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An example of a normed spaces non-isomorphic to its product by the real line

by

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Abstract. An example of a normed space non-isomorphic to its product by the real line.

Let X be an infinite-dimensional Banach space over reals and let R be the real line. The following problem is still open.

PROBLEM. *Is X isomorphic to $X\times R$?*

C. Bessaga, A. Pełczyński and the present author [1] gave an example of an infinite-dimensional nuclear complete locally convex metric space (nuclear B_0-space) non-isomorphic to its product by R.

The present note contains an example of a non-complete normed space with this property. The constructed example is a pre-hilbertian space.

1. Let Γ be a set of sequences of positive numbers such that

(*) *if $\{t_n\}\in\Gamma$ and $\{a_n\}$ is a bounded sequence of positive numbers, then $\{t_n a_n\}\in\Gamma$.*

Let X be a normed space with a norm $\|x\|$. The space X is said to be Γ-approximable if there is a sequence $\{L_n\}$ of finite-dimensional subspaces such that

(i) $\dim L_n = n-1 \quad (n = 1, 2, ..)$,

(ii) $L_n \subset L_{n+1} \quad (n = 1, 2, \ldots)$,

and

(iii) $\{\delta_n(x)\}\in\Gamma, \quad$ where $\delta_n(x) = \inf\{\|x-y\|: y\in L_n\}$.

The property (*) trivially implies

PROPOSITION 1. *Γ-approximability is an invariant of isomorphism, i. e. if two normed spaces X and Y are isomorphic, then X is Γ-approximable if and only if Y is Γ-approximable.*

2. Let X be the normed space of all sequences of reals $x = \{x_n\}$, such that

$$M_x = \sup_n 2^{2^n}|x_n| < +\infty$$

with the usual hilbertian norm $\|x\| = \sqrt{\left(\sum_{n=1}^{\infty}|x_n|^2\right)}$.

Let Γ be the set of sequences of positive numbers belonging to X.

PROPOSITION 2. *The space X is Γ-approximable.*

Proof. Let L_n be the space of elements of the type $\{a_1, \ldots, a_{n-1}, 0, \ldots\}$. Clearly, the sequence $\{L_n\}$ satisfies (i) and (ii). Moreover, we have

$$\delta_n(x) = \left(\sum_{i=1}^{\infty}|t_i|^2\right)^{1/2} \leqslant \frac{M_x}{2^{2^n}}\left(\sum_{i=n}^{\infty}\frac{1}{2^{2(2^i-2^n)}}\right)^{1/2} \leqslant 2\cdot\frac{M_x}{2^{2^n}}.$$

Thus $\{\delta_n(x)\}\in\Gamma$.
Hence the space X is Γ-approximable.

3. Now we shall show that the space $X\times R$ is not Γ-approximable. For this purpose we shall need some geometric properties of ellipsoids in a Hilbert space.

We recall that a compact ellipsoid in a Hilbert space H is a set

$$E = \left\{\sum_{i=1}^{\infty}a_ie_i^E:\ \sum_{i=1}^{\infty}|a_i|^2 \leqslant 1\right\},$$

where $(e_i^E, e_j^E) = 0$ for $i \neq j$ and $\lim_i \|e_i^E\| = 0$.

The vectors e_i^E are called the axes of the ellipsoid E. Of course, the ellipsoid E does not depend on the ordering of the axes in a sequence. We shall assume in the sequel that $\|e_1^E\| \geqslant \|e_2^E\| \ldots$ and we shall put $\lambda_i^E = \|e_i^E\|$, $i = 1, 2, \ldots$

Let us put

$$\dot{E} = \{x\in E:\ rx\notin E \text{ for all } r > 1\}.$$

LEMMA 1. *Let E be a compact ellipsoid in a Hilbert space H. For any orthogonal projection P the set PE is a compact ellipsoid. Moreover, if* $\dim\ker P = 1$ *then*

$$\lambda_i^{PE} \geqslant \lambda_{i+1}^E \quad i = 1, 2, \ldots$$

Proof. The first assertion is obvious. Let $0 \neq e_0\in\ker P$. Let $E_n = E\cap\operatorname{span}\{e_1, \ldots, e_n\}$. Let us observe that if $x\in E_n\cap\dot{E}$, then $\|x\| \geqslant \lambda_n^E$.

Now we shall show Lemma 1 by induction. Let e_1' be an element of $E_2\cap\dot{E}$ orthogonal to e_0. Thus $Pe_1' = e_1'$ and $\|e_1'\| \geqslant \lambda_2^E$. Hence $\lambda_1^{PE} = \|e_1^{PE}\| \geqslant \|e_1'\| \geqslant \lambda_2^E$. Let $H_n = \operatorname{span}\{e_0, e_1^{PE}, \ldots, e_{n-1}^{PE}\}$. Let e_n' be an element $E_{n+1}\cap\dot{E}$ orthogonal to H_n. Let us observe that $\lambda_n^{PE} = \|e_n^{PE}\| \geqslant \|e_n'\| \geqslant \lambda_{n+1}^E$.

This implies the conclusion.

LEMMA 2. *Let*

$$K = \left\{x\in E:\ \|x\| \geqslant \frac{\lambda_1^E}{2}\right\}.$$

Then conv K *contains an ellipsoid E' such that*

$$\lambda_1^{E'} = \lambda_1^E$$

and

$$\lambda_i^{E'} \geqslant \frac{\lambda_i^E}{\sqrt{2}} \qquad i = 2, 3, \ldots$$

Proof. Let us pick $x\in\dot{E}$ so that $(x, e_1) = 0$ and let $K_x = K\cap\operatorname{span}\{x, e_1\}$.

