For $\varepsilon > 0$ given, by the preceding arguments, the two last terms in the last member are less then $\varepsilon/3$ for i large enough. For such an i fixed, the first term in the last member is less then $\varepsilon/3$ for m large enough, because the τ_i define a weak decomposition of TE. Hence $(h_m)_{m\in N}$ converges weakly to Tf.

Moreover, $h_m - h_{m-1} \in L_m$ for each $m > 0$. Indeed, since $T'f^{(i)}$ converges to $(h_m)_{m\in N}$ in $l_\infty(F)$, the sequence $\tau_m Tf^{(i)}$ converges to $h_m - h_{m-1}$ for each $m > 0$. Thus, since the L_m are sequentially closed, $h_m - h_{m-1} \in L_m$.

By virtue of the unicity of the decomposition into the L_i, from $Tf = \sum_{m=1}^{\infty} (h_m - h_{m-1})$, we deduce that $h_m - h_{m-1} = \tau_m Tf$ for each m, hence $T'f = (h_m)_{m\in N}$.

c) By the closed graph theorem (see [2], p. 28), the map T' is continuous from E into $l_\infty(F)$. Thus, the sequence $\left(\sum_{i=1}^{m} \tau_i T\right)_{m\in N}$ is equicontinuous and, hence, each $\tau_i T$ is continuous.

Since $\left(\sum_{i=1}^{m} \tau_i T\right)_{m\in N}$ is equicontinuous, the last assumption follows from the Banach–Steinhaus theorem.

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A new definition of nuclear systems with applications to bases in nuclear spaces

by

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Abstract. The concept of nuclear system was introduced by the author in an earlier paper in the same journal as a means of constructing examples of nuclear Fréchet spaces. In this paper the definition is simplified by showing that a nuclear Fréchet space on which there exists a continuous norm can be represented as a projective limit of *injective* nuclear maps on a Hilbert space. This permits the solution of two problems given in the first paper and the improvement of some results in that paper.

These results are applied to the problem of existence of Schauder bases in nuclear Fréchet spaces. An equivalent condition for existence is given and also a test for when a given sequence is a basis. Also a theorem on perturbation of a system with a basis is proved.

A subsequent paper will apply these results to concrete examples of nuclear Fréchet spaces.

The concept of nuclear system was introduced in [2] as a means of constructing examples of Fréchet nuclear spaces and studying the basis problem. In the present paper we give a simplified definition of nuclear system, somewhat anticipated in [2] (see problem 4° of [2]) and obtain new results about nuclear systems and the existence of Schauder bases in Fréchet nuclear spaces.

In Section 1, the new definition is given and it is shown that nuclear systems still characterize full Fréchet nuclear spaces (see below for definitions).

In Section 2 the main results of [2] are restated according to the new definition and new results about nuclear systems are obtained.

In Section 3, we prove new results on the existence of a Schauder basis in the associated space of a nuclear system. We are able to derive

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a test for deciding when a given sequence is a basis. The test requires the computation of the norms of certain operators in l_2. Also we give sufficient conditions for a nuclear system generated by a perturbation of a diagonal matrix to have a basis.

The notation is generally standard and agrees with [2]. We denote by l_2 the usual separable Hilbert space of sequences. If $M \subset l_2$ we denote its orthogonal complement by $M^{\perp}$. We denote by φ the subset of l_2 consisting of all finitely non-zero sequences. By a diagonal operator D on l_2 we mean a linear continuous map defined by $De^n = \lambda_n e^n$, $n = 1, 2, \ldots$, where e^n represents the n^{th} coordinate vector and λ_n is a scalar. The adjoint of an operator A on l_2 is denoted by A^*, and the composition of two operators A, B is written AB. The end of a proof is indicated by the symbol ■.

1. New definition of nuclear system. In [2] we defined a nuclear system as follows. Let (A_k) be a sequence of nuclear operators in l_2 and define the *associated space* $\hat{E}(A_k)$ by

$$\hat{E}(A_k) = \{(x_k):\ x_k \in l_2 \ \text{ and } \ x_k = A_k x_{k+1},\quad k = 1, 2, \ldots\}.$$

With the induced product topology $\hat{E}$ is a locally convex space. Let $P_k\colon \hat{E} \to l_2$, $k = 1, 2, \ldots$, be the usual projection map. We said that (A_k) was a *nuclear system* provided that

(i) each A_k has dense range,

(ii) each P_k is injective.

Two nuclear systems are said to be *equivalent* if their associated spaces are isomorphic as locally convex spaces. A nuclear system is *injective* if each A_k is injective. It was shown in [2] that there exist nuclear systems which are not injective. However, the following result answers in the affirmative question 4° which was posed in [2].

