Proof of Theorem 3.1. Here the details are identical to the proof of the corresponding theorem in [1]; the method is due to Strichartz [8].

Let $f \in L^p_{a,\gamma}$. By (2.2), $f = H^\gamma_a \varphi$ for some $\varphi \in L^p$ and $\|f\|_{p,a,\gamma} \approx \|f\|_p + \|\varphi\|_p$. By (3.2), (3.3), and a density argument, we have $\|\varphi\|_p \approx \|S^{(\gamma)}_a f\|_p$.

On the other hand, suppose both f and $S^{(\gamma)}_a f$ are in L^p. Let g_n be a sequence of functions in C^∞_0 such that

(1) $g_n \geqslant 0$,

(2) $\|g_n\|_1 = 1$,

(3) $\|F * g_n - F\|_p \to 0$ for each $F \in L^p$.

A routine argument shows that both g_n and $f * g_n$ are in $L^p_{a,\gamma}$. By what we have shown,

$$\|f * g_n\|_{p,a,\gamma} \leqslant C\big(\|f * g_n\|_p + \|S^{(\gamma)}_a (f * g_n)\|_p\big).$$

But $\|f * g_n\|_p \leqslant \|f\|_p$, and since $g_n \geqslant 0$ Minkowski's inequality gives $S^{(\gamma)}_a (f * g_n) \leqslant g_n * S^{(\gamma)}_a f$ and thus $\|S^{(\gamma)}_a (f * g_n)\|_p \leqslant \|g_n S^{(\gamma)}_a f\|_p \leqslant \|S^{(\gamma)}_a f\|_p$. Hence $\{f * g_n\}$ forms a bounded sequence in $L^p_{a,\gamma}$. It is shown in [3] that $L^p_{a,\gamma}$ is reflexive; consequently, some subsequence $f * g_{n_k}$ is weakly convergent in $L^p_{a,\gamma}$. Since $f * g_{n_k} \to f$ in L^p, it follows that $f \in L^p_{a,\gamma}$.

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Convolution of functions in Lorentz spaces

by

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Abstract. The well-known results of M. Rajagopalan, W. Żelazko, and N. Rickert concerning the *L^p-conjecture* are extended to the Lorentz spaces $L(p, q)(G)$ defined on (non-compact) locally compact groups G. Related results for compact groups are also given. The theorems presented here are complementary to, and motivated by, earlier results on convolution of functions in Lorentz spaces by R. O'Neil and the author.

1. Introduction. In O'Neil [2] and Yap [8] convolution of functions in various Lorentz spaces $L(p, q)(X, \mu)$ (for definition, see (2.1) *infra*) were considered and interesting and useful results were obtained. For example, if G is a *unimodular* locally compact group with Haar measure λ and p_1, p_2 are real numbers such that $1/p_1 + 1/p_2 \geqslant 1$, then for f in $L(p_1, q_1)(G, \lambda)$ and g in $L(p_2, q_2)(G, \lambda)$, the usual convolution product $f * g$ is always defined (under some mild restriction on q_1 and q_2) and it has further pleasant properties [2], [8]. In this note we consider the case $1/p_1 + 1/p_2 < 1$ and thus answer some natural questions left open in the earlier papers. In addition to this we also consider the Banach algebras $L(p, q)(G, \lambda)$ when G is a compact group and $p > 1$. Our results are generalizations of theorems in [1], [3], [4], [7], [9].

2. Definitions and preliminary results. In this section we review the basic definitions, and give some preliminary results which are needed in the sequel.

Definitions 2.1. Let f be a measurable function defined on a measure space (X, μ). For $y \geqslant 0$, we define $m(f, y) = \mu\{x \in X \colon |f(x)| > y\}$, and let f^* be the function on $[0, \infty)$ which is inverse to $m(f, \cdot)$ and right-continuous. For $x > 0$, we define

$$f^{**}(x) = x^{-1} \int\limits_0^x f^*(t)\,dt.$$

We also define

$$\|f\|^*_{(p,q)} = \left\{\int\limits_0^\infty [x^{1/p} f^*(x)]^q \frac{dx}{x}\right\}^{1/q} \quad \text{if} \quad 0 < p < \infty, \ 0 < q < \infty;$$

(i)

$$\|f\|^*_{(p,\infty)} = \sup_{x>0} x^{1/p} f^*(x) \quad \text{if} \quad 0 < p < \infty;$$

$$L(p, q) = L(p, q)(G, \mu) = \{f\colon \|f\|^*_{(p,q)} < \infty\}.$$

If we replace $f^*(x)$ by $f^{**}(x)$ in (i) above, the resulting number will be denoted by $\|f\|_{(p,q)}$.

