Since $s_N(f, x) = o(\log N)$ for a.e. $x$, we see that $s_N(f, x)\lambda_N(\varphi, t) = o(1)$ for a.e. $t, x$ as $N \to \infty$. By summation by parts we have

$$\sum_{n=1}^{N} a_n(\varphi, t)A_n(f, x) = \sum_{n=1}^{N} A_{n+1}(\varphi, t)\varphi_n(f, x) + \sum_{n=0}^{N-1} A_{n+1}(\varphi, t)\varphi_n(\varphi, t),$$

and thus Theorem 3 completes the proof.

6. We conclude with two remarks.

Remark 4. If one were only interested in the validity of the existence of $\sum_{n=0}^{N} A_{n+1}(\varphi, t)\varphi_n(f, x)$ for a.e. $t, a.e. x$ (instead of every $t$, then an appeal to the Carleson–Hunt [1, 2] result would suffice. More precisely, if $f \in L^q$ and $\varphi \in BV$, or just $|a_n(\varphi)| = O\left(\frac{1}{n}\right)$, then since $s_n(f, x) = o(\log n)$ for a.e. $x$, the trigonometric series $\sum A_n(\varphi, t)\varphi_n(f, x)$ is for a.e. $x$ in $L^2(dt)$.

Hence $\lim \sum_{n=0}^{N} A_n(\varphi, t)\varphi_n(f, x)$ exists for a.e. $t, a.e. x$.

Remark 5. It is natural to ask for what subsequences $(t_j)$ it is true that $\lim \sum_{n=0}^{N} A_n(\varphi, t_j)\varphi_n(f, x)$ exists for every $t$ and a.e. $x$. Here $\varphi = \sum A_n(\varphi, t)$ is in $BV$ and $f \in L_1$. The reader does not know whether $t_j = j$ or $t_j = 2j$ are sufficient. However, it is easy to see that one cannot choose $t_j$ arbitrarily. For, if $\varphi(t) = \sum_{n=0}^{N} a_n(\varphi, t)$, which is in $BV$, and $t_j = 4j + 3$, then

$$\sum_{n=0}^{N} \sin \frac{t_j}{2} a_n = -\infty,$$

and the above limit does not exist for $f = 1$.

References


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(b) the sequence \( \sum_{i=1}^{n} \tau_i T_{t_{mN}} \) is equicontinuous in \( \mathcal{L}(E, F) \).

(c) \( T = \sum_{i=1}^{n} \tau_i T_i \), the series being convergent in \( \mathcal{L}_{mb}(E, F) \) [resp. in \( \mathcal{L}_{mb}(E, F) \)].

In the last assertion, \( \mathcal{L}_{mb}(E, F) \) denotes the space of linear continuous maps from \( E \) into \( F \), equipped with the topology of uniform convergence over the precompact sets of \( E \).

**Corollary 1.** If \( F \) is bornological and sequentially complete and admits a net of type \( \mathcal{V} \), any weak decomposition of \( F \) is a Schauder decomposition.

Since \( F \) is sequentially complete and bornological, it is ultrabornological.

Applying the theorem with \( E = F \) and \( T = I \), we obtain that the \( \tau_i \) are continuous. Moreover, the sequence \( \left( \sum_{i=1}^{n} \tau_i f \right)_{mN} \) is equicontinuous. Thus, by the Banach-Steinhaus theorem, it converges to \( f \) if \( \left( \sum_{i=1}^{n} \tau_i f \right)_{mN} \) converges to \( f \) for any \( f \) in a total subset of \( F \). Since the series \( \sum_{i=1}^{n} \tau_i f \) are weakly convergent, the set \( \bigcap_{m=1}^{\infty} L_m \) is weakly total, hence total in \( F \). Moreover, for any \( f \in L_m \), \( \tau_i f = \delta f \), hence \( \sum_{i=1}^{n} \tau_i f \) converges to \( f \).

**Corollary 2.** Let \( E \) be ultrabornological and let \( F \) be sequentially complete and admit a net of type \( \mathcal{V} \). If \( F \) admits a weak base \( \{ e_i \} \), \( i = 1, 2, \ldots \), such that

\[
q = \sum_{i=1}^{\infty} \lambda_i(g) e_i, \quad \forall g \in F,
\]

and if \( T \) is a continuous linear map from \( E \) into \( F \), then the \( \lambda_i(T \cdot) \) are continuous linear forms on \( E \).

It is an immediate consequence of the theorem, because each \( L_i \) is here the linear hull of \( e_i \) and hence it is closed.

**Proof of the theorem.** a) Assume that the topology of \( F \) is defined by the system of semi-norms \( Q \).

We denote by \( l_{mb}(F) \) the space of all bounded sequences \( (f_m)_{m \in \mathbb{N}} \) of \( F \), equipped with the system of seminorms

\[
g^*(f_m)_{m \in \mathbb{N}} = \sup_{m} g(f_m), \quad q*Q.
\]

Since \( F \) is sequentially complete and admits a net of type \( \mathcal{V} \), it follows from \([3]\) that \( l_{mb}(F) \) admits a net of type \( \mathcal{V} \).