THEOREM 1. *Every nuclear system is equivalent to an injective nuclear system.*

Proof. Let (A_k) be a nuclear system and define $B_k = A_1 \ldots A_k$, $M_k = \ker B_k$. Let $\pi_k\colon l_2 \to l_2$ be the orthogonal projection onto $M_k^{\perp}$. Since each A_k has dense range it follows that each B_k has dense range and hence $M_k^{\perp}$ is infinite dimensional so we can define an isometry $\eta_k\colon l_2 \to l_2$ with $\eta_k(l_2) = M_k^{\perp}$. Let $\tilde{B}_k = B_k \eta_k$. Clearly $\tilde{B}_k$ is injective and has the same image as B_k. Hence $\tilde{B}_k(l_2) \subseteq \tilde{B}_{k-1}(l_2)$ since this is true for B_k.

We consider the *functions* $\pi_k B_k^{-1}\colon B_k(l_2) \to l_2$ and $\tilde{B}_k^{-1}$; $B_k(l_2) \to l_2$. We claim that

$$\tilde{B}_k^{-1} = \eta_k^{-1}(\pi_k B_k^{-1}).$$

Indeed if $x = B_k z \in B_k(l_2)$, then $\pi_k z \in M_k^{\perp}$ so

$$\tilde{B}_k^{-1}\eta_k^{-1}\pi_k B_k^{-1}(x) = \tilde{B}_k \eta_k^{-1}\pi_k(z) = B_k \pi_k(z) = B_k(x)$$

and the result follows from the fact that $\tilde{B}_k$ is injective.

Now we set $\tilde{A}_1 = \tilde{B}_1$ and for each $k > 1$ we define $\tilde{A}_k = \tilde{B}_{k-1}^{-1}\tilde{B}_k$. Then we have

$$\tilde{A}_k = \eta_{k-1}^{-1}(\pi_{k-1}B_{k-1}^{-1})B_k\eta_k = \eta_{k-1}^{-1}\pi_{k-1}B_{k-1}^{-1}B_{k-1}A_k\eta_k = \eta_k^{-1}\pi_{k-1}A_k\eta_k.$$

Hence $\tilde{B}_k = \tilde{A}_1 \ldots \tilde{A}_k$ for all k so that $\ker \tilde{A}_k \subseteq \ker \tilde{B}_k$ so $\tilde{A}_k$ is injective. Moreover, we have

$$\tilde{A}_k = \eta_{k-1}^{-1}(\pi_{k-1}B_{k-1}^{-1})B_k\eta_k = \eta_{k-1}^{-1}\pi_{k-1}B_{k-1}^{-1}B_{k-1}A_k\eta_k = \eta_k^{-1}\pi_{k-1}A_k\eta_k$$

which implies that $\tilde{A}_k$ is nuclear since A_k is nuclear and $\eta_k, \eta_k^{-1}, \pi_k$ are continuous.

Next we observe that

$$\tilde{B}_k^{-1}B_k = \eta_k^{-1}(\pi_k B_k^{-1})B_k = \eta_k^{-1}\pi_k$$

so $\tilde{B}_k^{-1}B_k$ is an isomorphism of l_2 onto itself. Hence we have,

$$\tilde{A}_k(l_2) = \tilde{A}_k\tilde{B}_k^{-1}B_k(l_2) = \tilde{B}_{k-1}^{-1}\tilde{B}_k\tilde{B}_k^{-1}B_k(l_2) = \tilde{B}_{k-1}^{-1}B_{k-1}A_k(l_2),$$

and since $A_k(l_2)$ is dense and $\tilde{B}_k^{-1}B_{k-1}$ is an isomorphism it follows that $\tilde{A}_k$ has dense range. Thus we have shown that $(\tilde{A}_k)$ is an injective nuclear system.

It remains to show that $(\tilde{A}_k)$ is equivalent to (A_k) and for this purpose we verify conditions (i), (ii)′, (iii)′ of Theorem 2, [2] with $A_k, \tilde{A}_k$ interchanged and $n_k = k$. We define (f_k) by

$$f_1 = \text{identity},\qquad f_{k+1} = \tilde{B}_k^{-1}B_k,\qquad k = 1, 2, \ldots$$

Note that by the above argument, each f_k is an isomorphism.

Condition (i). For $k = 1$, we have

$$\tilde{A}_1 f_2 = \tilde{B}_1\tilde{B}_1^{-1}B_1 = B_1 = A_1 = f_1 A_1,$$

and for $k > 1$,

$$\tilde{A}_k f_{k+1} = \tilde{A}_k\tilde{B}_k^{-1}B_k = \tilde{B}_{k-1}^{-1}B_k = \tilde{B}_{k-1}^{-1}B_{k-1}A_k = f_k A_k.$$

Condition (ii)′. This is obvious since P_1 is by hypothesis injective and f_1 is injective so $f_1 P_1$ is injective.

