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# Linear operators and operational calculus, Part II

by

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**Abstract.** For $0 < s \leqslant \infty$ let $Q$ be the set of all functions in $C^\infty(-\infty, s)$ which vanish on $(-\infty, 0]$; under addition and convolution $Q$ is an algebra. We denote by $P(0, s)$ the collection of all mappings of $Q$ into itself which commute with convolution. The main result is that $P(0, s)$ is algebraically isomorphic to the space of distributions on $(-\infty, s)$ having support in $[0, s)$. From this follows the sequential completeness of $P(0, s)$ and the sequential continuity of multiplication in $P(0, s)$; convergence is defined simply in terms of the ordinary pointwise convergence of functions. We also deduce the structure property that every operator in $P(0, s)$ is of finite order on each subinterval $[0, x]$ of $[0, s)$. The major results are then obtained for the more general interval $[a, b)$, where $-\infty < a < b \leqslant \infty$.

We shall begin by showing that the space of operators considered in [5] and [9] is sequentially complete when topologized as in [5]. The remainder of this paper deals with an operational calculus corresponding to an arbitrary interval $[a, b)$, where $-\infty < a < b \leqslant \infty$.

The algebra $P(0, s)$ is defined in § 1. In § 2 we characterize the space of distributions on $(-\infty, s)$ whose supports are contained in the interval $[0, s)$ in terms of a class of distributionally convergent infinite series. In § 3 we deduce that each such distribution defines a unique operator in $P(0, s)$ and show in § 4 that this space of distributions is, in fact, isomorphic to $P(0, s)$. Using this isomorphism we are able to prove that $P(0, s)$ is sequentially complete; convergence is defined simply in terms of the ordinary pointwise convergence of functions. We also prove that "multiplication" is a continuous operation. Finally, we extend these results to the more general interval $[a, b)$.

Let $\mathscr{D}_+$ be the space of all infinitely differentiable functions on $\boldsymbol{R} = (-\infty, \infty)$ whose supports are bounded to the left. Denote by $P$ the set of all linear mappings $A$ of $\mathscr{D}_+$ into $\mathscr{D}_+$ such that $A(p_1 * p_2) = p_1 * Ap_2$ whenever $p_1$ and $p_2$ belong to $\mathscr{D}_+$.

Following the idea initiated in [5], let us endow $\mathscr{D}_+$ with the topology of pointwise convergence on $\boldsymbol{R}$; since $P$ consists of mapping into $\mathscr{D}_+$, let $P_\sigma$ denote the linear space $P$ endowed with the topology of simple convergence on $\mathscr{D}_+$. Thus, if $A_n$ $(n = 0, 1, 2, \ldots)$ is a sequence in $P$, then

$$A_0 = \lim_{n\to\infty} A_n \tag{1}$$

means that the equation

$$A_0 q(t) = \lim_{n\to\infty} A_n q(t) \tag{2}$$

holds for every $q$ in $\mathscr{D}_+$ and every $t$ in $\boldsymbol{R}$. It can be shown that $P_\sigma$ is a commutative algebra with

$$A_0 B_0 = \lim_{n\to\infty} A_n B_n \tag{3}$$

whenever (1) and $B_0 = \lim B_n$ (as $n \to \infty$). A result analogous to (3) will be proved in 6.11; from it the reader will be able to prove (3) for himself.

0.1. THEOREM. *Let $A_n$ $(n = 1, 2, \ldots)$ be a sequence in $P$. If the sequence $A_n q(t)$ $(n = 1, 2, \ldots)$ converges for all $t$ in $\boldsymbol{R}$ and all $q$ in $\mathscr{D}_+$, then there exists an element $A_0$ of $P$ such that $A_0 = \lim A_n$ (as $n \to \infty$).*

0.2. Let $F$ belong to the space $\mathscr{D}'_+$ of distributions having left-bounded support. We denote by $F^*$ the mapping $q \mapsto F*q$ of $\mathscr{D}_+$ into $\mathscr{D}_+$. Thus,

$$F*q(t) = F*q(t) = \langle F, q\circ\Gamma_t\rangle \tag{0.3}$$

for any $t$ in $\boldsymbol{R}$ and $q$ in $\mathscr{D}_+$. Here $\Gamma_t$ is the function $\Gamma_t(u) = t-u$.

0.4. The mapping $F \mapsto F^*$ is a bijection of $\mathscr{D}'_+$ onto $P$. See [9].

0.5. Thus, for any $B$ in $P$ there exists a unique $B'$ in $\mathscr{D}'_+$ such that

$$B'*q = Bq \quad (\text{all } q\in\mathscr{D}_+). \tag{0.6}$$

0.7. Proof of 0.1. Since $A_n \in P$ we have $A'_n \in \mathscr{D}'_+$ and

$$A_n q(t) = A'_n * q(t) = \langle A'_n, q\circ\Gamma_t\rangle.$$

Therefore, setting $t = 0$:

$$\lim_{n\to\infty} A_n q(0) = \lim_{n\to\infty}\langle A'_n, q\circ\Gamma_0\rangle. \tag{4}$$

If $\varphi\in\mathscr{D}_-$, then $\varphi\circ\Gamma_0\in\mathscr{D}_+$; we can set $q = \varphi\circ\Gamma_0$ in (4) to obtain the existence of the limit

$$\lim_{n\to\infty}\langle A'_n, \varphi\circ\Gamma_0\circ\Gamma_0\rangle = \lim_{n\to\infty}\langle A'_n, \varphi\rangle$$

for any $\varphi$ in $\mathscr{D}_-$. From the sequential completeness of $\mathscr{D}'_+$ (the dual of $\mathscr{D}_-$: see [8, Vol. II, p. 28]), there exists an $F$ in $\mathscr{D}'_+$ such that

$$\langle F, \varphi\rangle = \lim_{n\to\infty}\langle A'_n, \varphi\rangle \quad (\text{all } \varphi\in\mathscr{D}_-). \tag{5}$$

If $q\in\mathscr{D}_+$ and $t\in\boldsymbol{R}$, then $q\circ\Gamma_t\in\mathscr{D}_-$; substituting into (5):

$$\langle F, q\circ\Gamma_t\rangle = \lim_{n\to\infty}\langle A'_n, q\circ\Gamma_t\rangle. \tag{6}$$

From (6), (0.3) and (0.6) we obtain

$$\langle F, q\circ\Gamma_t\rangle = \lim_{n\to\infty} A_n q(t).$$



The conclusion $F* = \lim A_n$ (as $n \to \infty$) is now immediate from (0.3) and (1)–(2).

**§ 1. Introduction.** Throughout, $0 < s \leqslant \infty$. To any function $f$ on the open interval $(-\infty, s)$ there corresponds a largest number $\sigma f$ such that $f$ vanishes on the open interval $(-\infty, \sigma f)$.

1.1. DEFINITION. Let $\Lambda^s$ be the family of all complex-valued functions $f$ defined on $(-\infty, s)$, such that $\sigma f \geqslant 0$, and such that $f$ is continuous on the half-open interval $[0, s)$.

1.2. Remarks. Thus, if $f\in\Lambda^s$, then $\sigma f \geqslant 0$,

$$f \text{ is continuous on } [0, s) \tag{1.3}$$

and

$$f(t) = 0 \text{ whenever } t < \sigma f. \tag{1.4}$$

1.5. Convolution. Suppose that $f\in\Lambda^s$ and $g\in\Lambda^s$. The function $f*g$ in $\Lambda^s$ is defined by

$$f*g(t) = \int_{t-s}^{s} f(t-u)g(u)\,du \quad (\text{for } t < s).$$

Clearly, since $\sigma f \geqslant 0$ and $\sigma g \geqslant 0$,

$$f*g(t) = \int_0^t f(t-u)g(u)\,du \quad (\text{for } 0 \leqslant t < s).$$

It is not too hard to verify that

$$\sigma(f*g) \geqslant \sigma f \tag{1.6}$$

(see [4]).

1.7. Notation. We shall make occasional use of Heaviside's jump function $\mathbf{1}$:

$$\mathbf{1}(t) = \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \geqslant 0. \end{cases} \tag{1.8}$$

Let $Q$ be the linear space of all functions $q$ that are infinitely differentiable on $(-\infty, s)$ and such that $\sigma q \geqslant 0$. Let $Q_\sigma$ be the linear space $Q$ endowed with the topology of pointwise convergence on the interval $(-\infty, s)$. Consequently, the equation

$$q = \lim_s q_s \quad (q \text{ and } q_s \text{ in } Q) \tag{1.9}$$

means that

$$q(t) = \lim_s q_s(t) \quad (0 \leqslant t < s); \tag{1.10}$$

recall that $q(t) = q_s(t) = 0$ for all $t < 0$. It is easily verified that $Q_\sigma$ is a locally convex topological vector space.

If $A$ is a mapping of $Q$ into $Q$ we denote by $Aq$ the function that $A$ assigns to a given $q$ in $Q$. Let $P(0,s)$ be the family of all mappings $A$ (of $Q$ into $Q$) such that

(1.11) $$A(p_1 * p_2) = p_1 * Ap_2 \quad (\text{all } p_1 \text{ and } p_2 \text{ in } Q).$$

Clearly, a mapping $A$ of $Q$ into $Q$ belongs to $P(0,s)$ if (and only if) $A(p_1*p_2)(t) = p_1*Ap_2(t)$ for all $t$ in $[0,s)$ and all $p_1$ and $p_2$ in $Q$.

