STUDIA MATHEMATICA T. XLI. (1972)

On certain linear combinations of partial sums of Fourier series

by

C. J. NEUGEBAUER (Lafayette, Ind.)

Abstract. Associated with each function $\varphi(t)$ of bounded variation is a multiplier sequence $\{\lambda j(\varphi, t)\}$ which transforms $f(x) \in L\log L$ into L^1 and for which the transformed Fourier series converges for a.e. t, x.

1. For $f \in L^1$, we let $f \sim \sum A_n(f, x)$ be its Fourier series, where $A_n(f, x) = a_n \cos nx + b_n \sin nx$. We write $s_n(f, x) = \sum_{j=0}^{n} A_j(f, x)$. Let $\varphi(x) = \sum A_n(\varphi, x)$ be of bounded variation. The purpose of this paper is to study the linear combinations $\sum_{j=0}^{n} A_{2j+1}(\varphi, t) s_j(f, x)$. Expressions of this type were used in a study of differentiation of trigonometric series [3]. The main result which we obtain is the following:

If $f \in L \log L$, i.e., if $\int_{-\pi}^{\pi} |f| \log^+ |f|\, dx < \infty$, then

$$\lim_{n\to\infty} \sum_{0}^{n} A_{2j+1}(\varphi, t) s_j(f, x) = f_\varphi(t, x) \quad \text{for every } t \quad \text{and a.e. } x,$$

$f_\varphi(t, x) \in L_1(dx)$ for every t, and the Fourier series of $f_\varphi(t, x)$ with respect to x, being obtained from $\sum A_n(f, x)$ via the multiplier sequence $\lambda_n(\varphi, t) = \sum_{j \geqslant n} A_{2j+1}(\varphi, t)$, converges to $f_\varphi(t, x)$ for a.e. t and a.e. x.

2. If $f \sim \sum A_n(f, x)$ is in L^1, then

$$L(x) = \sum_{1}^{\infty} \frac{1}{n} (a_n \sin nx - b_n \cos nx) = \sum_{1}^{\infty} \frac{1}{n} B_n(f, x)$$

is apart from a linear term $\int_0^x f(t)\, dt$. We let $\tilde{s}_n(x) = \sum_{1}^{n} B_j(f, x)$, and observe that $|s_n(x)| + |\tilde{s}_n(x)| = o(\log n)$ for a.e. x [5_I, p. 66]. It is easy to see, using summation by parts, that for a.e. x the following formula holds

$$\frac{1}{2h}\,\Delta^2 L(x, 2h) = \frac{1}{2h}\,[L(x+2h)+L(x-2h)-2L(x)]$$

$$= -2\sum B_n(x)\frac{\sin^2 nh}{nh} = -2\sum \tilde{s}_n(x)\left\{\frac{\sin^2 nh}{nh} - \frac{\sin^2(n+1)h}{(n+1)h}\right\}$$

$$= -2\sum \frac{\tilde{s}_n(x)}{n(n+1)h}\sin^2 nh + \frac{2\sin h}{h}\sum\frac{\tilde{s}_n(x)}{n+1}\sin(2n+1)h.$$

If $f\in L\log L$, then the conjugate function $\tilde{f}\sim\sum B_n(f, x)$ is in L^1 [5_I, p. 266], and applying the above formula to $\tilde{f}$ we obtain with the obvious notation

$$\frac{1}{2h}\,\Delta^2\tilde{L}(x, 2h) = 2\sum\frac{s_n(f, x)}{n(n+1)h}\sin^2 nh - \frac{2\sin h}{h}\sum\frac{s_n(f, x)}{n+1}\sin(2n+1)h,$$

valid for a.e. x and $h>0$.

Let $\varphi = \sum a_n(\varphi)\cos nx + b_n(\varphi)\sin nx$ be in C^∞ and consider the bilinear functional $S(f, \varphi, x) = \sum_{j=0}^{\infty} a_{2j+1}(\varphi)s_j(f, x)$. We let $\|\varphi\| = \|\varphi\|_1 + \|\varphi'\|_1$.

THEOREM 1.

(i) $\|S(f, \varphi, \cdot)\|_p \leqslant A_p\|\varphi\|\,\|f\|_p$, $1<p<\infty$;

(ii) $\|S(f, \varphi, \cdot)\|_1 \leqslant A\|\varphi\|\,(\int_{-\pi}^{\pi}|f|\log^+|f|\,dx+1)$.