Condition (iii)′. If $x \in P_1(\hat{E}(\tilde{A}_k))$, then $\exists$ a sequence $(x_k) \in \hat{E}(\tilde{A}_k) \ni x_1 = x_k$. Let $y_k = f_k^{-1}x_k$. Then $f_k A_k y_{k+1} = \tilde{A}_k f_{k+1} y_{k+1} = \tilde{A}_k x_{k+1} = x_k = f_k y_k$ so $(y_k) \in \hat{E}(A_k)$ and $x = y_1 \in P_1(\hat{E}(A_k))$. ■

The above theorem permits a simplification of the definition of nuclear system. The result which governs this definition is Theorem 1 of [2] which states that *the associated space of a nuclear system is a full* (that is, admits a continuous norm) *nuclear Fréchet space and every full nuclear Fréchet space is isomorphic to the associated space of some nuclear system.* This result remains true under the following definition which we henceforth adopt.

A *nuclear system* is a sequence of nuclear maps in l_2 which are injective and have dense range. The associated space $\hat{E}$ and the canonical projections (P_k) are defined exactly as before.

PROPOSITION 1. *If E is a full nuclear Fréchet space, then there exists a fundamental system of neighborhoods of 0, (V_k), such that the canonical maps $\hat{E}_{V_{k+1}} \to \hat{E}_{V_k}$ (on the completed quotient spaces) are injective.*

Proof. Let (A_k) be a nuclear system whose associated space is isomorphic to E. Then by definition of the topology, a fundamental system of neighborhoods is given by the sets,

$$V_k = \left\{x = (x_n) \in \hat{E}:\ \|x_n\| \leqslant \frac{1}{k},\ n = 1, \ldots, k\right\}.$$

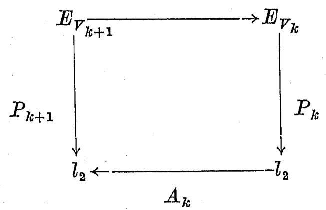
If ϱ_k is the gage of V_k, then we have,

$$\begin{aligned}\varrho_k(x) &= \inf\left\{\lambda > 0:\ \|x_n\| \leqslant \frac{\lambda}{k},\ n = 1, \ldots, k\right\}\\ &= \max\{k\|x_n\|:\ n = 1, \ldots, k\}\\ &= k\max\{\|A_1 \ldots A_{k-1}(x_k)\|, \ldots, \|A_{k-1}(x_k)\|, \|x_k\|\}\end{aligned}$$

and so we have

$$k\|x_k\| \leqslant \varrho_k(x) \leqslant K\|x_k\|,\ \text{where } K = \max\{\|A_1 \ldots A_{k-1}\|, \ldots, \|A_{k-1}\|, 1\}.$$

Thus we may conclude that ϱ_k is a norm and $P_k: \hat{E}_{V_k} \to l_2$ is bicontinuous. Since P_k has dense range ([2], Corollary) it follows that P_k is an isomorphism. Moreover, from the definition of nuclear system, the following commutes diagram



and the result then follows from the fact that A is injective. ■

We remark that the above result also shows that a full Fréchet nuclear space can be constructed so that the canonical maps are any preassigned sequence of injective nuclear operators with dense ranges.

2. Applications. In this section we observe that the new definition of nuclear system permits us to restate the major results of [2] in a simplified form. Also we are able to answer one more question raised in that paper.

The following result is a translation of Theorem 2 of [2].

THEOREM 2. *Two nuclear systems (A_k), $(\tilde{A}_k)$ are equivalent if and only if there exists a subsequence (n_k) of indices and continuous linear maps f_k: $l_2 \to l_2$ such that,*

(i) $\tilde{A}_k f_{k+1} = f_k A_{n_k} \ldots A_{n_{k+1}-1}$, $\quad k = 1, 2, \ldots,$

(ii) f_1 *maps* $\bigcap_{k=n_1}^{\infty} A_{n_1} \ldots A_{n_k}(l_2)$ *injectively onto* $\bigcap_{k=1}^{\infty} \tilde{A}_1 \ldots \tilde{A}_k(l_2)$.

The following result is a translation of the corollary to Theorem 2 of [2].

THEOREM 3. *The associated space of a nuclear system (A_k) has a Schauder basis if and only if there exist diagonal nuclear maps D_k: $l_2 \to l_2$ and continuous linear maps f_k: $l_2 \to l_2$, $k = 1, 2, \ldots,$ such that*

(i) $A_k f_{k+1} = f_k D_k$, $k = 1, 2, \ldots,$

(ii) f_1 *maps* $\bigcap_{k=1}^{\infty} D_1 \ldots D_k(l_2)$ *injectively onto* $\bigcap_{k=1}^{\infty} A_1 \ldots A_k(l_2)$.