It is easy to verify that conv K_x contains an ellipse (two-dimensional ellipsoid) E'' such that

$$\lambda_1^{E''} = \lambda_1^E$$

and

$$\lambda_2^{E''} \geqslant \frac{\|x\|}{\sqrt{2}}.$$

This trivially implies the conclusion of the lemma.

LEMMA 3. *Let $\{g_1, \ldots, g_n, \ldots\}$ be an orthonormal sequence of elements of Hilbert space H. Let E be a compact ellipsoid contained in H. Let $L_1 = \{0\}$,* $L_n = \operatorname{span}\{g_1, \ldots, g_{n-1}\}$ *and*

$$K_n = \left\{x\in E:\ \delta_i(x) \geqslant \frac{\lambda_i^E}{2^{i+1}}\quad i = 1, 2, \ldots, n\right\},$$

where as before

$$\delta_i(x) = \inf\{\|x-y\|:\ y\in L_i\}.$$

Then the sets K_n are not empty.

Proof. Let P_n be the orthogonal projection onto a subspace orthogonal to g_n. Let $\pi_n = \prod_{i=1}^{n} P_n$. We shall show by induction that (**) $\operatorname{conv}\pi_nK_n$ *contains an ellipsoid E_n such that* $\lambda_i^{E_n} \geqslant \dfrac{\lambda_{i+n}^E}{2^{n+1}}$.

By Lemma 2, $\operatorname{conv} K_1$ contains an ellipsoid E'' such that $\lambda_i^{E''} \geqslant \frac{\lambda_i^E}{\sqrt{2}} \geqslant \frac{\lambda_i^E}{2}$. Thus by Lemma 1 $\operatorname{conv} P_1 K_1 = P_1 \operatorname{conv} K_1$ contains an ellipsoid E_1 such that

$$\lambda_i^{E_1} \geqslant \frac{\lambda_{i+1}^E}{2}.$$

Therefore (**) hold for $n = 1$.

Let us suppose that (**) holds for $n = m$. By Lemma 1, $\operatorname{conv} \pi_{n+1} K_n = \operatorname{conv} P_{m+1}(\pi_m K_m) = P_{m+1}(\operatorname{conv} \pi_m K_m)$ contains an ellipsoid E_m' such that

$$\lambda_i^{E_m'} \geqslant \lambda_{i+1}^{E_m} \geqslant \frac{\lambda_{i+m+1}^E}{2^m}.$$

Let us observe that $\delta_{m+1}(x) = \|\pi_{m+1} x\|$. Thus

$$\pi_{m+1} K_{m+1} = \left\{ x \in \pi_{m+1} K_m : \ \|x\| \geqslant \frac{\lambda_{m+1}^E}{2^{m+1}} \right\}.$$

Therefore, by Lemma 2, $\operatorname{conv} \pi_{m+1} K_{m+1}$ contains an ellipsoid E_{m+1} such that

$$\lambda_i^{E_{m+1}} \geqslant \lambda_i^{E_m'} \cdot \frac{1}{\sqrt{2}} \geqslant \frac{\lambda_i^{E_m'}}{2} \geqslant \frac{\lambda_{i+m+1}^E}{2^{m+1}}.$$

This completes the proof.

COROLLARY. *Let E be a compact ellipsoid in a Hilbert space H. Let L_n be an arbitrary sequence of finite-dimensional subspaces satisfying* (i) *and* (ii). *Then there is an element $x \in E$ such that*

$$\delta_n(x) \geqslant \frac{\lambda_n^E}{2^{n+1}}.$$

Proof. The sets K_n are compact and $K_n \supset K_{n+1}$. Thus $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$. This trivially implies the conclusion of the corollary.

PROPOSITION 3. *The space $X \times R$ is not Γ-approximable.*

Proof. The space $X \times R$ is obviously isomorphic to the space X_1 of real sequences $x = \{x_n\}$ such that

$$\sup_n 2^{2^{n-1}} |x_n| < +\infty$$

with the usual Hilbert norm $\|x\| = (\sum_{n=1}^{\infty} x_n^2)^{1/2}$.

Let L_n be an arbitrary sequence of finite-dimensional subspaces satisfying (i) and (ii).

Let

$$e_n = \{0, \ldots, \underbrace{0, 2^{-2^{n-1}}}_{n\text{-th place}}, 0, \ldots\}$$

Obviously, $\|e_n\| = 2^{-2^{n-1}}$. Let us observe that the ellipsoid

$$E = \left\{ \sum_{n=1}^{\infty} a_n e_n : \ \sum_{n=1}^{\infty} |a_n|^2 \leqslant 1 \right\}$$

is contained in x_1. Thus by the Corollary there is an $x \in E \cap X_1$ such that

$$\delta_n(x) \geqslant \frac{2^{-2^{n-1}}}{2^n} = 2^{-2^{n-1}-n}.$$

Therefore,

$$2^{2^n} \delta_n(x) \geqslant 2^{2^n - 2^{n-1} - n} \to \infty.$$

This implies that the space X_1 is not Γ-approximable.

From Propositions 1–3 we immediately obtain the following

THEOREM. *The prehilbertian space X is not isomorphic to the space $X \times R$.*

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References

[1] C. Bessaga, A. Pełczyński, S. Rolewicz, *On diametral approximative dimension and linear homogeneity of F-spaces*, Bull. Acad. Pol. Sc. 9 (1961), pp. 677–683.

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