Since $P(0,s)$ consists of mappings into the topological space $Q_\sigma$, this space $P(0,s)$ can be endowed with the topology of pointwise convergence on $Q$: let $P_\sigma(0,s)$ denote the resulting topological space. In consequence, the equation

(1.12) $$A = \lim_s A_s \quad (A \text{ and } A_s \text{ in } P(0,s))$$

means that the equation

(1.13) $$Aq(t) = \lim_s A_s q(t)$$

holds for all $q$ in $Q$ and any $t$ in $[0,s)$. As a consequence of 1.21, each element of $P(0,s)$ is a linear mapping of $Q$ into itself; therefore $P(0,s)$ can be made into a vector space by defining addition and scalar multiplication in the usual way. Accordingly, $P_\sigma(0,s)$ is a locally convex topological vector space.

1.14. The algebra $P(0,s)$. If $A$ and $B$ are in $P(0,s)$, we denote by $AB$ the composition of $A$ with $B$; thus $ABq = A(Bq)$ for any $q$ in $Q$. By adjoining to the linear space $P(0,s)$ the multiplication $(A,B) \mapsto AB$ we obtain a commutative algebra since

(1.15) $$AB = BA \quad \text{for any } A \text{ and any } B \text{ in } P(0,s)$$

(see [4]).

1.16. Orientation. The locally convex space $P_\sigma(0,s)$ is sequentially complete: see 5.7. The multiplication $(A,B) \to AB$ is sequentially continuous in both variables: see 6.11. The most general element of $P(0,s)$ is characterized in 5.11. (see also 1.26).

1.17. Convolution and differentiation. If $f$ is a function on $(-\infty, s)$, we denote by $f^*$ the mapping $q \mapsto f*q$; thus

(1.18) $$f^* q = f*q \quad (\text{for all } q \in Q).$$

In [4] it is seen that $f^*$ belongs to $P(0,s)$ when $f \in \Lambda^s$. The differentiation operator $D$ is the mapping that assigns to any $q$ in $Q$ its derivative $q'$:

(1.19) $$Dq = q' \quad (\text{for all } q \in Q).$$

The equation

(1.20) $$D(f*q) = f*Dq \quad (\text{for all } q \in Q)$$

holds for any $f$ in $\Lambda^s$. From (1.20) it follows that $D \in P(0,s)$.



1.21. THEOREM. *There exists a sequence $q_k$ $(k = 1, 2, \ldots)$ in $Q$ such that*

(1.22) $$A = \lim_{k\to\infty} (Aq_k)^* \quad (\text{for all } A \in P(0,s)).$$

Proof. Choose $p_k$ in $Q$ with $0 \leqslant p_k \leqslant 1$ and such that $p_k = 1$ on the half-closed interval $[k^{-1}, s)$ (cf. [10, Theorem 16.4)]. Take any $q$ in $Q$; to prove (1.22) it will suffice to show that

(1) $$Aq(t) = \lim_{k\to\infty} (Ap_k') * q(t) \quad (0 \leqslant t < s).$$

Let $\mathbf{1}$ be defined by (1.8) and observe that

(2) $$p = \mathbf{1} * p' \quad (\text{all } p \in Q).$$

Further, we have

(3) $$Aq - (Ap_k') * q = Aq - p_k' * Aq \qquad \text{by (1.11)}$$

(4) $$= Aq - p_k * (Aq)' \qquad \text{by (1.20)}$$

(5) $$= (\mathbf{1} - p_k) * (Aq)';$$

the last equation is from (2) with $p = Aq$. Combining (3)–(5) with 1.5:

(6) $$[Aq - (Ap_k') * q](t) = \int_0^t [\mathbf{1} - p_k](u) [Aq]'(t-u)\, du;$$

but $[\mathbf{1} - p_k](u) = 0$ for $u \geqslant k^{-1}$ (since $p_k = 1$ on $[k^{-1}, s)$); consequently, (6) yields

$$|[Aq - (Ap_k') * q](t)| \leqslant \{\sup_{0 \leqslant u \leqslant t} |[Aq]'(t-u)|\} \int_0^{1/k} du.$$

Conclusion (1) is immediate by taking $k \to \infty$.

1.23. DEFINITION. A sequence $s_n$ $(n = 0, 1, 2, \ldots)$ is called a *subdivision* of $[0,s)$ if

$$0 = s_0 < s_1 < \ldots < s_n < s_{n+1} < \ldots < s$$

and

$$s = \lim_{n\to\infty} s_n.$$

1.24. THEOREM. *Let $f_n$ $(n = 0, 1, 2, \ldots)$ be a sequence in $\Lambda^s$ and $s_n$ $(n = 0, 1, 2, \ldots)$ be a subdivision of $[0,s)$ such that $s_n \leqslant \sigma f_n$. If $k_n$ $(n = 0, 1, 2, \ldots)$ is a sequence of non-negative integers, then the equation*

(1.25) $$A = \sum_{n=0}^{\infty} D^{k_n} f_n^*$$

*defines an element $A$ of $P(0,s)$.*

Proof. If $n$ is a positive integer, we set

(1) $$A_n = \sum_{\nu=0}^{n-1} D^{k_\nu} f_\nu^*.$$

Let us prove that

(2) $$Aq(t) = A_n q(t) \quad (\text{all } t < s_n, \text{ all } q \in Q).$$

From (1.20) and (1.18) we see that

(3) $$Aq(t) = A_n q(t) + \sum_{\nu=n}^{\infty} f_\nu * q^{(k_\nu)}(t).$$

Suppose that $\nu \geqslant n$: We may refer to (1.6) to obtain

(4) $$\sigma(f_\nu * q^{(k_\nu)}) \geqslant \sigma f_\nu \geqslant s_\nu \geqslant s_n:$$

the last inequality comes from $\nu \geqslant n$. From (4) it follows that $f_\nu * q^{(k_\nu)}(t) = 0$ for $t < s_n$ (whenever $\nu \geqslant n$). Conclusion (2) is now immediate. Next, we use the fact that $A_n \in P(0, s)$ (see 1.17 and 1.14) to infer that $A_n q \in Q$, whence $\sigma(A_n q) \geqslant 0$, so that (2) gives $\sigma(Aq) \geqslant 0$. We still have to prove that the equations

(5) $$[Aq]^{(k)}(t) = [A_n q]^{(k)}(t) \quad (k = 0, 1, 2, \ldots)$$

and

(6) $$A(p_1 * p_2)(t) = p_1 * Ap_2(t)$$

hold for any $t < s$ and any $p_1$ and $p_2$ in $Q$. Since $s_n$ $(n = 0, 1, 2, \ldots)$ is a subdivision of $[0, s)$, there exists an integer $n$ such that $t < s_n \leqslant \sigma f_n$; equation (5) is now an immediate consequence of $A_n q \in Q$ and equation (2). Since $t < s_n$ and $p_1 * p_2 \in Q$, we may apply (2):

(7) $$A(p_1 * p_2)(t) = A_n(p_1 * p_2)(t) = (p_1 * A_n p_2)(t):$$

the second equality is from (1.11) and $A_n \in P(0, s)$. In view of $\sigma p_1 \geqslant 0$ and 1.5, equations (7) imply that

(8) $$A(p_1 * p_2)(t) = \int_{t-s}^{t} p_1(t-u)[A_n p_2](u);$$

note that $u \leqslant t < s_n$; we may therefore use (2) to replace $[A_n p_2](u)$ by $[Ap_2](u)$ in (8):

$$A(p_1 * p_2)(t) = \int_{t-s}^{t} p_1(t-u)[Ap_2](u)\,du = (p_1 * Ap_2)(t).$$

This shows that (6) holds for any $t < s$ and any $p_1$ and $p_2$ in $Q$. Consequently, $A \in P(0, s)$.

1.26. Counter-example. It could be conjectured that any element of $P(0, s)$ is of the form $D^m f^*$, where $m$ is some integer and $f$ belongs to $\Lambda^s$. Consider a subdivision $s_n$ $(n = 0, 1, 2, \ldots)$ of $[0, s)$ and define $f_n$ by $f_n(t) = \mathbf{1}(t - s_n)$ (any $t < s$); from 1.24 it follows that the operator

$$A = \sum_{n=0}^{\infty} D^n f_n^*$$

belongs to $P(0, s)$; nevertheless, it can be shown that the equation $A = D^m f^*$ fails for all integers $m$ and for all functions $f$ in $\Lambda^s$. As it turns out (see 5.11), the most general element of $P(0, s)$ has the form (1.25).