The application of Theorem 3 to the basis problem would be considerably simplified if one could say anything more definite about the maps (f_k). In every example known to the author, one can choose the maps to be isomorphisms and usually isometries. We show now that they cannot always be chosen to be isometries.

LEMMA 1. *Let A be a bounded operator on l_2; U, W isometries; and D a diagonal operator with diagonal elements (λ_n) such that $D = UAW^{-1}$. Then the eigenvalues of $A^* A$ are precisely $(|\lambda_n|^2)$. If, in addition, each $\lambda_n \neq 0$ and $|\lambda_i| \neq |\lambda_j|$ for $i \neq j$ and A is symmetric, then $U = VW$, where V is a diagonal operator each of whose diagonal elements has modulus* 1.

Proof. For the first statement we have $A = U^{-1} DW = U^* DW$ so

$$A^* A = W^* D^* U U^{-1} DW = W^* D^* DW$$

and this implies that the eigenvalues of $A^* A$ are precisely the diagonal elements of the diagonal operator $D^* D$ which are $(|\lambda_n|^2)$.

Now incorporate the additional assumptions and set $V = UW^{-1}$. Then we have

$$VD^* V = UW^{-1} D^* UW^{-1} = U(U^* DW)^* W^{-1} = UAW^{-1} = D$$

and

$$V^*DV^* = WU^{-1}DWU^{-1} = WAU^{-1} = D^*.$$

Therefore it follows that $VD^* = DV^*$ and $V^*D = D^*V$.

Suppose that V has the matrix representation (v_{ij}) and take $i \neq j$. First suppose that $|\lambda_i| < |\lambda_j|$. Then if $v_{ji} \neq 0$ and $v_{ij} \neq 0$, it follows that $v_{ij}\lambda_j^* = \lambda_i v_{ji}$ and $v_{ji}\lambda_j = \lambda_i^* v_{ij}$, so

$$|v_{ij}| = \frac{|\lambda_i|}{|\lambda_j|}|v_{ji}| < |v_{ji}| = \frac{|\lambda_i|}{|\lambda_j|}|v_{ij}| < |v_{ij}|$$

which is a contradiction so one and hence both v_{ij}, v_{ji} are zero. A similar argument works if $|\lambda_i| > |\lambda_j|$ so we have shown that V is diagonal. Since V is clearly an isometry it follows that its diagonal elements all have modulus 1. ■

PROPOSITION 2. *There exist nuclear systems for which Theorem 3 does not apply with all of the maps (f_k) isometries.*

Proof. Let (A_k) be a nuclear system in which each A_k is symmetric, its eigenvalues are simple and have distinct moduli, and if $i \neq j$, then the set of eigenvectors of A_i is not identical to that of A_j.

Now suppose we have diagonal maps (D_k) and isometries (f_k) with

$$A_k f_{k+1} = f_k D_k, \quad k = 1, 2, \ldots$$

Then by the first part of Lemma 1, the eigenvalues of $A_k^2 = A_k^* A_k$ are precisely the moduli of the diagonal elements of D_k^2 and these are then the squares of the eigenvalues of A_k. Hence the diagonal elements of D_k have distinct moduli. Moreover, none of them are zero since (D_k) is a nuclear system so that $D_k(l_2)$ must be dense.

Thus we can apply the second statement in Lemma 1 and assert the existence of diagonal operators V_k with diagonal elements of modulus 1 and such that

$$f_k^{-1} = V_k f_{k+1}^{-1} \quad \text{or} \quad f_{k+1} = f_k V_k, \quad k = 1, 2, \ldots,$$

whence

$$f_{k+1} = f_1 V_1 \ldots V_k, \quad k = 1, 2, \ldots,$$

so

$$(V_1 \ldots V_{k-1})^{-1} f_1^{-1} A_k f_1 V_1 \ldots V_k = D_k, \quad k = 1, 2, \ldots \quad (V_0 = \text{identity})$$

or, since diagonal operators commute,

$$f_1^{-1} A_k f_1 = V_1 \ldots V_{k-1} D_k (V_1 \ldots V_k)^{-1} = D_k V_k^{-1}, \quad k = 1, 2, \ldots,$$

so that the eigenvectors of A_k consist of the images under f_1 of the coordinate vectors in l_2. In particular the set of eigenvectors of A_k is independent of k and this contradicts our choice of A_k. ■

PROBLEM 1. *Does Proposition 2 remain valid if we replace isometries by isomorphisms?*

As another application of the simplified definition of nuclear system we answer question 3° of [2] in Proposition 3. This result shows that if we consider a full nuclear Fréchet space to be a certain dense subspace of l_2 (indeed, $P_1(\hat{E})$), equipped with a certain metric topology, then we are dealing with a dense subspace somewhat more complicated than the image of a continuous operator.

