 z_1 z_3 + z_2^2) y^2 - z_4 y, \end{aligned}$$

and an easy induction shows that the general expression for $u_n$ is of the form

$$u_n = (-z_1 y)^n + \sum \lambda_{i_1, \ldots, i_s} z_{i_1}, \ldots, z_{i_s} y^s,$$

with

$$\sum |\lambda_{i_1, \ldots, i_s}| \leqslant 2^{n-1} - 1;$$

the summations being over all $s$-tuples $\{i_1, \ldots, i_s\}$ for $1 \leqslant i_1, \ldots, i_s \leqslant n$, $1 \leqslant s \leqslant n-1$. Hence

$$(*) \quad P_{ij}(u_n) \geqslant p_{ij}(z_1^n y^n) - (2^{n-1} - 1) \max \{ p_{ij}(z_{i_1} \ldots z_{i_s} y^s) : \\ 1 \leqslant i_1, \ldots, i_s \leqslant n, 1 \leqslant s \leqslant n-1 \}$$

for all $i, j, n$.

The set of $t \in [0,1]$ for which $z_1(t)$ is not almost nilpotent is of positive measure. Therefore, there is some set $S \subseteq [0,1]$ of positive measure, and some index $i$, such that $p_i(z_1(t)^n) \neq 0$ for all $t \in S$ and all $n$. In fact, we can choose $S$ so that there is a sequence of positive numbers $\varepsilon_n$ such that $p_i(z_1(t)^n) > \varepsilon_n$ for all $t \in S$ and all $n$. We can also (decreasing $S$ again, if necessary), arrange that there is a sequence of positive numbers $K_n$ such that $p_i(z_{i_1}(t) \ldots z_{i_s}(t)) < K_n$ $(t \in S,\ 1 \leqslant i_1, \ldots, i_s \leqslant n,\ 1 \leqslant s \leqslant n-1,\ n = 1, 2, 3, \ldots)$. We now define $y = 1 \otimes f$, where $f \in L^{\omega}[0,1]$ is chosen, by the lemma above, so that $f(t) = 0$ $(t \notin S)$, $f(t) \geqslant 1$ $(t \in S)$, and

$$\int_0^1 f(t)^n dt \geqslant \varepsilon_n^{-1} + (2^{n-1} - 1) K_n \varepsilon_n^{-1} \int_0^1 f(t)^{n-1} dt,$$



for all $n$. Then

$$\begin{aligned} p_{i1}(z_1^n y^n) &= \int_0^1 p_i(z_1(t)^n) f(t)^n dt \geqslant \varepsilon_n \int_0^1 f(t)^n dt \\ &\geqslant 1 + (2^{n-1} - 1) K_n \int_0^1 f(t)^{n-1} dt \\ &\geqslant 1 + (2^{n-1} - 1) \int_0^1 p_i(z_{i_1}(t) \ldots z_{i_s}(t)) f(t)^s dt \\ &= 1 + (2^{n-1} - 1) p_{i1}(z_{i_1} \ldots z_{i_s} y^s), \end{aligned}$$

for $1 \leqslant i_1, \ldots, i_s \leqslant n$, $1 \leqslant s \leqslant n-1$, $n = 1, 2, 3, \ldots$ Hence, with this choice of $i, y$, $(*)$ implies that $p_{i1}(u_n) \geqslant 1$ for all $n$. But then

$$\tilde{p}_{i1}(\tilde{u}) = \sum_{n=1}^{\infty} p_{i1}(u_n) = \infty,$$

which is the desired contradiction. Therefore $\tilde{z} \notin \mathscr{R}(A')$. The first assertion of the theorem follows immediately.

To prove the second assertion, suppose $A$ has no non-zero almost nilpotent elements. Then we have shown that every $\tilde{z} \in \mathscr{R}(A')$ must have $z_1 = 0$. Now suppose $z_1 = \ldots = z_{k-1} = 0$, $z_k \neq 0$. Again, let $\tilde{y} = (0, y, 0, 0, \ldots)$, for some $y \in A_A^{\omega}[0,1]$, and let $\tilde{u}$ be the quasi-inverse of $\tilde{z}\tilde{y}$. Then in place of $(*)$ we have:

$$p_{ij}(u_{nk}) \geqslant p_{ij}(z_k^n y^n) - (2^{nk-1} - 1) \max \{ p_{ij}(z_{i_1} \ldots z_{i_s} y^s) : \\ k \leqslant i_1, \ldots, i_s \leqslant nk,\ 1 \leqslant s \leqslant n-1 \};$$

a suitable choice of $y, i$ gives $p_{i1}(u_{nk}) \geqslant 1$, for all $n$, and a contradiction follows. Therefore $z \in \mathscr{R}(A')$ implies $z = 0$; i.e. $A'$ is semisimple.