## § 2. Distributions on $(-\infty, s)$.

2.1. Notation. If $\Omega$ is an open subset of the reals, $C(\Omega)$ denotes the family of all functions that are continuous on $\Omega$; further, $\mathscr{D}(\Omega)$ denotes the family of all functions that are infinitely differentiable on $\Omega$ and whose support is a compact subset of $\Omega$. As usual, $\mathscr{D}'(\Omega)$ denotes the space of distributions on $\Omega$ (that is, the dual of $\mathscr{D}(\Omega)$). If $f$ is a locally integrable function on $\Omega$, then $\partial^n f$ denotes the distribution defined by

$$\langle \partial^n f, \varphi \rangle = (-1)^n \int_\Omega f(u)\varphi^{(n)}(u)\,du \quad (\text{for all } \varphi \in \mathscr{D}(\Omega)).$$

In particular, $\partial^0 f$ is the regular distribution corresponding to the function $f$; observe that

(2.2) $$\langle \partial^n f, \varphi \rangle = \langle \partial^0 f, (-1)^n \varphi^{(n)} \rangle \quad (\text{all } \varphi \in \mathscr{D}(\Omega)).$$

If $J$ is a subset of $\Omega$, it will be convenient to denote by $\mathscr{E}'(\Omega; J)$ the space of all elements of $\mathscr{D}'(\Omega)$ whose support is a compact subset of $J$; as usual, $\mathscr{E}'(\Omega; \Omega)$ will be denoted simply by $\mathscr{E}'(\Omega)$. We recall that the *support* of an element $F$ of $\mathscr{D}'(\Omega)$ (denoted supp $F$) is the complement (with respect to $\Omega$) of the largest open set on which $F$ vanishes. Throughout,

$$I = (-\infty, s).$$

2.3. Remark. Given $J \subset \boldsymbol{R}$, the compact subsets of $J$ are the compact subsets of $\boldsymbol{R}$ which are contained in $J$.

2.4. Lemma. *If $K$ is a compact subset of $I$, then $K \subset (-\infty, a]$, where $a < s$.*

Proof. By 2.3 the set $K$ is a closed bounded subset of $\boldsymbol{R}$ which is contained in $(-\infty, s)$.

2.5. Lemma. *Let $s_n$ $(n = 0, 1, 2, \ldots)$ be a subdivision of $[0, s)$. If $F$ belongs to $\mathscr{D}'(I)$ and has support contained in $[0, s)$, there exists a sequence $F_n$ $(n = 0, 1, 2, \ldots)$ in $\mathscr{E}'(I)$ such that*

(2.6) $$F = F_0 + \sum_{n=1}^{\infty} F_n,$$

(2.7) $$F_0 \in \mathscr{E}'(I; [0, s))$$

and

(2.8) $$F_n \in \mathscr{E}'\big(I; (s_n, s)\big) \quad (n = 1, 2, \ldots).$$

Proof. Let $\Omega_0 = (-\infty, s_2)$ and $\Omega_n = (s_n, s_{n+2})$ $(n = 1, 2, \ldots)$. Let $\beta_n$ $(n = 0, 1, 2, \ldots)$ be a partition of unity in $C^\infty(I)$ subordinated to the locally finite open covering $\Omega_n$ $(n = 0, 1, 2, \ldots)$ of $I$ (see [10, Def. 16.1]). Set

(1) $$F_n = \beta_n F \quad (n = 0, 1, 2, \ldots)$$

and note that

(2) $$\operatorname{supp} F_n \subset (\operatorname{supp} \beta_n) \cap (\operatorname{supp} F) \quad (n = 0, 1, 2, \ldots)$$

(see [3, p. 348, Prop. 2]). From (1)–(2) and $\operatorname{supp} F \subset [0, s)$ it follows that

$$\operatorname{supp} F_0 \subset \Omega_0 \cap [0, s_2),$$

which proves (2.7). Again, conclusion (2.8) comes from (2) and $\operatorname{supp} \beta_n \subset \Omega_n$. Next, to prove (2.6), take any $\varphi \in \mathscr{D}(I)$; since $\operatorname{supp} \varphi$ is a compact subset of $I$ we can use 2.4 to assert the existence of a number $\sigma < s$ such that

(3) $$\operatorname{supp} \varphi \subset (-\infty, \sigma].$$

In view of $\sigma < s$ we can infer the existence of an integer $k$ such that $\sigma < s_k$; from (3) it results that

(4) $$\operatorname{supp} \varphi \subset (-\infty, s_n) \quad (\text{all } n \geqslant k).$$

But (2.8) implies that $F_n$ vanishes on $(-\infty, s_n)$, so that (4) gives

$$\langle F_n, \varphi \rangle = 0 \quad (\text{all } n \geqslant k),$$

which implies that

(5) $$\sum_{n=0}^{\infty} \langle F_n, \varphi \rangle = \sum_{n=0}^{k-1} \langle F_n, \varphi \rangle = \langle F, \sum_{n=0}^{k-1} \beta_n \varphi \rangle;$$

the second equality is from (1). Since $\operatorname{supp} \beta_n \subset (s_n, s_{n+2})$, we see from (4) that

$$\beta_n \varphi = 0 \quad (\text{all } n \geqslant k).$$

Consequently,

(6) $$\sum_{n=0}^{k-1} \beta_n \varphi = \sum_{n=0}^{\infty} \beta_n \varphi = \Big(\sum_{n=0}^{\infty} \beta_n\Big) \varphi = \varphi:$$

the last equality is from equation (16.6) in [10]. Conclusion (2.6) now comes from (5)–(6).



2.9. Lemma. *If $T$ belongs to $\mathscr{D}'(\mathbf{R})$, we denote by $T|I$ the functional which assigns to each $\varphi$ in $\mathscr{D}(I)$ the value $\langle T, \varphi \rangle$. The correspondence $T \mapsto T|I$ maps $\mathscr{D}'(\mathbf{R})$ into $\mathscr{D}'(I)$. If $f$ is a function which is locally integrable on $\mathbf{R}$, then*

(2.10) $$(\partial^k f)|I = \partial^k (f|I) \quad (k = 0, 1, 2, \ldots),$$

*where $f|I$ is the restriction to $I$ of the function $f$.*

Proof. See [2, p. 1649].

2.11. Lemma. *There is a mapping $S \mapsto \hat{S}$ of $\mathscr{E}'(I)$ into $\mathscr{E}'(\mathbf{R})$ such that $\operatorname{supp} \hat{S} = \operatorname{supp} S$ and $\hat{S}|I = S$.*

Proof. If $S \in \mathscr{E}'(I)$, then $\operatorname{supp} S$ is a compact subset of $I = (-\infty, s)$; from 2.4 there exists a number $a < s$ with $\operatorname{supp} S \subset (-\infty, a]$. We may therefore use [2, p. 1650] with $I_0 = \big(\frac{1}{2}(a+s), \infty\big)$.

2.12. Lemma. *Suppose that $T$ belongs to $\mathscr{D}'(\mathbf{R})$ and $G_k$ $(k = 1, 2, \ldots, m)$ is a finite sequence of locally integrable functions on $\mathbf{R}$ such that $\sigma G_k \geqslant x$, where $x > -\infty$. If*

(2.13) $$T = \sum_{k=0}^{m} \partial^k G_k,$$

*then there exists a locally integrable function $g$, with $\sigma g \geqslant x$, such that the equation*

(2.14) $$T = \partial^\nu g$$

*holds for some non-negative integer $\nu$. If each $G_k$ belongs to $C(\mathbf{R})$, then $g$ belongs to $C(\mathbf{R})$. If each $G_k$ belongs to $\Lambda^\infty$, then $g$ belongs to $\Lambda^\infty$.*

Proof. Set $Y_n(t) = \mathbf{1}(t) t^n / n!$ and note that

(1) $$\partial^n (F * Y_n) = F * \mathbf{1} \quad (\text{for all } F \in \mathscr{D}'_+):$$

this comes from the fact that

(2) $$\partial^k (F * G) = F * \partial^k G \quad (k = 0, 1, 2, \ldots)$$

(see [11, p. 132]). From (1)–(2) it follows that

(3) $$\partial^{n+1} (F * Y_n) = F * \partial \mathbf{1} = F \quad (\text{for all } F \in \mathscr{D}'_+);$$

the last equation is from $\partial \mathbf{1} = \delta$ and [11, p. 127, Ex. 5.4–2]. Consequently, 2.13 and (3) give

(4) $$T = \sum_{n=0}^{m} \partial^n G_n = \sum_{n=0}^{m} \partial^n \partial^{m-n+1} (G_n * Y_{m-n}) = \partial^{m+1} g,$$

where

(5) $$g = \sum_{n=0}^{m} G_n * Y_{m-n}.$$

By [11, p. 126, ex. 5.4–1] we see that $g$ is a locally integrable function. By hypothesis, $\sigma G_n \geqslant x$; on the other hand it is clear that $Y_{m-n} \in \Lambda^\infty$. We may therefore conclude from [11, p. 125 Theorem 5.4–2] that $\sigma(G_n * Y_{m-n}) \geqslant x$. The conclusion $\sigma g \geqslant x$ is now immediate from (5). Note that (2.12) is immediate from (4). If each $G_n \in C(\boldsymbol{R})$, then each $G_n * Y_{m-n} \in C(\boldsymbol{R})$ (see [5]), from which we may conclude that $g \in C(\boldsymbol{R})$. If each $G_n \in \Lambda^\infty$, then each $G_n * Y_{m-n} \in \Lambda^\infty$ (by 1.5) and therefore $g \in \Lambda^\infty$.