Proof. We will first prove (i). By [5_I, p. 266] we have $\|s_n\|_p+\|\tilde{s}_n\|_p \leqslant A_p\|f\|_p$.

We consider $g(x, h) = \sum_0^\infty \frac{s_n(f, x)}{2n+1}\sin(2n+1)h$, and we note that $g(x, h)\in L^2(dh)$ for a.e. x, since $s_n(f, x) = o(\log n)$ for a.e. x. Consequently, by Parseval's formula [5_I, p. 157],

$$|S(f, \varphi, x)| = \frac{1}{\pi}\left|\int_{-\pi}^{\pi} g(x, h)\varphi'(h)\,dh\right|, \quad \text{a.e. } x.$$

Hence

$$\|S(f, \varphi, \cdot)\|_p \leqslant \frac{1}{\pi}\int_{-\pi}^{\pi}\|g(\cdot, h)\|_p|\varphi'(h)|\,dh.$$

The expression $\|g(\cdot, h)\|_p$ can easily be estimated since

$$2|g(x, h)| \leqslant \frac{|h|}{2\sin h}\left\{\left|\frac{\Delta^2\tilde{L}(x, 2h)}{2h}\right| + 2\left|\sum_1^\infty\frac{s_n(f, x)}{n(n+1)h}\sin^2 nh\right|\right\} +$$

$$+\left|\sum_1^\infty s_n(f, x)\left(\frac{2}{2n+1}-\frac{1}{n+1}\right)\sin(2n+1)h\right|.$$

The $L^p(dx)$-norm of the last term is $\leqslant A_p\|f\|_p$, and since $\left|\sum\frac{\sin^2 nh}{n(n+1)h}\right| \leqslant K<\infty$ by [5_I, p. 10 (4.17)], the $L^p(dx)$-norm of the middle term is $\leqslant A_p\|f\|_p$. Finally,

$$\left\|\frac{\Delta^2\tilde{L}(\cdot, 2h)}{2h}\right\|_p \leqslant 2\|\tilde{f}\|_p \leqslant A_p\|f\|_p.$$

This establishes (i). The proof of (ii) is the same using, instead of the L^p-inequalities, $\|s_n\|_1+\|\tilde{s}_n\|_1 \leqslant A\left(\int_{-\pi}^{\pi}|f|\log^+|f|\,dx+1\right)$ and

$$\left\|\frac{\Delta^2\tilde{L}(\cdot, 2h)}{2h}\right\|_1 \leqslant 2\|\tilde{f}\|_1 \leqslant A\left(\int_{-\pi}^{\pi}|f|\log^+|f|\,dx+1\right)$$

[5_I, p. 266].

Remark 1. The constant A_p appearing in (i) satisfies $A_p\sim Ap$ as $p\to\infty$ and $A_p = A_{p'}$, $\frac{1}{p}+\frac{1}{p'} = 1$ [5_I, p. 261, p. 295, Problem 2]. Consequently, as $p\downarrow 1$, $A_p = O\left((p-1)^{-1}\right)$. Hence [$5_{II}$, p. 119 (4.41)] could also have been used to establish (ii).

The next theorem can be viewed as the $p=\infty$ version of Theorem 1.

THEOREM 2. *Let $f\in L\log L$. Then for a.e. x there is $0<M_x<\infty$ such that $|S(f, \varphi, x)| \leqslant M_x\|\varphi\|$, $\varphi\in C^\infty$.*

Proof. We know that $\tilde{f}\in L^1$ and that $\tilde{L}(x) = \sum\frac{A_n(f, x)}{n}$ is apart from a linear term $-\int_0^x\tilde{f}(t)\,dt$. Hence $\tilde{L}'(x)$ exists a.e., and $\tilde{L}(x)$ is continuous. With the same notation as in Theorem 1 we have

$$|S(f, \varphi, x)| \leqslant \frac{1}{\pi}\int_{-\pi}^{\pi}|g(x, h)|\,|\varphi'(h)|\,dh.$$

We estimate $|g(x, h)|$. We only need to consider $h>0$, and since $g(x, h) = g(x, \pi-h)$, we may further restrict ourselves to $0<h\leqslant\pi/2$. Clearly, $\left|\frac{\Delta^2\tilde{L}(x, 2h)}{2h}\right| \leqslant K_x<\infty$, a.e. x, and since $s_n(f, x) = o(\log n)$ for a.e. x,

$$\left|\sum_1^\infty s_n(f, x)\left(\frac{2}{2n+1}-\frac{1}{n+1}\right)\sin(2n+1)h\right| \leqslant K_x<\infty \quad \text{for a.e. } x.$$

To handle the term $\sum \frac{s_n(f,x)}{n(n+1)h} \sin^2 nh$ we use a suggestion of the referee (the writer's original proof only applied to $1 < p < \infty$ and relied upon the Carleson–Hunt theorem of the convergence of the Fourier series a.e.).

