Since $s_N(f,x) = o(\log N)$ for a.e. x, we see that $s_N(f,x)\lambda_N(\varphi,t) = o(1)$ for a.e. t, x as $N\to\infty$. By summation by parts we have

$$\sum_0^N \lambda_n(\varphi,t)A_n(f,x) = \sum_{j=0}^{N-1} A_{2j+1}(\varphi,t)s_j(f,x)+s_N(f,x)\cdot\lambda_N(\varphi,t),$$

and thus Theorem 3 completes the proof.

6. We conclude with two remarks.

Remark 4. If one were only interested in the validity of the existence of $\lim\limits_{n\to\infty}\sum\limits_0^n A_{2j+1}(\varphi,t)\cdot s_j(f,x)$ for a.e. t, a.e. x (instead of every t), then an appeal to the Carleson–Hunt [1, 2] result would suffice. More precisely, if $f\in L^1$ and $\varphi\in BV$, or just $|a_n(\varphi)| = O\left(\frac{1}{n}\right)$, then, since $s_n(f,x) = o(\log n)$ for a.e. x, the trigonometric series $\sum A_n(\varphi,t)s_n(f,x)$ is for a.e. x in $L^2(dt)$. Hence $\lim\limits_{n\to\infty}\sum\limits_0^n A_n(\varphi,t)s_n(f,x)$ exists for a.e. t, a.e. x.

Remark 5. It is natural to ask for what subsequences $\{i_j\}$ is it true that $\lim\limits_{n\to\infty}\sum\limits_0^n A_{i_j}(\varphi,t)s_j(f,x)$ exists for every t and a.e. x. Here $\varphi = \sum A_n(\varphi,t)$ is in BV and $f\in L\log L$. The writer does not know whether $i_j = j$ or $i_j = 2j$ are sufficient. However, it is easy to see that one cannot choose $\{i_j\}$ arbitrarily. For, if $\varphi(t) = \sum\frac{\sin nt}{n}$, which is in BV, and $i_j = 4j+3$, then $\sum\frac{\sin i_j\pi/2}{i_j} = -\infty$, and the above limit does not exist for $f\equiv 1$.

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STUDIA MATHEMATICA T. XLI. (1972)

On weak and Schauder decompositions

by

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Abstract. We prove a general result about weak and Schauder decompositions which extends the well-known equivalence between weak and Schauder bases in Fréchet spaces.

All the spaces considered in this paper are locally convex topological vector spaces.

Let F be such a space. Consider a vector subspace L of F and assume that there exists a sequence of vector subspaces L_i such that, for each $f\in L$, we have

$$f = \sum_{i=1}^{\infty} f_i,\quad f_i\in L_i,$$

where the f_i are uniquely determined by f and the series converges in the topology (or the weak topology) of F. We write F_w for F equipped with its weak topology.

The maps τ_i defined from L into L_i by $\tau_i f = f_i$ are trivially linear. Moreover, each τ_i is the identity in $L\cap L_i$ and $\tau_i = 0$ in $L\cap L_j$ for each $j\neq i$.

The sequence L_i is called a *decomposition* (resp. a *weak decomposition*) of L into the L_i. It is a *Schauder decomposition* if the τ_i are continuous.

In this paper, we shall use the closed graph theorem of [2], which states that *if T is a linear map from an ultrabornological space E into a space F admitting a net of type $\mathscr{C}$ and if the graph of T is sequentially closed in $E\times F$, then T is continuous.*

We refer to [2] for the definition of nets.

THEOREM. *Let E be an ultrabornological space and let F be sequentially complete and admit a net of type $\mathscr{C}$.*

Let T be continuous from E into F and assume that its range TE admits a weak decomposition (resp. a decomposition) into sequentially closed subspaces L_i of F,

$$g = \sum_{i=1}^{\infty}\tau_i g,\quad \tau_i g\in L_i,\quad \forall g\in TE.$$

Then,

(α) *the $\tau_i T$ are continuous from E into F,* (β) *the sequence* $(\sum_{i=1}^{m} \tau_i T)_{m\in N}$ *is equicontinuous in* $\mathscr{L}(E, F)$,

(γ) $T = \sum_{i=1}^{\infty} \tau_i T$, *the series being convergent in* $\mathscr{L}_{pc}(E, F_w)$ [*resp. in* $\mathscr{L}_{pc}(E, F)$].