2.15. Lemma. *Let $s_n$ $(n = 0, 1, 2, \ldots)$ be a subdivision of $[0, s)$. If $F$ belongs to $\mathscr{D}'(I)$ and has support contained in $[0, s)$, there exists a sequence $f_n$ $(n = 1, 2, \ldots)$ in $\Lambda^s$ with $\sigma f_n \geqslant s_n$, such that the equation*

$$F = F_0 + \sum_{n=1}^{\infty} \partial^{k_n} f_n \tag{2.16}$$

*holds for some element $F_0$ of $\mathscr{E}'(I; [0, s))$ and for some sequence $k_n$ $(n = 1, 2, \ldots)$ of non-negative integers.*

Proof. From 2.5 we see that

$$f = F_0 + \sum_{n=1}^{\infty} F_n$$

with $F_0 \in \mathscr{E}'(I; [0, s))$ and $F_n \in \mathscr{E}'(I; (s_n, s))$ $(n = 1, 2, \ldots)$; the proof will therefore be concluded by finding a function $g_n$ in $\Lambda^\infty$ and an integer $k_n$ such that

$$\sigma g_n \geqslant s_n \quad \text{and} \quad F_n = \partial^{k_n}(g_n | I). \tag{1}$$

Since $F_n \in \mathscr{E}'(I; (s_n, s))$, we may use 2.9 to infer that

$$\hat{F}_n \in \mathscr{E}'(\boldsymbol{R}; (s_n, s)). \tag{2}$$

In consequence of (2) we may use [8, Vol. I, p. 90, XXVI] to infer that the equation

$$\hat{F}_n = \sum_{k=0}^{n} \partial^k G_k \tag{3}$$

holds for $G_k \in C(\boldsymbol{R})$ with $\operatorname{supp} G_k \subset (s_n, s)$; therefore, $\sigma G_k \geqslant s_n$ and $G_k \in \Lambda^\infty$. We may therefore use (3) and (2.12) to infer that the equation

$$\hat{F}_n = \partial^{k_n} g_n \tag{4}$$

holds for some element $g_n$ of $\Lambda^\infty$ such that $\sigma g_n \geqslant s_n$ and for some non-negative integer $k_n$. To complete the proof of (1) it now suffices to observe that the equations

$$F_n = \hat{F}_n | I = \partial^{k_n} g_n | I = \partial^{k_n}(g_n | I)$$

come from (2.11), (4) and (2.10).



2.17. Theorem. *Let $s_n$ $(n = 0, 1, 2, \ldots)$ be a subdivision of $[0, s)$. If $F$ belongs to $\mathscr{D}'(I)$ and has support contained in $[0, s)$, there exists a sequence $f_n$ $(n = 0, 1, 2, \ldots)$ in $\Lambda_s$ with $\sigma f_n \geqslant s_n$, such that the equation*

$$F = \sum_{n=0}^{\infty} \partial^{k_n} f_n \tag{2.18}$$

*holds for some sequence $k_n$ $(n = 0, 1, 2, \ldots)$ of non-negative integers.*

Proof. In view of 2.15 we need only prove that the equation

$$F_0 = \partial^\nu f_0 \tag{5}$$

holds for some non-negative integer $\nu$ and for some $f_0$ in $\Lambda^s$ (recall that $s_0 = 0$). From 2.15 we see that $F_0 \in \mathscr{E}'(I; [0, s))$; we may therefore use 2.11 to obtain $\hat{F}_0$ in $\mathscr{E}'(\boldsymbol{R})$ and

$$\operatorname{supp} \hat{F}_0 \subset [0, s). \tag{6}$$

From [8, Vol. I, p. 90, XXVI] we now infer that the equation

$$\hat{F}_0 = \sum_{k=0}^{n} \partial^k G_k \tag{7}$$

holds for some $G_k \in C(\boldsymbol{R})$ with $\operatorname{supp} G_k$ contained in the neighborhood $(-1, s)$ of $[0, s)$. From 2.12 and (7) if follows that the equation

$$\hat{F}_0 = \partial^j g \tag{8}$$

holds for some non-negative integer $j$ and some $g \in C(\boldsymbol{R})$ with $\sigma g > -1$. From (6) we see that $\hat{F}_0$ vanishes on $(-\infty, 0)$, so that (8) gives

$$\partial^j g \quad \text{vanishes on} \quad (-\infty, 0). \tag{9}$$

Let $\mathbf{1}g$ be the pointwise product of the function $\mathbf{1}$ (defined by (1.8)) with the function $g$; clearly,

$$\mathbf{1}g = 0 \quad \text{on} \quad (-\infty, 0). \tag{10}$$

Consequently $\partial^j(\mathbf{1}g)$ vanishes on $(-\infty, 0)$ so that (9) gives

$$\partial^j(g - \mathbf{1}g) \quad \text{vanishes on} \quad (-\infty, 0),$$

which implies

$$\operatorname{supp} \partial^j(g - \mathbf{1}g) \subset [0, \infty). \tag{11}$$

On the other hand, $g = \mathbf{1}g$ on $(0, \infty)$ (by (1.8)); consequently, $\partial^j(g - \mathbf{1}g)$ vanishes on $(0, \infty)$, whence

$$\operatorname{supp} \partial^j(g - \mathbf{1}g) \subset (-\infty, 0]. \tag{12}$$

We may conclude from (11)–(12) that the support of $\partial^j(g - \mathbf{1}g)$ consists of at most the point $t = 0$. Consequently [8, Vol. I, p. 99, XXXV] asserts the existence of a finite set $c_n$ $(n = 0, 1, \ldots, m)$ of complex numbers such that

$$\partial^j(g - \mathbf{1}g) = \sum_{n=0}^{m} c_n \partial^n \delta = \sum_{n=0}^{m} c_n \partial^{n+1}\mathbf{1},$$

whence

$$\partial^j g = \partial^j(\mathbf{1}g) + \sum_{k=1}^{m+1} \partial^k (c_{k-1}\mathbf{1}). \tag{13}$$

From (8) and (13) it now follows that

$$\hat{F}_0 = \partial^j(\mathbf{1}g) + \sum_{k=1}^{m+1} \partial^k (c_{k-1}\mathbf{1}).$$

Note that $\mathbf{1}g$ and $c_{k-1}\mathbf{1}$ belong to $\Lambda^\infty$; Lemma 2.12 therefore asserts that the equation $\hat{F}_0 = \partial^\nu h$ holds for some $h$ in $\Lambda^\infty$ and some non-negative integer $\nu$. Appealing to (2.10):

$$F_0 = \hat{F}_0 | I = (\partial^\nu h) | I = \partial^\nu (h | I).$$

Conclusion (5) is now obtained by setting $f_0 = h | I$.

**§ 3. The sliding units.** This section is of crucial importance; we shall begin by describing the "sliding units" that will enable us to inject the space of distributions in $\mathscr{D}'(I)$ whose support is contained in $[0, s)$ into the space $P(0, s)$. As before

$$I = (-\infty, s) \quad \text{and} \quad \boldsymbol{R} = (-\infty, \infty).$$

3.1. Definition. A sliding unit is an infinitely differentiable function on $\boldsymbol{R}$ that assumes the value 1 on a neighborhood of $[0, \infty)$. If $x < s$ we denote by $[x]$ the set of all sliding units $e$ such that $\sigma e > x - s$.

3.2. Remarks. Suppose that $x < s$. If $e \in [x]$, then

$$-\infty \leqslant x - s < \sigma e < 0 \quad \text{and} \quad 0 < \sigma e < \infty. \tag{3.3}$$

Observe that the set $[x]$ is not void; indeed, we can apply [10, Theorem 16.4] with $F = [0, \infty)$ and $U = \left(\tfrac{1}{2}(x - s), \infty\right)$.

3.4. Lemma. *If $0 \leqslant x < s$ and $e \in [x]$, then*

$$-\infty < \sigma e < x < x - \sigma e < s. \tag{3.5}$$

Proof. From (3.3) we see that $-\infty < \sigma e < 0$; since $0 \leqslant x$ we have

$$-\infty < \sigma e < x < x - \sigma e.$$

The last inequality comes from the fact that $-\sigma e > 0$ (see (3.3)). It only remains to show that

$$x - \sigma e < s. \tag{1}$$



In case $s < \infty$ this is immediate from (3.3); if $s = \infty$ inequality (1) comes from $x < s$ and from the fact that $-\sigma e < \infty$ (see (3.3)).

3.6. Lemma. *If $t_1 \leqslant t_2 < s$, then $[t_2] \subset [t_1]$.*

Proof. If $e \in [t_2]$, then $e$ is a sliding unit such that $\sigma e > t_2 - s$; since $t_2 - s \geqslant t_1 - s$ we may conclude that $\sigma e > t_1 - s$ and therefore that $e \in [t_1]$.

3.7. Definitions. Let $f$ be a function with domain dom $f$. If $e$ is a sliding unit the function $ef$ is defined by

$$ef(t) = \begin{cases} 0 & \text{if } t \notin \operatorname{dom} f \\ e(t)f(t) & \text{if } t \in \operatorname{dom} f. \end{cases} \tag{3.8}$$

If $K \subset \operatorname{dom} f$ we shall use the notation

$$\|f\|_K = \sup_{t \in K} |f(t)|. \tag{3.9}$$

3.10. Remarks. Suppose that $0 \leqslant x < s$ and $e \in [x]$. If $f \in C^\infty(x - s, \infty)$, then $ef \in C^\infty(x - s, \infty)$. If $f = 0$ on $(x, \infty)$, then

$$\operatorname{supp} ef \subset [\sigma e, x]; \tag{3.11}$$

in consequence, $ef \in \mathscr{D}(I)$.