Since $\sum A_n(f,x)$ is strongly summable a.e. [5_{II}, p. 184], we see that $\frac{1}{n}\sum_1^n |s_n(f,x)| \leqslant K_x < \infty$, for a.e. x, $n = 1, 2, \ldots$ For $0 < h \leqslant \pi/2$, we let $N = \left[\frac{1}{h}\right]$, and write

$$\sum_1^\infty \frac{|s_n(f,x)|}{n(n+1)h} \sin^2 nh = \sum_1^N + \sum_{N+1}^\infty = A_N + B_N.$$

Since $|\sin nh| \leqslant nh$, we see that $A_N \leqslant K_x < \infty$ for a.e. x. For B_N we use summation by parts to obtain

$$\sum_{N+1}^M \frac{|s_n(f,x)|}{n^2} \leqslant 2\cdot K_x \sum_{N+1}^{M-1} \frac{1}{n^2} + K_x \cdot \frac{1}{M}, \quad \text{for a.e. } x.$$

Hence $\frac{1}{h}\sum_{N+1}^M \frac{|s_n(f,x)|}{n^2} \leqslant (N+1)\sum_{N+1}^M \frac{|s_n(f,x)|}{n^2} \leqslant 3\cdot K_x$. Hence for a.e. x, $|g(x,t)| \leqslant M_x < \infty$, and the proof is complete.

3. The completion of C^∞ with respect to $\|\varphi\| = \|\varphi\|_1 + \|\varphi'\|_1$ is the space W of all absolutely continuous functions. Consequently, $S(f, \varphi, x)$ can be extended to $\varphi \in W$ so as to preserve Theorems 1 and 2. The purpose of this section is to extend $S(f, \varphi, x)$ to $\varphi \in BV$, where BV denotes the space of all 2π-periodic functions φ of bounded variation. We let $V(\varphi)$ be the variation of φ over a period.

LEMMA 1. *If $f \sim \sum A_n(f,x)$ is in $L\log L$ and $\varphi = \sum a_n(\varphi)\cos nx + b_n(\varphi)\sin nx$ is in BV, then $\sum_0^n a_{2j+1}(\varphi) s_j(f,x) = O(1)$ for a.e. x as $n \to \infty$.*

Proof. Let $\sigma_n(\varphi, x) = \sum_0^n a_{j,n}\cos jx + b_{jn}\sin jx$ be the $(C, 1)$ means of φ. Since $|a_j| \leqslant \frac{V(\varphi)}{\pi\cdot j}$, it follows easily that $|a_j - a_{j,n}| \leqslant \frac{K}{n+1}$. From Theorem 2 we obtain

$$\left|\sum_0^n a_{2j+1,n} s_j(f,x)\right| \leqslant M_x(f)\|\sigma_n(\varphi)\|, \quad \text{for a.e. } x.$$

Since

$$\|\sigma_n(\varphi,\cdot)\|_1 \leqslant \|\varphi\|_1 \quad \text{and} \quad \|\sigma_n'(\varphi,\cdot)\|_1 = V[\sigma_n(\varphi)] \leqslant V(\varphi) = \|\varphi'\|_1,$$

we see that $\left|\sum_0^n a_{2j+1,n} s_j(f,x)\right| \leqslant M_x(f)\|\varphi\|$. Next,

$$\left|\sum_0^n (a_{2j+1} - a_{2j+1,n}) s_j(f,x)\right| \leqslant \frac{K}{n+1}\sum_0^n |s_j(f,x)| = O(1) \quad \text{for a.e. } x$$

by strong summability. This completes the proof.