For any \( f \in E \), the series \( \sum_{n} \tau_n T f \) is weakly convergent to \( T f \). Therefore, the sequence of partial sums is weakly bounded, thus bounded, by Mackey's theorem. Hence that sequence belongs to \( l_{mb}(E) \).

Let \( T' \) be the map which, to each \( f \in E \), associates the sequence \( T'f = \left( \sum_{n} \tau_n T f \right)_{m \in \mathbb{N}} \). It is a linear map from \( E \) into \( l_{mb}(F) \).

b) The graph of \( T' \) is sequentially closed in \( E \times l_{mb}(F) \).

Assume that \( T'f \) converges to \( f \) in \( E \) and that \( T'f(m) \) converges to \( \left( h_m \right)_{m \in \mathbb{N}} \) in \( l_{mb}(F) \). Let us prove that, if we write \( h_0 = 0 \), then \( \tau_n T f = h_n - h_{n-1} \) for each \( m > 0 \). It follows therefrom that \( T'(f) = (h_m)_{m \in \mathbb{N}} \), hence the conclusion.

The sequence \( (h_m)_{m \in \mathbb{N}} \) converges or converges weakly to \( y \) according as the \( \tau_i \) define a decomposition or a weak decomposition in \( F \). Indeed, let \( q \) be fixed. We have

\[
g(T_f - h_m) = g(T_f - T'_f(m) + \sum_{n} \tau_n T f(m) - h_m) + \sum_{n} \tau_n T f(m) - h_m) \leq \varepsilon/3, \quad \forall m, \quad \forall k \geq k_0.
\]

Since \( T f \) converges to \( (h_m)_{m \in \mathbb{N}} \) in \( l_{mb}(F) \), we have

\[
\sup_{m} g(\sum_{n} \tau_n T f(m) - h_m) \to 0
\]

if \( k \to \infty \). Moreover, since \( T \) is continuous, \( T f \) converges to \( T f \) in \( F \). Therefore, for \( \varepsilon > 0 \) given, there exists \( k_0 \) such that

\[
\sup_{m} g(\sum_{n} \tau_n T f(m) - h_m) \leq \varepsilon/3, \quad \forall m, \quad \forall k \geq k_0.
\]

Let us take \( k = k_0 \). If the \( \tau_i \) defines a decomposition of \( TE \), there exists \( m_0 \) such that

\[
g(T f - T f(m)) \leq \varepsilon/3, \quad \forall m \geq m_0,
\]

hence

\[
g(T f - h_m) \leq \varepsilon, \quad \forall m \geq m_0,
\]

and \( (h_m)_{m \in \mathbb{N}} \) converges to \( T f \).

Let us consider now the case of a weak decomposition. Let \( S \) be an arbitrary continuous linear form on \( E \). There exist then \( C > 0 \) and \( q \) such that \( |S(g)| \leq C q(g) \) for each \( g \in F \). We have

\[
|S(T f - h_m)| \leq |S(T f - T f(m))| + |S(T f - T f(m) - \sum_{n} \tau_n T f(m)| + |S(\sum_{n} \tau_n T f(m) - h_m)|
\]

\[
\leq |S(T f - h_m)| + C q(T f - T f(m)) + C q(\sum_{n} \tau_n T f(m) - h_m).
\]
For \( \varepsilon > 0 \) given, by the preceding arguments, the two last terms in the last member are less than \( \varepsilon /3 \) for \( i \) large enough. For such an \( i \) fixed, the first term in the last member is less than \( \varepsilon /3 \) for \( m \) large enough, because the \( \tau_i \) define a weak decomposition of \( TX \). Hence \( (h_m)_{m=1}^N \) converges weakly to \( T \).

Moreover, \( h_m - h_{m-1} \in I_m \) for each \( m > 0 \). Indeed, since \( T'f^{(i)} \) converges to \( (h_m)_{m=1}^N \) in \( L_i (F) \), the sequence \( \tau_m T'f^{(i)} \) converges to \( h_m - h_{m-1} \) for each \( m > 0 \). Thus, since the \( I_m \) are sequentially closed, \( h_m - h_{m-1} \in I_m \).

By virtue of the unicity of the decomposition into the \( I_i \), from \( T \sum \tau_i T' \) we deduce that \( h_m - h_{m-1} = \tau_m T' \) for each \( m \), hence \( T' = (h_m)_{m=1}^N \).

o) By the closed graph theorem (see [2], p. 28), the map \( T' \) is continuous from \( E \) into \( I_0 (F) \). Thus, the sequence \( \sum \tau_i T' \) is equicontinuous and, hence, each \( \tau_i T' \) is continuous.

Since \( \sum \tau_i T' \) is equicontinuous, the last assumption follows from the Banach–Steinhaus theorem.

References


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