**5.** It is, perhaps, worth noting that this provides an example where the tensor product of two locally convex algebras is semisimple, even though one of the algebras is not. The apparent discrepancy between this and the results of Mallios [4] is merely due to his rather unusual use of the word 'semisimple'.

**6.** Our theorem also gives an example of a $B_0$-algebra $A'$ such that $A'/\mathscr{R}(A')$ is not semisimple; as follows. Let $A$ be the Banach algebra of all continuous, complex-valued functions on $[0,1]$, with the supremum norm:

$$\|f\| = \sup \{ |f(t)| : 0 \leqslant t \leqslant 1 \} \quad (f \in A),$$

and convolution multiplication:

$$(fg)(t) = \int_0^t f(t-s) g(s) ds \quad (f, g \in A,\ 0 \leqslant t \leqslant 1).$$

It follows from a theorem of Titchmarsh ([6], Theorem VII) that the set $N$ of all nilpotent elements of $A$ consists of just those $f \in A$ for which there exists $\varepsilon > 0$ such that $f(t) = 0$ $(0 \leqslant t \leqslant \varepsilon)$. So the closure $\overline{N}$ of $N$ is the set of $f \in A$ such that $f(0) = 0$, and we observe that for all $g \in A$, $g^2 \in \overline{N}$.

Now let $A_1 = A \oplus C.1$ be the algebra obtained by adjoining an identity to $A$, and let $A' = A_1 \otimes L^\infty [0,1] \otimes \alpha^1(N)$, as above. Since the set of all nilpotent elements of $A_1$ is $N$, every $\tilde{z} \in \mathscr{R}(A')$ has $z_1(t) \in N$ a.e. (almost everywhere). So, for every $\tilde{z} \in \overline{\mathscr{R}(A')}$, $z_1(t) \in \overline{N}$ a.e. Thus, if $\tilde{x} = x \otimes \otimes 1 \otimes 1$, with $x \in A \setminus \overline{N}$, then $\tilde{x} \notin \overline{\mathscr{R}(A')}$. However, $(\tilde{x})^2 = x^2 \otimes 1 \otimes 1$ is in the closure of the set of nilpotent elements of $A'$, and hence is in $\overline{\mathscr{R}(A')}$. Thus $A'/\overline{\mathscr{R}(A')}$ has non-zero nilpotent elements. Since it is commutative, it cannot be semisimple.


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# An application of interpolation theory to Fourier series

by

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**Abstract.** In this note we introduce a generalization of the weak interpolation theory of Lions and Peetre. With the help of this generalization we present a unified account of some theorems in the theory of Fourier series with positive coefficients.

The generalization consists in considering interpolation not of subspaces of a topological vector space, but of what we call quasi-cones of it (see Definition I.1). We shall in this note present only the minimal amount of interpolation theory of quasi-cones needed for the application to the problem at hand, and hope to return to the general theory in a subsequent paper. We shall assume familiarity with the notion of $L(p,q)$ spaces, as well as with the terminology of the Lions-Peetre interpolation theory.

## I. Interpolation of quasi-cones.

DEFINITION 1. Let $V$ be a (real or complex) vector space. A subset $Q$ of $V$ will be called a *quasi-cone* (QC) iff $Q+Q \subset Q$. $Q$ is a *cone* iff we also have $\lambda Q \subset Q$ for all $0 < \lambda$. We shall apply our results to cones, but since no additional work is involved, we shall state the results for quasi-cones. Two cones which will be important in the applications we give are:

$$Q_1 = \{\{x_n\}_1^\infty \mid x_n \downarrow 0\} \quad \text{and} \quad Q_2 = \{\{x_n\}_1^\infty \mid \text{for some } \beta, n^{-\beta} x_n \downarrow 0\}.$$

DEFINITION 2. Let $B$ be a vector space over $C$. A *quasi-norm on* $B$ is a function $\| \ \|: B \to R^+$ satisfying:

(a) $\|b\| = 0$ iff $b = 0$.

(b) For all $\lambda \in C$, $b \in B$: $\|\lambda b\| = |\lambda| \, \|b\|$.

(c) A number $k = k(B)$ exists, so that

$$\|b_1 + b_2\| \leqslant k(\|b_1\| + \|b_2\|), \quad \text{for all} \quad b_1, b_2 \in B.$$

A quasi-normed space is a topological vector space, whose topology is given by a quasi-norm.

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