PROPOSITION 3. *If (A_k) is a nuclear system, then there exists no continuous operator $A\colon l_2 \to l_2$ such that*

$$\bigcap_{k=1}^{\infty} A_1 \ldots A_k(l_2) = A(l_2).$$

Proof. If we have such an A, then we can replace it by an injective operator which has the same range so we may assume that A is injective. Because each A_k is injective, we can define $T\colon l_2 \to \hat{E}(A_k)$ by

$$Tx = \big(Ax, A_1^{-1}Ax, (A_1A_2)^{-1}Ax, \ldots\big).$$

Moreover, if $Tx = 0$, then in particular, $Ax = 0$ so $x = 0$. Thus T is injective.

Next we show that T is continuous. By definition of the product topology, it suffices to show that each map $(A_1 \ldots A_k)^{-1}A$ is continuous. By our hypothesis we know that $(A_1 \ldots A_k)^{-1}A(l_2) \subset l_2$. Now suppose that (x^ν) is a sequence in l_2 such that it converges to x and the sequence $\big((A_1 \ldots A_k)^{-1}Ax^\nu\big)_\nu$ converges to y. Then we have,

$$\begin{aligned} A_1 \ldots A_k(A_1 \ldots A_k)^{-1}Ax = Ax &= A\lim_\nu x^\nu = \lim_\nu Ax^\nu \\ &= \lim_\nu A_1 \ldots A_k(A_1 \ldots A_k)^{-1}Ax^\nu \\ &= A_1 \ldots A_k \lim_\nu (A_1 \ldots A_k)^{-1}Ax^\nu = A_1 \ldots A_k y. \end{aligned}$$

Since $A_1 \ldots A_k$ is injective it follows that $(A_1 \ldots A_k)^{-1}Ax = y$ so by the closed graph theorem, $(A_1 \ldots A_k)^{-1}A$ is continuous and hence T is continuous.

Finally let $y = (y_k) \in \hat{E}$. Then $y_1 \in \bigcap_k A_1 \ldots A_k(l_2) = A(l_2)$ so there is an $x \in l_2$ with $y_1 = Ax$. But then

$$Tx = \big(y_1, A_1^{-1}y_1, (A_1A_2)^{-1}y_1, \ldots\big) = (y_1, y_2, y_3, \ldots) = y.$$

Hence T is onto so T is an isomorphism but this is impossible since l_2 is a Banach space and $\hat{E}$ is a nuclear space and both are infinite dimensional. ■

3. Some basis theorems. In this section we give some additional theorems characterizing the existence of a Schauder basis in the associated space of a nuclear system. We shall make use of the following well-known result. For a recent proof see [3].

LEMMA 2. *If (x_n) is a total sequence in a Fréchet space E and (ϱ_k) is a fundamental system of seminorms for E, then (x_n) is a Schauder basis for E if and only if for each k there exists $j \geqslant k$ and $M > 0$ such that for any sequence t of scalars and integers $p \leqslant q$, we have*

$$\varrho_k(t_1x_1 + \ldots + t_px_p) \leqslant M\varrho_j(t_1x_1 + \ldots + t_qx_q).$$

Let (A_k) be a nuclear system and write $B_k = A_1 \ldots A_k$, $k = 1, 2, \ldots$, $B_0 =$ identity. If (b_n) is a linearly independent total sequence in $\hat{E} = \hat{E}(A_k)$ and $(s_{\mu\nu})$ is an infinite matrix of scalars we say that $(s_{\mu\nu})$ is a *regularizer* for (b_n) if the following conditions are satisfied.

(R$_1$): *For each ν, $(s_{\mu\nu})_\mu \in \varphi$.*

In view of (R$_1$) we may define the operator $S\colon \varphi \to l_2$ by

$$S(t) = \sum_\mu \sum_\nu s_{\mu\nu} t_\nu b^{1\mu}, \qquad t \in \varphi,$$

where we use the notation, $b^{kn} = P_k b_n \in l_2$. Note that $S(\varphi)$ is contained in the vector subspace generated by the sequence $(b^{1\mu})_\mu$ and hence

$$S(\varphi) \subset \bigcap_k B_k(l_2).$$

This permits us to write,

(R$_2$): *$B_k^{-1}S(\varphi)$ is dense in l_2 for each $k \geqslant 0$.*

(R$_3$): *S is injective.*

Now for each $p = 1, 2, \ldots$, let $\pi_p\colon l_2 \to \varphi$ be the projection onto the first p coordinates. If $\tau \in B_j^{-1}S(\varphi)$, then by (R$_3$) there exists unique $t \in \varphi$ with $\tau = B_j^{-1}S(t)$. Hence if j, k are non-negative integers and p is a positive integer, we can apply this and the fact that $S(\varphi) \subset B_k(l_2)$ to define a linear map $T = T_{k,p,j}\colon B_j^{-1}S(\varphi) \to l_2$ by the relation,

$$TB_j^{-1}S(t) = B_k^{-1}S\pi_p(t), \qquad t \in \varphi.$$

We may then write the final condition which completes the definition,

(R$_4$): *For each $k \geqslant 0$ there exists $j \geqslant k$ such that each $T_{k,p,j}$ can be extended to a continuous operator on l_2 and the sequence $(T_{k,p,j})_p$ is a bounded sequence of operators in l_2.*