Remark. It is known that

(a) for $1 < p < \infty$, we have $\|f\|^*_{(p,q)} \leqq \|f\|_{(p,q)} \leqq C(p,q)\|f\|^*_{(p,q)}$, where $C(p,q)$ is a constant depending only on p and q ([8], p. 652);

(b) for $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $\|\cdot\|_{(p,q)}$ is a norm for $L(p, q)$ ([2], p. 136);

(c) for $1 < p < \infty$, we have $L(p,p) = L^p$, and $\|\cdot\|_p$ is equivalent to $\|\cdot\|_{(p,q)}$ ([2], p. 136).

We now state without proof a simple fact for later reference.

LEMMA 2.2. *If (X, μ) is a measure space with $\mu(X) = 1$ and $r > p \geqq 1$, then $L(r, q) \subset L(p, q)$ and $\|f\|^*_{(p,q)} \leqq \|f\|^*_{(r,q)}$, $\|f\|_{(p,q)} \leqq \|f\|_{(r,q)}$.*

3. The closure problem. Throughout this section G will denote a locally compact group with left Haar measure λ, and $L(p, q)$ will denote the Lorentz space $L(p, q)(G, \lambda)$.

THEOREM 3.1. *Let p_1, p_2, and q_1, q_2 be positive real numbers satisfying $1 < p_i < \infty$, $1/p_1 + 1/p_2 < 1$. Then we have*

(i) *if G is non-compact, then there exists an open set U in G, and functions $f \in L(p_1, q_1)$, $g \in L(p_2, q_2)$ such that $f*g(x)$ is not defined for x in U;*

(ii) *If G is compact, then for $f \in L(p_1, q_1)$ and $g \in L(p_2, q_2)$, $f*g(x)$ exists and is finite for λ-almost all x in G, and it is in some $L(r, s)$.*

Proof. (i) We begin with the case of a unimodular group G. Let V be a compact symmetric neighborhood of the identity element e in G and choose a sequence $\{x_n\}$ of distinct points in G such that

$$V^2 x_n \cap V^2 x_m = \emptyset \quad \text{if} \quad m \neq n.$$

Since $y \to \lambda(y^{-1}V \cap V)$ is a continuous function, there is an open set $U \subseteq V$ such that $\lambda(y^{-1}V \cap V) > 0$ for $y \in U$. Define r by $r(1/p_1 + 1/p_2) = 1$. Choose $\varepsilon_n = \pm 1$, so that $\sum \varepsilon_n/n$ is divergent. Now define

$$f = \sum_{n=2}^\infty c_n \chi_{Vx_n}, \qquad g = \sum_{n=2}^\infty b_n \chi_{Vx_n},$$

where $c_n = 1/n^{r/p_1}$, $b_n = \varepsilon_n/n^{r/p_2}$, and χ_A denotes the characteristic function of A. We assert that f and g have the desired properties. Indeed it is easy to see that

$$m(f, y) = (n-1)\lambda(V) \quad \text{for} \quad c_{n+1} \leqq y < c_n,$$
$$f^*(x) = c_{n+1} \quad \text{for} \quad (n-1)\lambda(V) \leqq x < n\lambda(V).$$

Now to show that $f \in L(p_1, q_1)$. By Calderón's theorem [2], [8], we may assume that $q_1 \leqslant 1$ in the following calculations. We have

$$\int\limits_0^\infty x^{q_1/p_1 - 1} f^*(x)^{q_1} dx = \int\limits_0^{\lambda(V)} x^{q_1/p_1 - 1} f^*(x)^{q_1} dx + \sum_{n=2}^\infty \int\limits_{(n-1)\lambda(V)}^{n\lambda(V)} x^{q_1/p_1 - 1} c_n^{q_1} dx$$

$$= C + \sum_{n=2}^\infty c_n^{q_1} \int\limits_{(n-1)\lambda(V)}^{n\lambda(V)} x^{q_1/p_1 - 1} dx \qquad (C = \text{Constant})$$

$$= C + \sum_{n=2}^\infty c_n^{q_1} (p_1/q_1) \lambda(V)^{q_1/p_1} [n^{q_1/p_1} - (n-1)^{q_1/p_1}]$$

$$= C + \sum_{n=2}^\infty n^{-rq_1/p_1} \lambda(V)^{q_1/p_1} \sigma_n^{q_1/p_1 - 1}, \qquad n-1 < \sigma_n < n$$

(by applying the Mean-Value Theorem to the function $x \to x^{q_1/p_1}$)

$$\leqq C + \sum_{n=2}^\infty (n-1)^{-rq_1/p_1} (n-1)^{q_1/p_1 - 1} \lambda(V)^{q_1/p_1} < \infty.$$

Thus $f \in L(p_1, q_1)$. Similarly, we have $g \in L(p_2, q_2)$. But for $x \in U$,

$$f^*g(x) = \int f(xy) g(y^{-1}) d\lambda(y) = \lambda(x^{-1}V \cap V) \sum \varepsilon_n/n,$$

which is not defined.