THEOREM 3. *Let $f \sim \sum A_n(f,x)$ be in $L\log L$, and let $\varphi(x) = \sum a_n(\varphi)\cos nx + b_n(\varphi)\sin nx$ be in BV. Then $\lim_{n\to\infty}\sum_0^n a_{2j+1}(\varphi) s_j(f,x)$ exists for a.e. x.*

Proof. We consider the sequence of operators defined by

$$(T_n f)(x) = \sum_0^n a_{2j+1}(\varphi) s_j(f,x).$$

It is our purpose to show that the hypothesis of Theorem 3 of [4] is satisfied with $\Phi(t) = (t+1)\log(t+1)$, $t \geqslant 0$. Since T_n is translation invariant and by the lemma, $\limsup_{n\to\infty}|T_n f(x)| < \infty$ for a.e. x, we only need to verify that T_n is of type (Φ, Φ), i.e., there is a constant A such that

$$\int_{-\pi}^{\pi} \Phi(|T_n f(x)|)\,dx \leqslant \int_{-\pi}^{\pi} \Phi(A|f(x)|)\,dx.$$

To prove this, we write $T_n f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\left\{\sum_0^n a_{2j+1}D_j(t)\right\}dt$, where $D_j(t)$ is the Dirichlet kernel. Clearly $\sum_0^n |a_{2j+1}|\,|D_j(t)| \leqslant C_n < \infty$. If we apply now Jensen's inequality we obtain

$$\Phi(|T_n f(x)|) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi\{2\cdot C_n|f(x+t)|\}\,dt.$$

From this the desired inequality follows with $A = 2\cdot C_n$.

By Theorem 3 of [4] there is a constant A such that

$$|\{x\colon T^* f(x) > \alpha\}| \leqslant \int_{-\pi}^{\pi} \Phi\left(\frac{A}{\alpha}|f(x)|\right)dx,$$

where $T^* f(x) = \sup_{n\geqslant 0}|T_n f(x)|$.

This weak type estimate implies convergence a.e., and this can be verified in the following way. Let $\overline{\varphi}(f, x) = \limsup T_n f(x)$, $\underline{\varphi}(f, x) = \liminf T_n f(x)$. If t is a trigonometric polynomial we see that

$$0 \leqslant \overline{\varphi}(f, x) - \underline{\varphi}(f, x) = \overline{\varphi}(f-t, x) - \underline{\varphi}(f-t, x) \leqslant 2T^*(f-t, x).$$

Let $a > 0$ be given. Then

$$|\{x\colon \overline{\varphi}(f, x) - \underline{\varphi}(f, x) > a\}| \leqslant \int_{-\pi}^{\pi} \Phi\left(\frac{2A}{a}|f-t|\right) dx.$$

The last integral can be made as small as we please by taking for t the $(C, 1)$ means of f [5_{I}, p. 146].

Remark. In the next section Theorem 3 will be used in the following way. Let $\varphi(x) = \sum A_n(\varphi, x)$ be in BV. Then, for each t, $\tau_t\varphi(x) = \varphi(x+t)$ is in BV and $a_n(\tau_t\varphi) = A_n(\varphi, t)$. Thus, if $f \sim A_n(f, x)$ is in $L\log L$, then for each t, $\lim_{n\to\infty} \sum_0^n A_{2j+1}(\varphi, t) s_j(f, x)$ exists for a.e. x. We denote this limit by $f_\varphi(t, x)$.

4. We wish to study the integrability properties of $f_\varphi(t, x)$. For that purpose it will prove useful to have the following easily verified properties of a function $\varphi \in BV$. There exists a sequence $\{\varphi_n\} \subseteq C^\infty$, e.g., $\varphi_n(x) = \sigma_n(\varphi, x)$ the $(C, 1)$ mean of φ, such that

(i) $\varphi_n(x) \to \frac{1}{2}[\varphi(x+0) + \varphi(x-0)]$ for each x,

(ii) $\|\varphi_n\|_\infty < \|\varphi\|_\infty$ and $V(\varphi_n) \leqslant V(\varphi)$,

(iii) $\sum_{j\geqslant k} a_{2j+1}(\varphi_n) \to \sum_{j\geqslant k} a_{2j+1}(\varphi)$, as $n \to \infty$, for each k.

We denote by $\|\varphi\|_{BV} = \|\varphi\|_\infty + V(\varphi)$.

LEMMA 2. *Let T be a trigonometric polynomial of degree n and let $\varphi \in BV$. Then for each t*

$$\Big\|\sum_0^n A_{2j+1}(\varphi, t) s_j(T, \cdot)\Big\|_p \leqslant A_p \|\varphi\|_{BV} \|T\|_p + c_n \|T\|_p, \quad 1 < p < \infty,$$

where $c_n = o(1)$.