The following result might be considered as analogous to the Gram–Schmidt orthogonalization process for constructing orthonormal bases in separable Hilbert spaces.

THEOREM 4. *If (A_k) is a nuclear system, then $\hat{E}$ has a Schauder basis if and only if there exists a total linearly independent sequence which possesses a regularizer. In this case, every total linearly independent sequence possesses a regularizer.*

Proof. Assume that E has a basis and (b_n) is a total linearly independent sequence. By the Krein–Milman–Rutman theorem for full Fréchet spaces (see [1], Theorem 1), the vector subspace generated by (b_n) contains a basis (c_ν). Hence we have an infinite matrix $(s_{\mu\nu})$ satisfying (R$_1$) such that

$$c_\nu = \sum_\mu s_{\mu\nu} b_\mu, \qquad \nu = 1, 2, \ldots$$

Let $k \geqslant 0$ and $t \in \varphi$. Then

$$B_k^{-1}S(t) = \sum_\mu \sum_\nu s_{\mu\nu} t_\nu b^{k+1,\mu} = \sum_\nu t_\nu \sum_\mu s_{\mu\nu} b^{k+1,\mu} = \sum_\nu t_\nu c^{k+1,\nu}.$$

But (c_ν) is total in $\hat{E}$ and $P_{k+1}(\hat{E})$ is dense in l_2 so $(c^{k+1,\nu})_\nu$ is total in l_2 and hence by the above equation, $B_k^{-1}S(\varphi)$ is dense in l_2 so (R_2) is satisfied.

Next, if $S(t) = 0$, we have

$$\sum_\nu t_\nu c^{1\nu} = \sum_\nu \sum_\mu t_\nu s_{\mu\nu} b^{1\nu} = S(t) = 0$$

and since P_1 is injective and (c_ν) is linearly independent it follows that $(c^{1\nu})_\nu$ is linearly independent so $t = 0$. Hence (R$_3$) is satisfied.

Finally we apply Lemma 2 to (c_n) to assert that for each $k \geqslant 0$ there exists $j \geqslant k$ and $M > 0$ such that for $t \in \varphi$ and integers $p \leqslant q$ we have

$$\varrho_{k+1}(t_1c_1 + \ldots + t_pc_p) \leqslant M\varrho_{j+1}(t_1c_1 + \ldots + t_qc_q),$$

where (ϱ_k) is any fundamental sequence of seminorms for $\hat{E}$. By the argument used in Proposition 1, we can take ϱ_k to be defined by $\varrho_k(x) = \|x_k\|$. Hence we have,

$$\|t_1c^{k+1,1} + \ldots + t_pc^{k+1,p}\| \leqslant M\|t_1c^{j+1,1} + \ldots + t_qc^{j+1,q}\|$$

or

$$\Big\|B_k^{-1}\Big(\sum_{\nu=1}^{p} t_\nu c^{1\nu}\Big)\Big\| \leqslant M\Big\|B_j^{-1}\Big(\sum_{\nu=1}^{q} t_\nu c^{1\nu}\Big)\Big\|$$

so that for any $t \in \varphi$ we have

$$\|B_k^{-1}S\pi_p(t)\| \leqslant M\|B_j^{-1}S(t)\|$$

or

$$\|T_{k,p,j}B_j^{-1}S(t)\| \leqslant M\|B_j^{-1}S(t)\|$$

and this inequality immediately implies (R$_4$). Hence $(s_{\mu\nu})$ is a regularizer for (b_n).

Conversely, let (b_n) be a total linearly independent sequence in $\hat{E}$ which possesses a regularizer $(s_{\mu\nu})$. Define the sequence (c_ν) in $\hat{E}$ by

$$c_\nu = \sum_\mu s_{\mu\nu} b_\mu .$$

Then for each $k \geqslant 0$ and $t \in \varphi$, we have

$$B_k^{-1} S(t) = \sum_\nu t_\nu c^{k+1,\nu}$$

so by (R_2) we conclude that for each $k \geqslant 0$, $(c^{k+1,\nu})_\nu$ is total in l_2 and this implies that (c_ν) is total in $\hat{E}$.