3.12. Definition. Given $x < s$, let $\Gamma_x$ be the function defined by $\Gamma_x(t) = x - t$.

3.13. Remarks. The function $\Gamma_x$ maps an interval $(a, b)$ onto the interval $(x - b, x - a)$:

$$\Gamma_x(a, b) = (x - b, x - a). \tag{3.14}$$

If $g$ is a function, then $(g \circ \Gamma_x)(t) = g(x - t)$ and

$$(g \circ \Gamma_x) \circ \Gamma_x = g. \tag{3.15}$$

3.16. Lemma. *Suppose that $0 \leqslant x < s$ and $e \in [x]$. If $q \in Q$, then $e(q \circ \Gamma_x)$, belongs to $\mathscr{D}(I)$ and*

$$\operatorname{supp} e(q \circ \Gamma_x) \subset [\sigma e, x]. \tag{3.17}$$

Proof. Note that $\Gamma_x(x - s, \infty) = (-\infty, s)$ and $q \in C^\infty(-\infty, s)$; consequently, $q \circ \Gamma_x \in C^\infty(x - s, \infty)$. Since $\Gamma_x(x, \infty) = (-\infty, 0)$ and since $q = 0$ on $(-\infty, 0)$, it follows that $q \circ \Gamma_x = 0$ on $(x, \infty)$. Consequently, we can apply 3.10 with $f = q \circ \Gamma_x$ to obtain the desired conclusion.

3.18. Lemma. *Suppose that $0 \leqslant x < s$ and $e \in [x]$. If $p \in Q$, there exists a number $N_e(x, p) < \infty$ such that*

$$\|p \circ \Gamma_t\|_{[\sigma e, x]} \leqslant N_e(x, p) \quad (\textit{all } t \leqslant x). \tag{3.19}$$

Proof. From 3.4 it follows that $-\infty < \sigma e < x < \infty$; consequently, $\Gamma_t[\sigma e, x] = [t - x, t - \sigma e]$, whence

$$\|p \circ \Gamma_t\|_{[\sigma e, x]} = \|p\|_{[t-x, t-\sigma e]} = \|p\|_{[0, t-\sigma e]}. \tag{1}$$

The second equality is obtained by combining $t-x \leqslant 0$ with the fact that $p = 0$ on $(-\infty, 0)$. We now set

(2) $$N_e(x,p) = \|p\|_{[0,x-\sigma e]}$$

and observe that $t-\sigma e \leqslant x-\sigma e$, so that

(3) $$\|p\|_{[0,t-\sigma e]} \leqslant N_e(x,p).$$

Conclusion (3.19) is immediate from (1) and (3). To prove that $N_e(x,p) < \infty$, observe that $x-\sigma e < s$ (by (3.5)); since $p$ is continuous on $[0,s)$ we have that $p$ is continuous on $[0, x-\sigma e]$, whence the conclusion $N_e(x,p) < \infty$ now comes from (2).

3.20. Theorem. *If $f$ belongs to $\Lambda^s$ and $m$ is a non-negative integer, then the equation*

(3.21) $$\langle \partial^m f, e(q\circ \Gamma_t)\rangle = D^m f^* q(t) \quad (\textit{all } t < s)$$

*holds for any $e$ in $[t]$ and any $q$ in $Q$.*

Proof. Take any $q \in Q$ and $t < s$. If $e \in [t]$, then

(1) $$\langle \partial^m f, e(q\circ\Gamma_t)\rangle = \int_{-\infty}^{s} (-1)^m f(u)[e(q\circ\Gamma_t)]^{(m)}(u)\,du.$$

Since $\sigma f \geqslant 0$ we have

(2) $$\langle \partial^m f, e(q\circ\Gamma_t)\rangle = \int_0^s (-1)^m \sum_{\nu=0}^{m} \binom{m}{\nu} f e^{(m-\nu)}[q\circ\Gamma_t]^{(\nu)}.$$

Since $e = 1$ on $[0,s)$ we have $e^{(m-\nu)} \neq 0$ on $[0,s)$ only when $\nu = m$; consequently (2) gives

(3) $$\langle \partial^m f, e(q\circ\Gamma_t)\rangle = \int_0^s (-1)^m f(u)[q\circ\Gamma_t]^{(m)}(u)\,du.$$

But $[q\circ\Gamma_t]' = -[q'\circ\Gamma_t]$, so that $[q\circ\Gamma_t]^{(m)} = (-1)^m[q^{(m)}\circ\Gamma_t]$; equation (3) therefore becomes

(4) $$\langle \partial^m f, e(q\circ\Gamma_t)\rangle = \int_{t-s}^{s} f(u)\,q^{(m)}(t-u)\,du:$$

we have replaced the lower limit by $t-s$ (since $t-s < 0 \leqslant \sigma f$). We may now use 1.5 to write

$$\langle \partial^m f, e(q\circ\Gamma_t)\rangle = f * q^{(m)}(t) = D^m f^* q(t):$$

the second equality is from (1.20) and (1.18).

3.22. Theorem. *If $F$ belongs to $\mathscr{D}'(I)$ and has support contained in $[0,s)$, there exists an element $A$ of $P(0,s)$ such that the equation*

(3.23) $$\langle F, e(q\circ\Gamma_t)\rangle = Aq(t) \quad (\textit{all } t < s)$$

*holds for any $e$ in $[t]$ and any $q$ in $Q$.*



Proof. Let $s_n$ $(n = 0,1,2,\ldots)$ be any subdivision of $[0,s)$. From 2.17 we see that there exists a sequence $f_n$ $(n = 0,1,2,\ldots)$ in $\Lambda^s$ such that $\sigma f_n \geqslant s_n$ and such that the equation

$$\langle F, \varphi\rangle = \sum_{n=0}^{\infty} \langle \partial^{k_n} f_n, \varphi\rangle \quad (\text{all } \varphi \in \mathscr{D}(I))$$

holds for some sequence $k_n$ $(n = 0,1,2,\ldots)$ of non-negative integers. Since $e(q\circ\Gamma_t) \in \mathscr{D}(I)$ (see 3.16), we obtain

$$\langle F, e(q\circ\Gamma_t)\rangle = \sum_{n=0}^{\infty} \langle \partial^{k_n} f_n, e(q\circ\Gamma_t)\rangle,$$

and, by 3.20:

$$\langle F, e(q\circ\Gamma_t)\rangle = \sum_{n=0}^{\infty} D^{k_n} f_n^* q(t) \quad (\text{all } t < s).$$

Conclusion (3.23) now comes from (1.24).

3.24. Corollary. *Suppose that $F$ belongs to $\mathscr{D}'(I)$ and has support contained in $[0,s)$. If $q \in Q$ and $t < s$, the family*

$$\{\langle F, e(q\circ\Gamma_t)\rangle : e \in [t]\}$$

*contains a unique element, which will be denoted by $F^* q(t)$. Consequently,*

(3.25) $$F^* q(t) = \langle F, e(q\circ\Gamma_t)\rangle \quad (\textit{all } e \in [t]).$$

3.26. Definition. Let $F$ and $q$ be as in 3.24; we denote by $F^* q$ the function that assigns to any $t$ in $I$ the number $F^* q(t)$. Further, let $F^*$ be the mapping that assigns to any $q$ in $Q$ the function $F^* q$.

3.27. Corollary. *If $F \in \mathscr{D}'(I)$ and $\operatorname{supp} F \subset [0,s)$, then $F^* \in P(0,s)$.*

Proof. Combine 3.26 with 3.22.

3.28. Lemma. *If $f \in \Lambda^s$, then*

(3.29) $$(\partial^m f)^* = D^m f^* \quad (m = 0,1,2,\ldots).$$

Proof. Immediate from (3.21) and (3.25)–3.26.

**§ 4. The isomorphism.** Let $\mathscr{D}'(I; I_0)$ be the space of all $F$ in $\mathscr{D}'(I)$ (with $I = (-\infty, s)$) such that $\operatorname{supp} F \subset [0,s)$. From 3.24–(3.25) and (3.23) it follows that the correspondence $F \mapsto F^*$ is a linear mapping of $\mathscr{D}'(I; I_0)$ into $P(0,s)$. In this § 4 we shall prove that the correspondence $F \mapsto F^*$ is a one-to-one mapping onto $P(0,s)$.

4.0. Lemma. *If $\varphi \in \mathscr{D}(I)$ there exists a number $t < s$ with $\varphi\circ\Gamma_t \in Q$; further*

(4.1) $$\langle F, \varphi\rangle = F^*(\varphi\circ\Gamma_t)(t) \quad (\textit{all } F \in \mathscr{D}'(I; I_0)).$$

Proof. Since $\operatorname{supp}\varphi$ is a compact subset of $I$, we can use 2.4 to infer the existence of a number $t < s$ such that $\operatorname{supp}\varphi \subset (-\infty, t]$; therefore,

$$(1) \qquad \varphi \quad \text{vanishes on} \quad (t, \infty).$$

Since $\varphi \in C^\infty(\boldsymbol{R})$, we see that $\varphi \circ \Gamma_t \in C^\infty(\boldsymbol{R})$: It only remains to prove that $\varphi \circ \Gamma_t$ vanishes on $(-\infty, 0)$. Since $\Gamma_t(-\infty, 0) = (t, \infty)$, the fact that $\varphi \circ \Gamma_t$ vanishes on $(-\infty, 0)$ follows immediately from (1). To prove (4.1), take $e \in [t]$ and note that the equations

$$F^*(\varphi \circ \Gamma_t)(t) = \langle F, e(\varphi \circ \Gamma_t \circ \Gamma_t)\rangle = \langle F, e\varphi\rangle$$

come directly from (3.25) and (3.15). Conclusion 4.1 now comes immediately from the fact that $e\varphi = \varphi$ on a neighbourhood of $[0, s) \supset \operatorname{supp} F$ (recall that $e = \mathbf{1}$ on a neighborhood of $[0, \infty)$: see 3.1).