In the last assertion, $\mathscr{L}_{pc}(E, F)$ denotes the space of linear continuous maps from E into F, equipped with the topology of uniform convergence over the precompact sets of E.

COROLLARY 1. *If F is bornological and sequentially complete and admits a net of type $\mathscr{C}$, any weak decomposition of F is a Schauder decomposition.*

Since F is sequentially complete and bornological, it is ultrabornological.

Applying the theorem with $E = F$ and $T = I$, we obtain that the τ_i are continuous. Moreover, the sequence $(\sum_{i=1}^{n} \tau_i)_{m\in N}$ is equicontinuous. Thus, by the Banach–Steinhaus theorem, it converges to I if $(\sum_{i=1}^{m} \tau_i f)_{m\in N}$ converges to f for any f in a total subset of F. Since the series $\sum_{i=1}^{\infty} \tau_i f$ are weakly convergent, the set $\bigcup_{i=1}^{\infty} L_i$ is weakly total, hence total in F. Moreover, for any $f \in L_i$, $\tau_j f = f\delta_{ij}$, hence $\sum_{j=1}^{\infty} \tau_j f$ converges to f.

COROLLARY 2. *Let E be ultrabornological and let F be sequentially complete and admit a net of type $\mathscr{C}$. If F admits a weak basis e_i, $i = 1, 2, \ldots$, such that*

$$g = \sum_{i=1}^{\infty} \lambda_i(g) e_i, \qquad \forall g \in F,$$

and if T is a continuous linear map from E into F, then the $\lambda_i(T\cdot)$ are continuous linear forms in E.

It is an immediate consequence of the theorem, because each L_i is here the linear hull of e_i and hence it is closed.

Proof of the theorem. a) Assume that the topology of F is defined by the system of semi-norms Q.

We denote by $l_\infty(F)$ the space of all bounded sequences $(f_m)_{m\in N}$ of F, equipped with the system of seminorms

$$q'[(f_m)_{m\in N}] = \sup_m q(f_m), \qquad q \in Q.$$

Since F is sequentially complete and admits a net of type $\mathscr{C}$, it follows from [3] that $l_\infty(F)$ admits a net of type $\mathscr{C}$.

For any $f \in E$, the series $\sum_{i=1}^{\infty} \tau_i Tf$ is weakly convergent to Tf. Therefore, the sequence of partial sums is weakly bounded, thus bounded, by Mackey's theorem. Hence that sequence belongs to $l_\infty(F)$.

Let T' be the map which, to each $f \in E$, associates the sequence $T'f = (\sum_{i=1}^{m} \tau_i Tf)_{m\in N}$. It is a linear map from E into $l_\infty(F)$.

b) The graph of T' is sequentially closed in $E \times l_\infty(F)$.

Assume that $f^{(i)}$ converges to f in E and that $T'f^{(i)}$ converges to $(h_m)_{m\in N}$ in $l_\infty(F)$. Let us prove that, if we write $h_0 = 0$, then $\tau_m Tf = h_m - h_{m-1}$ for each $m > 0$. It follows therefrom that $T'(f) = (h_m)_{m\in N}$, hence the conclusion.

The sequence $(h_m)_{m\in N}$ converges or converges weakly to Tf according as the τ_i define a decomposition or a weak decomposition in F. Indeed, let $q \in Q$ be fixed. We have

$$q(Tf - h_m) \leqslant q(Tf - Tf^{(i)}) + q\Big(Tf^{(i)} - \sum_{j=1}^{m} \tau_j Tf^{(i)}\Big) + q\Big(\sum_{j=1}^{m} \tau_j Tf^{(i)} - h_m\Big).$$

Since $T'f^{(i)}$ converges to $(h_m)_{m\in N}$ in $l_\infty(F)$, we have

$$\sup_m q\Big(\sum_{j=1}^{m} \tau_j Tf^{(i)} - h_m\Big) \to 0$$

if $i \to \infty$. Moreover, since T is continuous, $Tf^{(i)}$ converges to Tf in F. Therefore, for $\varepsilon > 0$ given, there exists i_0 such that

$$q(Tf - Tf^{(i)}), \quad q\Big(\sum_{j=1}^{m} \tau_j Tf^{(i)} - h_m\Big) \leqslant \varepsilon/3, \qquad \forall m, \qquad \forall i \geqslant i_0.$$

Let us take $i = i_0$. If the τ_i defines a decomposition of TE, there exists m_0 such that

$$q\Big(Tf^{(i_0)} - \sum_{j=1}^{m} \tau_j Tf^{(i_0)}\Big) \leqslant \varepsilon/3, \qquad \forall m \geqslant m_0,$$

hence

$$q(Tf - h_m) \leqslant \varepsilon, \qquad \forall m \geqslant m_0,$$

and $(h_m)_{m\in N}$ converges to Tf.