Finally, using the same seminorms as in the first half of the proof, we conclude from (R_4) that if $k \geqslant 0$, then there exists $j \geqslant k$ and $M > 0$ such that

$$\|T_{k,p,j} B_j^{-1} S(t)\| \leqslant M \|B_j^{-1} S(t)\|.$$

Reversing the steps in the last portion of the first half of the proof, we obtain the inequality required by Lemma 2 so we may conclude that (c_ν) is a basis for E. ∎

There are two interpretations of Theorem 4 which are useful in applications. The first, given in Proposition 4 relates the existence of a basis directly to an operator independently of matrices and the second, given in Proposition 5 is a test for a given sequence to be a basis.

Concrete examples illustrating these applications will appear in a subsequent paper.

PROPOSITION 4. *If (A_k) is a nuclear system, then $\hat{E}$ has a basis if and only if there exists a linear injective map, $S\colon \varphi \to \bigcap_k B_k(l_2)$ with $B_k^{-1}S(\varphi)$ dense in l_2 for each $k \geqslant 0$ and such that for each $k \geqslant 0$ there exists $j \geqslant k$ such that*

$$\sup_p \|B_k^{-1} S \pi_p S^{-1} B_j|_{B_j^{-1}S(\varphi)}\| < \infty .$$

In this case, if we consider $\hat{E}$ to be represented by $\bigcap_k B_k(l_2)$ via $P_1(\hat{E})$, then the basis is the sequence $\big(S(e^\nu)\big)_\nu$.

Proof. Suppose we have S as above. Since $S(e^n) \in \bigcap_k B_k(l_2) = P_1(\hat{E})$, we can define $b_n = P_1^{-1} S(e^n)$. Since S is injective, (b_n) is linearly independent. Also for any $k \geqslant 0$ we have

$$b^{k,n} = P_k b_n = P_k P_1^{-1} S(e^n) = B_{k-1}^{-1} S(e^n)$$

so $(b^{k,n})_n$ is total in l_2 for each k. Hence (b_n) is total in l_2.

Now let $(s_{\mu\nu})$ be the identity matrix. Hence

$$S(t) = \sum_n t_n S(e^n) = \sum_\mu \sum_\nu s_{\mu\nu} t_\nu b^{1\mu}, \quad t \in \varphi ,$$

so that S is the same as in the definition of regularizer. Thus (R_1), (R_2), (R_3) are obviously satisfied and for (R_4) we observe that for $t \in \varphi$,

$$B_k^{-1} S \pi_p S^{-1} B_j B_j^{-1} S(t) = B_k^{-1} S \pi_p (t)$$

so $T_{k,p,j} = B_k^{-1} S \pi_p S^{-1} B_j|_{B^{-1}S(\varphi)}$ and (R_4) is satisfied. The conclusion then follows from Theorem 4 and by the proof of Theorem 4, the basis is (b_n) so the last statement is proved.

Conversely if (b_n) is a basis for $\hat{E}$, then by the proof of Theorem 4 the identity matrix is a regularizer for (b_n) and this gives $S\colon \varphi \to \bigcap_k B_k(l_2)$ with $S(e^n) = b^{1,n}$. Then (R_2), (R_3) and (R_4) immediately yield the desired properties for S. ∎

PROPOSITION 5. *Let (A_k) be a nuclear system and (b_n) a total linearly independent sequence in $\hat{E}$. Then (b_n) is a Schauder basis for $\hat{E}$ if and only if for each $k \geqslant 0$ there exists $j \geqslant k$ such that*

$$\sup_p \|B_k^{-1} S \pi_p S^{-1} B_j|_{B_j^{-1}S(\varphi)}\| < \infty ,$$

where $S\colon \varphi \to l_2$ is defined by $S(e^n) = b^{1n}$.

Proof. It suffices to show that this condition is equivalent to the identity matrix being a regularizer for (b_n). Clearly S is the map given in the definition of regularizer when $(s_{\mu\nu})$ is the identity matrix. (R_1) and (R_3) are obvious. (R_2) follows from the fact that $\big(B_k^{-1}S(e^n)\big)_n = (b^{k+1,n})_n$ which is total in l_2 since (b_n) is total and P_{k+1} has dense range. Finally, (R_4) is clearly equivalent to the given condition. ∎

Remark. Without going into details, we observe that the results of this section can be adjusted to take into consideration the well-known fact that in a nuclear Fréchet space every Schauder basis is an unconditional basis. Using a perfectly straightforward generalization of Lemma 2, we can replace, in (R_4), the index p, running through the positive integers by the index σ, running through the collection of finite subsets of the positive integers and π_σ is then the orthogonal projection onto the subspace of l_2 generated by $\{e^n\}_{n\in\sigma}$. Appropriate adjustments in Propositions 4, 5 can then be easily made.