4.2. Lemma. *If $A \in P(0, s)$, there exists an $F$ in $\mathscr{D}'(I; I_0)$ such that $A = F^*$.*

Proof. Take any $\varphi$ in $\mathscr{D}(I)$; from 4.0 we infer the existence of a number $t < s$ such that $\varphi \circ \Gamma_t \in Q$. We may therefore apply 1.21 to obtain

$$(2) \qquad A(\varphi \circ \Gamma_t)(t) = \lim_{k\to\infty} [(Aq_k)^*(\varphi \circ \Gamma_t)](t).$$

Setting $m = 0$ in 3.28 we obtain

$$(3) \qquad [(Aq_k)^*(\varphi \circ \Gamma_t)](t) = \left[(\partial^0(Aq_k))^*(\varphi \circ \Gamma_t)\right](t)$$

$$(4) \qquad = \langle \partial^0(Aq_k), \varphi\rangle:$$

the last equation is from (4.1). Combining (2) with (3)–(4) we see that

$$(5) \qquad A(\varphi \circ \Gamma_t)(t) = \lim_{k\to\infty} \langle \partial^0(Aq_k), \varphi\rangle \qquad (\text{any } \varphi \in \mathscr{D}(I)).$$

We may now use [3, p. 315, Prop. 2] to infer the existence of $F$ in $\mathscr{D}'(I)$ such that

$$(6) \qquad F = \lim_{k\to\infty} \partial^0(Aq_k).$$

Since $Aq_k \in Q$ we see that $\sigma(Aq_k) \geqslant 0$, so that $\partial^0(Aq_k)$ vanishes on $(-\infty, 0)$; consequently, it is easy to infer from (6) that $F$ vanishes on $(-\infty, 0)$, whence

$$\operatorname{supp} F \subset [0, s).$$

Therefore, $F \in \mathscr{D}'(I; I_0)$: it only remains to prove that $F^* = A$. To that effect, take any $q \in Q$ and $t < s$; we have

$$(7) \qquad F^* q(t) = \langle F, e(q \circ \Gamma_t)\rangle \qquad \text{by (3.25)}$$

$$(8) \qquad = \lim_{k\to\infty} \langle \partial^0(Aq_k), e(q \circ \Gamma_t)\rangle \qquad \text{by (6)}$$

$$(9) \qquad = \lim_{k\to\infty} \left[(\partial^0(Aq_k))^* q\right](t) \qquad \text{by (3.25).}$$



We may now combine (7)–(9) with 3.28 to obtain

$$F^* q(t) = \lim_{k\to\infty} [(Aq_k)^* q](t) = Aq(t):$$

the second equation is from (1.22). Since $q \in Q$ and $t < s$, this concludes the proof of $F^* = A$.

4.3. Theorem. *The mapping $F \mapsto F^*$ is a linear bijection of $\mathscr{D}'(I; I_0)$ onto $P(0, s)$.*

Proof. It is a surjection by 4.2; it only remains to prove the bijectivity. To that effect, assume

$$(10) \qquad F^* = 0.$$

Since $F \mapsto F^*$ is a linear mapping it will suffice to prove that $F = 0$. Take any $\varphi \in \mathscr{D}(I)$; from 4.0 we know that the equation

$$\langle F, \varphi\rangle = F^*(\varphi \circ \Gamma_t)(t)$$

holds for some $t < s$. From (10) it now follows $\langle F, \varphi\rangle = 0$ for any $\varphi \in \mathscr{D}(I)$, whence our conclusion $F = 0$.

4.4. Theorem. *The mapping $F \mapsto F^*$ is sequentially continuous.*

Proof. Let $F_n$ $(n = 1, 2, \ldots)$ be a sequence in $\mathscr{D}'(I; I_0)$; it is a question of proving that, if

$$(1) \qquad 0 = \lim_{n\to\infty} F_n \qquad (\text{in the sense of } \mathscr{D}'(I)),$$

then $0 = \lim F_n^*$ (as $n \to \infty$). To prove this, take $q \in Q$ and $t < s$; we must show that

$$(2) \qquad 0 = \lim_{n\to\infty} F_n^* q(t).$$

Since $e(q \circ \Gamma_t) \in \mathscr{D}(I)$ for $e \in [t]$ (see 3.16), it follows from (1) that

$$0 = \lim_{n\to\infty} \langle F_n, e(q \circ \Gamma_t)\rangle,$$

whence (2) is now immediate from (3.25).

## § 5. The completeness property and the representation theorem.

5.1. Definition. Given $A \in P(0, s)$ let $A'$ denote the unique $F$ in $\mathscr{D}'(I; I_0)$ such that $F^* = A$.

5.2. Remarks. The existence and uniqueness of $F$ comes from 4.3. In consequence of 5.1, we have

$$(5.3) \qquad A' \in \mathscr{D}'(I; I_0) \quad \text{and} \quad A'^* = A.$$

Further, if $p \in Q$ and $t < s$, then

$$(5.4) \qquad Ap(t) = \langle A', e(p \circ \Gamma_t)\rangle \qquad (\text{all } e \in [t]):$$

this comes from (3.25). If $\varphi \in \mathcal{D}(I)$, then there exists a number $t < s$ such that

$$\varphi \circ \Gamma_t \in Q \tag{5.5}$$

and

$$\langle A', \varphi \rangle = A(\varphi \circ \Gamma_t)(t) \tag{5.6}$$

(see 4.0).

5.7. THEOREM. *Let $\lambda$ be an accumulation point of a set $J \subset \boldsymbol{R}$. Suppose that $\{A_\varepsilon:\ \varepsilon \in J\}$ is a family in $P(0, s)$ such that*

$$\lim_{\varepsilon \to \lambda} A_\varepsilon \tag{5.8}$$

*exists in the sense of (1.12). Then there exists an element $F$ of $\mathcal{D}'(I; I_0)$ such that*

$$F^* = \lim_{\varepsilon \to \lambda} A_\varepsilon \tag{5.9}$$

*and*

$$F = \lim_{\varepsilon \to \lambda} A'_\varepsilon, \tag{5.10}$$

*this last limit being taken in the topology of $\mathcal{D}'(I)$.*

Proof. Let $\varepsilon_n$ $(n = 1, 2, \ldots)$ be a sequence in $J$ such that $\varepsilon_n \to \lambda$ when $n \to \infty$. By hypothesis, the sequence $A_{\varepsilon_n}$ $(n = 1, 2, \ldots)$ converges; setting $B_n = A_{\varepsilon_n}$, this means that for every $q \in Q$ and every $t < s$ there exists a number $q_\lambda(t)$ such that

$$q_\lambda(t) = \lim_{n \to \infty} B_n q(t).$$

Take $\varphi \in \mathcal{D}(I)$; from (5.5) we know the existence of $t < s$ such that

$$\begin{aligned} (\varphi \circ \Gamma_t)_\lambda(t) &= \lim_{n \to \infty} B_n(\varphi \circ \Gamma_t)(t) \\ &= \lim_{n \to \infty} \langle B'_n, \varphi \rangle \qquad \text{by (5.6).} \end{aligned}$$

Therefore we may again use [3, p. 315, Prop. 2] to infer the existence $F$ in $\mathcal{D}'(I; I_0)$ such that

$$F = \lim_{n \to \infty} B'_n = \lim_{n \to \infty} A'_{\varepsilon_n}: \tag{1}$$

the second equality comes from $B_n = A_{\varepsilon_n}$. Equation (5.8) implies

$$F^* = \lim_{n \to \infty} A_{\varepsilon_n} \tag{2}$$

(by 4.4 and $A'^*_{\varepsilon_n} = A_{\varepsilon_n}$). Let us verify (5.9) (resp., (5.10)): if (5.9) (resp., (5.10)) is false, some neighborhood $N$ of $F^*$ in the topological space $P_\sigma(0, s)$ (resp., of $F$ in the space $\mathcal{D}'(I)$) could be found such that for all $n > 0$ the relation

$$A_{\varepsilon_n} \notin N \qquad (\text{resp., } A'_{\varepsilon_n} \notin N)$$

would hold for $\varepsilon_n \to \lambda$ as $n \to \infty$. Since $A_{\varepsilon_n} = B_n$, this contradicts (2) (resp., (1)).



5.11. THEOREM. *Let $s_n$ $(n = 0, 1, 2, \ldots)$ be any subdivision of $[0, s)$ and $A$ belong to $P(0, s)$. There exists a sequence of functions $f_n$ $(n = 0, 1, 2, \ldots)$ in $\Lambda^s$ such that $\sigma f_n \geqslant s_n$ and such that the equation*

$$A = \sum_{n=0}^{\infty} D^{k_n} f_n^* \tag{5.12}$$

*holds for some sequence $k_n$ $(n = 0, 1, 2, \ldots)$ of non-negative integers.*

Proof. Since $A' \in \mathcal{D}'(I; I_0)$, we infer from 2.17 the existence of a sequence $f_n$ $(n = 0, 1, 2, \ldots)$ in $\Lambda^s$ such that $\sigma f_n \geqslant s_n$ and such that the equation

$$A' = \sum_{n=0}^{\infty} \partial^{k_n} f_n \tag{1}$$

holds for some sequence $k_n$ $(n = 0, 1, 2, \ldots)$ of non-negative integers. From (1) and 4.4 it follows that

$$A'^* = \sum_{n=0}^{\infty} [\partial^{k_n} f_n]^* = \sum_{n=0}^{\infty} D^{k_n} f_n^*:$$

the second equality is from 3.28. Conclusion (5.12) is now immediate from (5.3).