Let us consider now the case of a weak decomposition. Let $\mathscr{T}$ be an arbitrary continuous linear form in F. There exist then $C > 0$ and $q \in Q$ such that $|\mathscr{T}(g)| \leqslant Cq(g)$ for each $g \in F$. We have

$$|\mathscr{T}(Tf - h_m)| \leqslant |\mathscr{T}(Tf - Tf^{(i)})| + \Big|\mathscr{T}\Big(Tf^{(i)} - \sum_{j=1}^{m} \tau_j Tf^{(i)}\Big)\Big| + \Big|\mathscr{T}\Big(\sum_{j=1}^{m} \tau_j Tf^{(i)} - h_m\Big)\Big|$$

$$\leqslant \Big|\mathscr{T}\Big(Tf^{(i)} - \sum_{j=1}^{m} \tau_j Tf^{(i)}\Big)\Big| + Cq(Tf - Tf^{(i)}) + Cq\Big(\sum_{j=1}^{m} \tau_j Tf^{(i)} - h_m\Big).$$

For $\varepsilon > 0$ given, by the preceding arguments, the two last terms in the last member are less then $\varepsilon/3$ for i large enough. For such an i fixed, the first term in the last member is less then $\varepsilon/3$ for m large enough, because the τ_i define a weak decomposition of TE. Hence $(h_m)_{m\in N}$ converges weakly to Tf.

Moreover, $h_m - h_{m-1} \in L_m$ for each $m > 0$. Indeed, since $T'f^{(i)}$ converges to $(h_m)_{m\in N}$ in $l_\infty(F)$, the sequence $\tau_m Tf^{(i)}$ converges to $h_m - h_{m-1}$ for each $m > 0$. Thus, since the L_m are sequentially closed, $h_m - h_{m-1} \in L_m$.

By virtue of the unicity of the decomposition into the L_i, from $Tf = \sum_{m=1}^{\infty} (h_m - h_{m-1})$, we deduce that $h_m - h_{m-1} = \tau_m Tf$ for each m, hence $T'f = (h_m)_{m\in N}$.

c) By the closed graph theorem (see [2], p. 28), the map T' is continuous from E into $l_\infty(F)$. Thus, the sequence $\left(\sum_{i=1}^{m} \tau_i T\right)_{m\in N}$ is equicontinuous and, hence, each $\tau_i T$ is continuous.

Since $\left(\sum_{i=1}^{m} \tau_i T\right)_{m\in N}$ is equicontinuous, the last assumption follows from the Banach–Steinhaus theorem.

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A new definition of nuclear systems with applications to bases in nuclear spaces

by

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Abstract. The concept of nuclear system was introduced by the author in an earlier paper in the same journal as a means of constructing examples of nuclear Fréchet spaces. In this paper the definition is simplified by showing that a nuclear Fréchet space on which there exists a continuous norm can be represented as a projective limit of *injective* nuclear maps on a Hilbert space. This permits the solution of two problems given in the first paper and the improvement of some results in that paper.

These results are applied to the problem of existence of Schauder bases in nuclear Fréchet spaces. An equivalent condition for existence is given and also a test for when a given sequence is a basis. Also a theorem on perturbation of a system with a basis is proved.

A subsequent paper will apply these results to concrete examples of nuclear Fréchet spaces.

The concept of nuclear system was introduced in [2] as a means of constructing examples of Fréchet nuclear spaces and studying the basis problem. In the present paper we give a simplified definition of nuclear system, somewhat anticipated in [2] (see problem 4° of [2]) and obtain new results about nuclear systems and the existence of Schauder bases in Fréchet nuclear spaces.

In Section 1, the new definition is given and it is shown that nuclear systems still characterize full Fréchet nuclear spaces (see below for definitions).

In Section 2 the main results of [2] are restated according to the new definition and new results about nuclear systems are obtained.

In Section 3, we prove new results on the existence of a Schauder basis in the associated space of a nuclear system. We are able to derive

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