Finally, suppose that G is a non-unimodular group. Then we define $G_0 = \{x \in G\colon \Delta(x) = 1\}$, where Δ is the modular function of G, and apply the preceding argument to G_0.

(ii) Suppose that G is compact. Let $f \in L(p_1, q_1)$, $g \in L(p_2, q_2)$. Define α by $\alpha(1/p_1 + 1/p_2) = 1$, so that $1 < \alpha < p_i$. Choose k so that $1 < \alpha < k < p_i$. Hence $p_i > p_i/k > 1$ and $(p_1/k)^{-1} + (p_2/k)^{-1} = k/p_1 + k/p_2 > 1$. By Lemma (2.2) we have $f \in L(p_1/k, q_1)$, $g \in L(p_2/k, q_2)$. Finally, by Theorem (3.5) of [8], $f*g(x)$ exists and is finite for λ-almost all $x \in G$, and $f*g \in L(r, s)$, where $1/r = k/p_1 + k/p_2 - 1$, and s is any positive number with $1/q_1 + 1/q_2 \geqq 1/s$.

Remark. A portion of the proof of (i) in the preceding theorem is borrowed from Rickert [4]. Rickert's result is an extension of those in

Rajagopalan [3] and Żelazko [9]. Our result is a generalization of the results of these authors (see Theorem (2.7) of [7] and the remarks immediately following (2.1) in this connection).

4. The case of a compact group. Throughout this section G will denote a compact group with Haar measure λ, and $L(p,q)$ will denote the Lorentz space $L(p,q)(G,\lambda)$.

THEOREM 4.1. *Let $1<p<\infty$ and $1\leqslant q<\infty$, then $L(p,q)$ is a Banach algebra with respect to some norm which is equivalent to the norm $\|\cdot\|_{(p,q)}$ (multiplication in $L(p,q)$ is the usual convolution of functions).*

Proof. Define r by $r=2p/(p+1)$, so that $1/r+1/r-1=1/p$ and $p>r$. Thus $L(p,q)\subset L(r,q)$, by Lemma (2.2). Now let f, g be in $L(p,q)$, then f, g are in $L(r,q)$ and hence

$$\|f*g\|_{(p,s)}\leqq C\,\|f\|_{(r,q)}\|g\|_{(r,q)},$$

by Theorem (3.5) of [8], where s is any positive number satisfying $1/s\leqq 2/q$. (C is a constant depending only on the indices of the spaces involved and it need not be the same at different occurrences.) In particular, $f*g$ is in $L(p,q)$. Furthermore

$$\begin{aligned}\|f*g\|_{(p,q)} &\leqq C\,\|f*g\|_{(p,q/2)} \quad \text{(by Calderón's theorem, see [8], p. 653)}\\ &\leqq C\,\|f\|_{(r,q)}\|g\|_{(r,q)} \quad \text{(by Theorem (3.5) of [8])}\\ &\leqq C\|f\|_{(p,q)}\|g\|_{(p,q)} \quad \text{(by Lemma (2.2)).}\end{aligned}$$

This last inequality and a well known theorem of Gelfand complete the proof.

COROLLARY 4.2. *Let $1<p,q<\infty$ and suppose that G is infinite. Then the Banach algebra $L(p,q)$ fails to have the factorization property, i. e., not every function in $L(p,q)$ can be written in the form $f*g$, with f, g in $L(p,q)$.*

Proof. Let $s=(q+1)/2$, so that $1<s<q$ and $1/s\leqq 2/q$. It follows from the preceding proof and [7], (2.7), that

$$L(p,q)*L(p,q)\subset L(p,s)\subsetneqq L(p,q).$$

Remark. This corollary is a generalization of results of Edwards ([1], p. 93) and the author [7], (2.8). It is not difficult to see that $L(p,q)*L(p,q)\equiv\{f*g\colon f,g\in L(p,q)\}$ is a first category subset of $L(p,q)$.

Remark. In view of the preceding theorem, it is interesting to note that if G is infinite and $1<q\leqslant\infty$, then there exist f, g in $L(1,q)$ such that $f*g\equiv\infty$.

Proof. Since G is non-discrete, there exists a sequence $\{V_n\}$ of measurable sets such that $\lambda(V_n)=1/n(n+1)$. Now define $f=\sum_{n=3}^{\infty}(n/\log n)\chi_{V_n}$, $g\equiv 1$. These two functions have the desired properties.

Remark. If $1<p<\infty$, $1\leqslant q<\infty$ and G is Abelian, then it can be shown that the maximal ideal space of $L(p,q)$ is the dual group $\hat{G}$, and that the Shilov–Wiener Tauberian theorem holds in $L(p,q)$. See [5], [6] for related results.

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