## § 6. The continuity of multiplication.

Suppose that $0 \leqslant x < s$. If $m$ is a non-negative integer and if $p \in Q$, the equation

$$\mu_x^{(m)}(p) = \sup_{0 \leqslant k \leqslant m} \|p^{(k)}\|_{[-m, x]} \tag{6.1}$$

defines a seminorm $\mu_x^{(m)}$ on the space $Q$; it is the mapping that assigns to any $p$ in $Q$ the number $\mu_x^{(m)}(p)$. Let $Q_\mu$ denote the space $Q$ endowed with the topology determined by the family of seminorms

$$\{\mu_x^{(m)}:\ m = 0, 1, 2, \ldots \ \text{ and } \ 0 \leqslant x < s\}. \tag{1}$$

Consequently,

$$q = Q_\mu \lim_{\varepsilon} q_\varepsilon \qquad (q \text{ and } q_\varepsilon \text{ in } Q) \tag{6.2}$$

means that

$$\lim_{\varepsilon} \mu_x^{(m)}(q - q_\varepsilon) = 0 \qquad (\text{all } m \geqslant 0, \ \text{ all } x < s). \tag{6.3}$$

If $p \in Q$, the equation

(6.4) $$\mu_x^{(m)}(p) = \sup_{0 \leqslant k \leqslant m} \sup_{0 \leqslant t \leqslant x} |p^{(k)}(t)|$$

is an immediate consequence of the fact that $p = 0$ on $(-\infty, 0)$.

6.5. THEOREM. *The space $Q_\mu$ is a Fréchet space; it is also barreled.*

Proof. Since Fréchet spaces are barreled (see [10, p. 347]), it suffices to show that $Q_\mu$ is a Fréchet space. Let $C^\infty(I)$ be the space of all infinitely differentiable functions defined on $I$; let $\mathscr{E}(I)$ be the result of endowing $C^\infty(I)$ with the topology determined by the seminorms (1). This space $\mathscr{E}(I)$ is a Fréchet space [10, pp. 85–89]. Since $Q \subset C^\infty(I)$ and since $Q_\mu$ is the space $Q$ endowed with the topology induced by $\mathscr{E}(I)$, it will suffice to prove that $Q$ is closed in $\mathscr{E}(I)$. To that effect, we can imitate the reasoning found at the bottom of p. 131 of [10]; let $q_s$ be any net in $Q$ that converges in $\mathscr{E}(I)$. Equation (6.2) then holds for some $q \in C^\infty(I)$; since (6.2) implies that

$$q(t) = \lim_s q_s(t) \qquad (-\infty < t < s),$$

we see that $q = 0$ on $(-\infty, 0)$ (since $\sigma q_n \geqslant 0$). Therefore, $\sigma q \geqslant 0$ and (since $q \in C^\infty(I)$) we have that $q \in Q$. Thus $Q$ is closed in $\mathscr{E}(I)$; consequently $Q_\mu$ is a Fréchet space.

6.6. LEMMA. *Suppose that $q \in Q$ and let $m$ be a non-negative integer. If $0 \leqslant x < s$ and $e \in [x]$, we denote by $H_m(x, e, q)$ the family*

$$\{e(q^{(k)} \circ \Gamma_t):\ 0 \leqslant k \leqslant m \text{ and } 0 \leqslant t \leqslant x\}.$$

*This family $H_m(x, e, q)$ is a bounded subset of $\mathscr{D}(I)$.*

Proof. If $t \leqslant x$ we have $[x] \subset [t]$ (by 3.6); it therefore follows from our hypothesis $e \in [x]$ that $e \in [t]$. From 3.17 and $e \in [t]$ it results immediately that

$$\operatorname{supp} e(q \circ \Gamma_t) \subset [\sigma e, t] \subset [\sigma e, x].$$

On the other hand, $[\sigma e, x]$ is a bounded subset of $\boldsymbol{R}$ (see (3.5)). Set $K = [\sigma e, x]$ and let $\nu$ be any non-negative integer; from [3, p. 166] we see that it will suffice to find a number $N_m(x, e, q)_\nu > 0$ (independent of $k$ and $t$) such that the inequality

(2) $$\|[e(q^{(k)} \circ \Gamma_t)]^{(\nu)}\|_K \leqslant N_m(x, e, q)_\nu$$

holds for $0 \leqslant k \leqslant m$ and $0 \leqslant t \leqslant x$. To that effect, note that

(3) $$|[e(q^{(k)} \circ \Gamma_t)]^{(\nu)}| \leqslant \sum_{i=0}^{\nu} \binom{\nu}{i} \|e^{(\nu-i)}\|_K \|[q^{(k)} \circ \Gamma_t]^{(i)}\|_K.$$

But

(4) $$\|[q^{(k)} \circ \Gamma_t]^{(i)}\|_K = \|q^{(k+i)} \circ \Gamma_t\|_{[\sigma e, x]} \leqslant N_e(x, q^{(k+i)}):$$



the inequality is from (3.19) with $p = q^{(k+i)}$. From (3) and (4) we now obtain

$$\|[e(q^{(k)} \circ \Gamma_t)]^{(\nu)}\|_K \leqslant \sum_{i=0}^{\nu} \binom{\nu}{i} \|e^{(\nu-i)}\|_K \sum_{a=0}^{m} N_e(x, q^{(a+i)}).$$

Thus (2) obtains with

$$N_m(x, e, q)_\nu = \sum_{i=0}^{m} \sum_{a=0}^{m} \binom{\nu}{i} \|e^{(\nu-i)}\|_K N_e(x, q^{(a+i)}).$$

6.7. THEOREM. *Let $T_n$ $(n = 1, 2, \ldots)$ be a sequence in $P(0, s)$. The following statements are equivalent:*

(i) $$\lim_{n\to\infty} T_n = 0;$$

(ii) $$Q_\mu \lim_{n\to\infty} T_n q = 0 \qquad (\text{all } q \in Q).$$

Proof. Since (ii) obviously implies (i), it will suffice to prove that (i) implies (ii). From (i) and 5.7 it follows the existence of an element $F$ of $\mathscr{D}'(I; I_0)$ such that

(1) $$F = \lim_{n\to\infty} T'_n \qquad (\text{in } \mathscr{D}'(I))$$

and

(2) $$F^* = \lim_{n\to\infty} T_n.$$

From (i) and (2) we see that $F^* = 0$, which implies (by 4.3) that $F = 0$; consequently (1) gives

(3) $$0 = \lim_{n\to\infty} T'_n$$

in the topology of $\mathscr{D}'(I)$; from [10, p. 358, Cor. 2] we see that (3) holds in the strong topology; thus

(4) $$0 = \lim_{n\to\infty} \sup_{\varphi \in H} |\langle T'_n, \varphi\rangle|$$

whenever $H$ is a bounded subset of $\mathscr{D}(I)$: see Example IV of [10, p. 198]. Take $0 \leqslant x < s$ and $e \in [x]$; for any non-negative integer $m$ we substitute in (4) for $H$ the bounded subset $H_m(x, e, q)$ (see 6.6):

(5) $$0 = \lim_{n\to\infty} \sup_{0 \leqslant k \leqslant m} \sup_{0 \leqslant t \leqslant x} |\langle T'_n, e(q^{(k)} \circ \Gamma_t)\rangle|.$$

If $0 \leqslant k \leqslant m$ and $0 \leqslant t \leqslant x$, then $e \in [t]$ (by 3.6 and since $e \in [x]$ by hypothesis); consequently we may use (5.4):

(6) $$T_n q^{(k)}(t) = \langle T'_n, e(q^{(k)} \circ \Gamma_t)\rangle.$$

On the other hand, since $T_n q^{(k)} = T_n D^k q = D^k T_n q$, we may combine (6) with (5) to obtain

$$0 = \lim_{n\to\infty} \sup_{0\leqslant k\leqslant m} \sup_{0\leqslant t\leqslant x} |D^k T_n q(t)|;$$

that is, by (6.4):

$$0 = \lim_{n\to\infty} \mu_x^{(m)}(T_n q) \quad (m = 0, 1, 2, \ldots \text{ and } 0 \leqslant x < s).$$

Conclusion (ii) is now immediate from (6.2)–(6.3).

6.8. Definition. If $E_1$ and $E_2$ are Fréchet spaces, we denote by $\mathscr{L}(E_1, E_2)$ the space of continuous linear mappings of $E_1$ into $E_2$.