Proof. Let $\varphi_k \in C^\infty$ with the above properties. By (i) of Theorem 1 with φ replaced by $\tau_t\varphi$,

$$\Big\|\sum_0^n A_{2j+1}(\varphi_k, t) s_j(T, \cdot) + T(\cdot)\sum_{n+1}^\infty A_{2j+1}(\varphi_k, t)\Big\|_p \leqslant A_p \|T\|_p \|\varphi_k\|.$$

Since $\|\varphi_k\| = \|\varphi_k\|_1 + \|\varphi_k'\|_1 \leqslant 2\pi\|\varphi\|_{BV}$, we obtain

$$\Big\|\sum_0^n A_{2j+1}(\varphi, t) s_j(T, \cdot)\Big\|_p \leqslant A_p \|T\|_p \|\varphi\|_{BV} + \|T\|_p \Big|\sum_{n+1}^\infty A_{2j+1}(\varphi_k, t)\Big|.$$

We let $k \to \infty$ and obtain the lemma with $c_n = \Big|\sum_{n+1}^\infty A_{2j+1}(\varphi, t)\Big|$.

THEOREM 4. *Let $\varphi = \sum A_n(\varphi, t)$ be in BV and let $f_\varphi(t, x)$ be as in the previous section. Then for each t,*

(1) $\|f_\varphi(t, \cdot)\|_p \leqslant A_p \|\varphi\|_{BV} \|f\|_p$, $\quad 1 < p < \infty$,

(2) $\|f_\varphi(t, \cdot)\|_1 \leqslant A \|\varphi\|_{BV} \left(\int_{-\pi}^{\pi} |f| \log^+ |f|\, dx + 1\right)$.

Proof. We apply the lemma to $T = s_n(f, x)$ and obtain

$$\Big\|\sum_0^n A_{2j+1}(\varphi, t) s_j(f, \cdot)\Big\|_p \leqslant A_p \|f\|_p \big(\|\varphi\|_{BV} + o(1)\big).$$

This proves (1). In the remark after Theorem 1, we have noted that $A_p = 0\big((p-1)^{-1}\big)$ as $p \downarrow 1$. Hence application of [5_{II}, p. 119, (4.41)] establishes (2) since we may assume, in view of the linearity of f_φ in φ, that $\|\varphi\|_{BV} = 1$.

Remark 3. If $\lambda_n(\varphi, t) = \sum_{j\geqslant 1} A_{2j+1}(\varphi, t)$, then the Fourier series of $f_\varphi(t, x)$ is $\sum \lambda_n(\varphi, t) A_n(f, x)$, where $f \sim \sum A_n(f, x)$. This shows that $\{\lambda_n\}$ is a multiplier sequence. Since $|A_j(\varphi, t)| \leqslant \frac{c\|\varphi\|_{BV}}{j}$, inequality (1) of Theorem 4 could also have been obtained by the Marcinkiewicz multiplier theorem [5_{II}, p. 232].

5. We wish to investigate the convergence of the Fourier series of $f_\varphi(t, x)$. We need the following lemma:

LEMMA 3. *Assume that $f \in L^1$ and $|f(x_0 \pm t) - f(t)| = o\left(\frac{1}{|\log t|^2}\right)$ as $t \to 0$. If the Fourier coefficients of f are $O(n^{-\delta})$ for some $\delta > 0$, then $s_n(f, x_0) - f(x_0) = o\left(\frac{1}{\log n}\right)$ as $n \to \infty$.*

Proof. The proof is, apart from obvious modifications, the same as [5_I, p. 63, theorem 10.7].

THEOREM 5. *Let $f \sim \sum A_n(f, x)$ be in $L\log L$, and let $\varphi(t) = \sum A_n(\varphi, t)$ be in BV. Then for a.e. t the Fourier series of $\lim_{n\to\infty} \sum_0^n A_{2j+1}(\varphi, t) s_j(f, x) = f_\varphi(t, x)$ converges for a.e. x.*

Proof. From Theorem 4, $f_\varphi(t, x) \sim \sum \lambda_n(\varphi, t) A_n(f, x)$, where $\lambda_n(\varphi, t) = \sum_{j\geqslant n} A_{2j+1}(\varphi, t)$. It is easy to see that $\psi(t) = \frac{1}{2}[\varphi(t) - \varphi(t+\pi)] = \sum_{j=0}^\infty A_{2j+1}(\varphi, t)$. Since $\psi'(t)$ exists a.e. we can apply the lemma and obtain that

$$\lambda_N(\varphi, t) = \psi(t) - s_{N-1}(\psi, t) = o\left(\frac{1}{\log N}\right) \quad \text{for a.e. } t.$$

Since $s_N(f, x) = o(\log N)$ for a.e. x, we see that $s_N(f, x)\lambda_N(\varphi, t) = o(1)$ for a.e. t, x as $N \to \infty$. By summation by parts we have

$$\sum_0^N \lambda_n(\varphi, t) A_n(f, x) = \sum_{j=0}^{N-1} A_{2j+1}(\varphi, t) s_j(f, x) + s_N(f, x)\cdot\lambda_N(\varphi, t),$$

and thus Theorem 3 completes the proof.