We end this section with some basis theorems of an entirely different nature. For simplicity, we consider only singly generated nuclear systems. It is obvious that if A is a diagonal matrix, then $\hat{E}$ has a basis. Moreover, if A is similar to a diagonal matrix, that is, if there exists an isomorphism U such that $U^{-1}AU$ is diagonal, then $\hat{E}$ has a basis. Indeed one can apply either Theorem 3 with $f_k = U$ or Theorem 4 with $b_n = P_1^{-1}U(e^n)$ and $(s_{\mu\nu})$ the identity matrix (so that $S = U$). A natural question that one might consider is what happens if A is a "perturbation" of a matrix that is similar to a diagonal matrix. We can apply Proposition 4 to obtain the following results.

PROPOSITION 6. *Let $A = T+J$ be a nuclear map, where $T = UDU^{-1}$ for some isomorphism U and injective diagonal map D. Assume further that for each $k \geqslant 0$ we have*

$$\|D^{-k}(D+U^{-1}JU)^k - I\| < 1.$$

Then A generates a nuclear system whose associated space possesses a Schauder basis.

Proof. In order to show that A generates a nuclear system, it suffices to show that A and A^* are injective.

Suppose $A(x) = 0$. Let $y = U^{-1}(x)$ so that we have

$$0 = T(x)+J(x) = UDU^{-1}(x)+J(x) = UD(y)+JU(y)$$

and hence since D is injective,

$$y = -D^{-1}U^{-1}JU(y).$$

But, applying our hypothesis for $k = 1$, we obtain, if $y \neq 0$,

$$\|y\| = \big\|\big(D^{-1}(D+U^{-1}JU)-I\big)(y)\big\| < \|y\|$$

so it follows that $y = 0$ and hence, $x = 0$.

Now suppose $A^*(x) = 0$. Let $y = D^*U^*(x)$ and we have for each $z \in l_2$, $z \neq 0$,

$$\Big|\Big(\big(D^{-1}(D+U^{-1}JU)-I\big)(U^{-1}z), y\Big)\Big| < \big|\big(U^{-1}(z), y\big)\big|$$

so

$$\big|\big((A-UDU^{-1})(z), x\big)\big| < \big|\big(U^{-1}(z), y\big)\big|$$

so

$$\big|\big(UDU^{-1}(z), x\big)\big| < \big|\big(UDU^{-1}(z), x\big)\big|,$$

and this implies that x is orthogonal to $UDU^{-1}(l_2)$. But since D is an injective diagonal operator, it has dense range so it follows that $x = 0$.

Thus we have shown that A generates a nuclear system. To obtain the existence of a basis, we apply Proposition 4 with $S = U$. By hypothesis, if we define for each $k \geqslant 1$ the map

$$E_k = D^{-k}(D+U^{-1}JU)^k - I,$$

then E_k is a continuous operator and $\|E_k\| < 1$ so $I+E_k$ is invertible. Moreover, we have,

$$U^{-1}B_kU = (U^{-1}AU)^k = (D+U^{-1}JU)^k = D^k(I+E_k).$$

Since e^n is in the range of D^k and $I+E_k$ is an isomorphism it follows that φ is contained in the range of $U^{-1}B_kU$ which equals the range of $U^{-1}B_k$. Hence $S(\varphi) = U(\varphi) \subset \bigcap_k B_k(l_2)$. Moreover, we have,

$$B_k^{-1}S(\varphi) = A^{-k}U(\varphi) = U(U^{-1}AU)^{-k}(\varphi) = U(I+E_k)^{-1}D^{-k}(\varphi)$$
$$= U(I+E_k)^{-1}(\varphi)$$

and since $U(I+E_k)^{-1}$ is an isomorphism, it follows that $B_k^{-1}S(\varphi)$ is dense.

Finally, we compute for $j \geqslant k$ and any p,

$$\|B_k^{-1}S\pi_pS^{-1}B_j\| = \|A^{-k}U\pi_pU^{-1}A^j\| = \|U(I+E_k)^{-1}D^{-k}\pi_pD^j(I+E_k)U^{-1}\|$$
$$\leqslant \|U\|\,\|(I+E_k)^{-1}\|\,\|D^{j-k}\|\,\|I+E_k\|\,\|U^{-1}\| < \infty$$

so the result follows from Proposition 4. ■

COROLLARY 1. *If $A = D+J$ is a nuclear map, where D is an injective diagonal map and for each k we have*

$$\|(I+JD^{-1})^k - I\| < 1,$$

then A generates a nuclear system whose associated space has a Schauder basis.

Proof. This is immediate from Proposition 6 if we take U to be the identity map. ■

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