6.9. Lemma. *If $p \in Q$, then $p^* \in \mathscr{L}(Q_\mu, Q_\mu)$.*

Proof. Take any $q$ in $Q$; we can use (1.20) to obtain

(1) $$[p^* q]^{(k)} = p * q^{(k)} \quad (k = 1, 2, \ldots).$$

If $0 \leqslant t \leqslant x < s$, then (1) and 1.5 give

(2) $$|[p^* q]^{(k)}(t)| \leqslant \int_0^t |p(t-u) q^{(k)}(u)|\, du.$$

From (2) it follows that

$$|[p^* q]^{(k)}(t)| \leqslant [\sup_{0\leqslant u\leqslant t} |q^{(k)}(u)|] \int_0^t |p(t-u)|\, du.$$

Setting

$$M_t(p) = \int_0^t |p(t-u)|\, du = \int_0^t |p(t)|\, dt,$$

equation (3) implies that for $0 \leqslant k \leqslant m$,

(4) $$|[p^* q]^{(k)}(t)| \leqslant M_x(p) \sup_{0\leqslant k\leqslant m} \|q^{(k)}\|_{[0,x]}.$$

From (4) it therefore follows that

(5) $$\mu_x^{(m)}(p^* q) \leqslant M_x(p)\, \mu_x^{(m)}(q) \quad (m = 0, 1, 2, \ldots \text{ and } 0 \leqslant x < S)$$

for any $q$ in $Q$. It is easily verified that $p^*$ is a linear mapping; from (5) and [3, p. 97, Prop. 2] it now follows that $p^* \in \mathscr{L}(Q_\mu, Q_\mu)$.

6.10. Theorem. $P(0, s) \subset \mathscr{L}(Q_\mu, Q_\mu)$.

Proof. Suppose that $A \in P(0, s)$. Combining 1.21 with 6.7, we obtain

$$Aq = Q_\mu \lim_{k\to\infty} p_k^* q \quad (\text{all } q \in Q),$$

where $p_k = Aq_k$. Thus, $A$ is the pointwise limit of a sequence $p_k^*$ $(k = 1, 2, \ldots)$ in $\mathscr{L}(Q_\mu, Q_\mu)$. Since $Q_\mu$ is barreled we may now use the Banach–Steinhaus theorem [10, p. 348] to conclude that $A \in \mathscr{L}(Q_\mu, Q_\mu)$.



6.11. Theorem. *Suppose that $A_n$ $(n = 0, 1, 2, \ldots)$ and $B_n$ $(n = 0, 1, 2, \ldots)$ are sequences in $P(0, s)$. If $A_0 = \lim A_n$ and $B_0 = \lim B_n$, then $A_0 B_0 = \lim A_n B_n$.*

Proof. From 6.7 it follows that

(6) $$A_0 q = Q_\mu \lim_{n\to\infty} A_n q \quad (\text{all } q \in Q)$$

and

(7) $$B_0 q = Q_\mu \lim_{n\to\infty} B_n q \quad (\text{all } q \in Q).$$

From 6.10 we known that $A_n \in \mathscr{L}(Q_\mu, Q_\mu)$ and $B_n \in \mathscr{L}(Q_\mu, Q_\mu)$. Let $\mathscr{L}_\sigma(Q_\mu, Q_\mu)$ denote the space $\mathscr{L}(Q_\mu, Q_\mu)$ endowed with the topology of pointwise convergence on $Q_\mu$. Equations (6)–(7) state that the sequences approach their respective limits in $\mathscr{L}_\sigma(Q_\mu, Q_\mu)$; we may therefore apply [1, p. 43, Cor. 2] to infer that the sequence $A_n \circ B_n$ $(n = 1, 2, \ldots)$ approaches the limit $A_0 \circ B_0$ in $\mathscr{L}_\sigma(Q_\mu, Q_\mu)$. That is,

(8) $$A_0 B_0 q = Q_\mu \lim_{n\to\infty} A_n B_n q \quad (\text{all } q \text{ in } Q).$$

The conclusion $A_0 B_0 = \lim A_n B_n$ now comes directly from 6.7 and (1.12)–(1.13).

## § 7. The interval $[a, b)$.

7.1. Henceforth, $-\infty < a < b \leqslant \infty$. Let $Q(a, b)$ be the space of all functions $q$ that are infinitely differentiable on the half-closed interval $[a, b)$ and such that $q^{(k)}(a) = 0$ for $k = 0, 1, 2, \ldots$

7.2. The translator. Given a number $x$ and a function $f$, let $T_x f$ be the function defined by $T_x f(t) = f(t-x)$. Let $Q^s$ be the space (previously denoted by $Q$) of all infinitely differentiable functions on $(-\infty, s)$ that vanish on $(-\infty, 0)$. If $\varphi \in Q^{b-a}$, then $T_a \varphi \in Q(a, b)$; consequently, if $V \in P(0, b-a)$, then the composition $T_a \circ V$ is a linear mapping of $Q^{b-a}$ into $Q(a, b)$.

7.3. Definition. Let $P(a, b)$ denote the linear space

$$\{T_a \circ V\colon\ V \in P(0, b-a)\}.$$

Let $L_0(a, b)$ be the set of all functions which are absolutely integrable on each interval $(a, x)$ with $a < x < b$ and which vanish on $(-\infty, a)$.

7.4. Theorem. *If $f$ belongs to $L_0(a, b)$, the equation*

$$f^* \varphi = T_a(T_{-a} f * \varphi) \quad (\text{all } \varphi \in Q^{b-a})$$

*defines a function $f^* \varphi$ in $Q(a, b)$. Let $f^*$ be the mapping $\varphi \mapsto f^* \varphi$. The mapping $f \mapsto f^*$ is a linear injection of $L_0(a, b)$ into $P(a, b)$.*

Proof. It is easily seen that the mapping $f \mapsto T_{-a} f$ is a linear injection of $L_0(a, b)$ into $L_0(0, b-a)$. By [4, (5.17)] the mapping $F \mapsto F^*$ is a linear injection of $L_0(0, b-a)$ into $P(0, b-a)$. And, it is immediate from 7.3 that the mapping $V \mapsto T_a \circ V$ is a linear injection of $P(0, b-a)$ into $P(a, b)$. Therefore, the mapping $f \mapsto T_a \circ (T_{-a}f)^*$ is a linear injection of $L_0(a, b)$ into $P(a, b)$. But, by (1.18) and 7.2,

$$T_a \circ (T_{-a}f)^* \varphi = T_a(T_{-a}f * \varphi) \quad (\text{all } \varphi \in Q^{b-a}),$$

which completes the proof.

7.5. Theorem. *Let $Q(a, b)_\sigma$ be the space $Q(a, b)$ endowed with the topology of pointwise convergence; let $P_\sigma(a, b)$ be the space $P(a, b)$ endowed with the topology of pointwise convergence. Then the sequence $V_n$ $(n = 1, 2, \ldots)$ converges in $P_\sigma(0, b-a)$ if and only if the sequence $T_a \circ V_n$ $(n = 1, 2, \ldots)$ converges in $P_\sigma(a, b)$; consequently, the space $P_\sigma(a, b)$ is sequentially complete and*

$$\lim_{n\to\infty} T_a \circ V_n = T_a \circ (\lim_{n\to\infty} V_n).$$

Proof. By 5.7 the space $P_\sigma(0, b-a)$ is sequentially complete. And we may infer from (1.12)–(1.13) and 7.2 that the sequence $V_n$ converges to $V$ in $P_\sigma(0, b-a)$ if and only if the equation

$$(T_a \circ V)\varphi(t) = \lim_{n\to\infty} (T_a \circ V_n)\varphi(t)$$

holds for all $\varphi$ in $Q^{b-a}$ and any $t$ in $[a, b)$.

7.6. Multiplication. If $V \in P(0, b-a)$, the equation

$$\tilde{V} = T_a \circ V \circ T_{-a} \tag{7.7}$$

defines a mapping of $Q(a, b)$ into itself; consequently, if $A \in P(a, b)$, then $\tilde{V} \circ A$ is a mapping of $Q^{b-a}$ into $Q(a, b)$. We shall write

$$VA = \tilde{V} \circ A \quad \text{and} \quad AV = A \circ V. \tag{7.8}$$

Note that the equation

$$T_a \circ V \circ W = V(T_a \circ W) \tag{7.9}$$

holds for any $W$ in $P(0, b-a)$.

7.10. Theorem. *If $V$ belongs to $P(0, b-a)$ and $A$ belongs to $P(a, b)$, then $AV$ belongs to $P(a, b)$ and $AV = VA$.*

Proof. By 7.3, $A = T_a \circ W$ for some $W \in P(0, b-a)$. Therefore

$$AV = (T_a \circ W) \circ V = T_a \circ (WV), \tag{1}$$

from which it follows that $AV \in P(a, b)$. By (1.15),

$$WV = VW; \tag{2}$$

combining (1)–(2) with (7.9) we have

$$AV = T_a \circ (VW) = V(T_a \circ W) = VA.$$



7.11. Theorem. *Suppose that $A_n$ $(n = 0, 1, 2, \ldots)$ is a sequence in $P(a, b)$ and $V_n$ $(n = 0, 1, 2, \ldots)$ is a sequence in $P(0, b-a)$. If $A_0 = \lim A_n$ and $V_0 = \lim V_n$, then $A_0 V_0 = \lim A_n V_n$.*

Proof. By 7.3, $A_n = T_a \circ W_n$ for some $W_n \in P(0, b-a)$. By hypothesis and 7.5 we have that $W_n$ converges to $W_0$. Therefore, by 6.11,

$$W_0 V_0 = \lim_{n\to\infty} W_n V_n;$$

consequently, by 7.5 again,

$$T_a \circ W_0 V_0 = \lim_{n\to\infty} T_a \circ W_n V_n.$$

We conclude the proof by observing that $T_a \circ W_n V_n = A_n V_n$.

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