6. We conclude with two remarks.

Remark 4. If one were only interested in the validity of the existence of $\lim_{n\to\infty} \sum_0^n A_{2j+1}(\varphi, t)\cdot s_j(f, x)$ for a.e. t, a.e. x (instead of every t), then an appeal to the Carleson–Hunt [1, 2] result would suffice. More precisely, if $f \in L^1$ and $\varphi \in BV$, or just $|a_n(\varphi)| = O\left(\frac{1}{n}\right)$, then, since $s_n(f, x) = o(\log n)$ for a.e. x, the trigonometric series $\sum A_n(\varphi, t) s_n(f, x)$ is for a.e. x in $L^2(dt)$. Hence $\lim_{n\to\infty} \sum_0^n A_n(\varphi, t) s_n(f, x)$ exists for a.e. t, a.e. x.

Remark 5. It is natural to ask for what subsequences $\{i_j\}$ is it true that $\lim_{n\to\infty} \sum_0^n A_{i_j}(\varphi, t) s_j(f, x)$ exists for every t and a.e. x. Here $\varphi = \sum A_n(\varphi, t)$ is in BV and $f \in L\log L$. The writer does not know whether $i_j = j$ or $i_j = 2j$ are sufficient. However, it is easy to see that one cannot choose $\{i_j\}$ arbitrarily. For, if $\varphi(t) = \sum \frac{\sin nt}{n}$, which is in BV, and $i_j = 4j+3$, then $\sum \frac{\sin i_j \pi/2}{i_j} = -\infty$, and the above limit does not exist for $f \equiv 1$.

References

[1] L. Carleson, *On convergence and growth of partial sums of Fourier series*, Acta Math. 116 (1966), pp. 135–157.

[2] R. A. Hunt, *On the convergence of Fourier series*, Proceedings of the Conference on Orthogonal Expansions and their Continuous Analogues, Southern Illinois University Press, 1968, pp. 234–255.

[3] C. J. Neugebauer, *Differentiation of trigonometric series*, this fascicle, pp. 121–125.

[4] E. M. Stein, *On limits of sequences of operators*, Ann. Math. 74 (1961), pp. 140–170.

[5] A. Zygmund, *Trigonometric series*, 2 vols., 2nd ed., Cambridge 1959.

Received August 10, 1970 (231)

STUDIA MATHEMATICA T. XLI. (1972)

On weak and Schauder decompositions

by

M. DE WILDE (Liège)

Abstract. We prove a general result about weak and Schauder decompositions which extends the well-known equivalence between weak and Schauder bases in Fréchet spaces.

All the spaces considered in this paper are locally convex topological vector spaces.

Let F be such a space. Consider a vector subspace L of F and assume that there exists a sequence of vector subspaces L_i such that, for each $f \in L$, we have

$$f = \sum_{i=1}^{\infty} f_i, \quad f_i \in L_i,$$

where the f_i are uniquely determined by f and the series converges in the topology (or the weak topology) of F. We write F_w for F equipped with its weak topology.

The maps τ_i defined from L into L_i by $\tau_i f = f_i$ are trivially linear. Moreover, each τ_i is the identity in $L \cap L_i$ and $\tau_i = 0$ in $L \cap L_j$ for each $j \neq i$.

The sequence L_i is called a *decomposition* (resp. a *weak decomposition*) of L into the L_i. It is a *Schauder decomposition* if the τ_i are continuous.

In this paper, we shall use the closed graph theorem of [2], which states that *if T is a linear map from an ultrabornological space E into a space F admitting a net of type $\mathscr{C}$ and if the graph of T is sequentially closed in $E \times F$, then T is continuous.*

We refer to [2] for the definition of nets.

THEOREM. *Let E be an ultrabornological space and let F be sequentially complete and admit a net of type $\mathscr{C}$.*

Let T be continuous from E into F and assume that its range TE admits a weak decomposition (resp. a decomposition) into sequentially closed subspaces L_i of F,

$$g = \sum_{i=1}^{\infty} \tau_i g, \quad \tau_i g \in L_i, \quad \forall g \in TE.$$

Then,

(α) *the $\tau_i T$ are continuous from E into F,*