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A difference quotient norm for spaces of quasi-homogeneous Bessel potentials

by

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Abstract. A generalized class of Sobolev spaces has been defined using Fourier transforms. It is shown that the natural norm is equivalent to an integral norm of a mixed difference quotient, thus generalizing a result of Strichartz.

Introduction. The purpose of this paper is to obtain a more useful norm for the spaces of quasi-homogeneous Bessel potentials introduced in Cotlar and Sadosky [3]. The desired norm is analogous to that obtained by Strichartz [8] for fractional Sobolev spaces and by the author [1] for parabolic spaces. The same general method could presumably be used here, but an examination of [1] reveals that the calculations would be hopelessly complicated. In this paper the crucial inequality is established by using an inductive argument to extend Strichartz's result. The key to the inductive step is given in § 2.

Some of the results presented here were discovered during conversations with Professor C. C. Tu of Case Western Reserve University; I am grateful for his help.

1. Preliminaries. Points in R^n will be denoted $x = (x_1, \ldots, x_n)$. Throughout the paper $a = (a_1, \ldots, a_n)$ will be a fixed multi-index, where each a_i is rational with $a_1 = 1$, $a_i \geqslant 1$ for $i = 2, \ldots, n$. Let m be the smallest positive integer such that each of $\frac{m}{a_1}, \ldots, \frac{m}{a_n}$ is an even integer. We define a quasi-homogeneous metric on R^n by

$$[x] = (x_1^{m/a_1} + \ldots + x_n^{m/a_n})^{1/m}.$$

For λ a positive real number, define

$$\lambda^{(a)} x = (\lambda^{a_1} x_1, \ldots, \lambda^{a_n} x_n).$$

Then $[\lambda^{(a)} x] = \lambda [x]$.

For arbitrary complex γ, define an operator J_a^γ by

$$(J_a^\gamma f)^\wedge = (1 + [x]^m)^{-\gamma/m} \hat{f}.$$

Here ^ denotes the Fourier transform, understood in the sense of distributions [6]; where no confusion arises the dual variable is also denoted by x.

For $1 \leqslant p \leqslant \infty$, we define a Banach space $L^p_{a,\gamma} = \{J^\gamma_a f\colon f \in L^p\}$. $L^p_{a,\gamma}$ is normed by $\|J^\gamma_a f\|_{p,a,\gamma} = \|f\|_p$.

An operator closely related to J^γ_a and $L^p_{a,\gamma}$ is defined by

$$(H^\gamma_a f)^\wedge = [x]^{-\gamma}\hat{f},$$

defined for all f such that $[x]^{-\gamma}\hat{f}$ is a tempered distribution. In many contexts, H^γ_a is an unbounded operator.

In order for equations to be valid in the sense of functions, we will occasionally restrict ourselves to the class L^∞_0 of essentially bounded, measurable functions vanishing a. e. outside some compact set.

For the induction argument, we will need to work in R^{n+1} as well as R^n. Points in R^{n+1} will be denoted as $\tilde{x} = (x, t)$, where $x \in R^n$. $A = (a, b) = (a_1, \ldots, a_n, b)$ will be a fixed $(n+1)$-index of the same type as a. We define

$$[\tilde{x}] = (x_1^{M/a_1} + \ldots + x_n^{M/a_n} + t^{M/b})^{1/M}$$

where M is the smallest positive integer such that each of $M/a_1, \ldots, M/a_n$, M/b is an even integer. $\lambda^{(A)}x$, J^γ_A, $L^p_{A,\gamma}$, and H^γ_A, are given the obvious definitions.

Occasionally, operators indexed by "a" will be applied to functions in R^{n+1}. We adopt the convention $T_a f(\tilde{x}) = \big(T_a f(\cdot, t)\big)(x)$ if all the quantities involved are functions; we understand the analogous definition in the case of distributions.

2. Tools. The following theorem will be used repeatedly. It has appeared in various forms in a number of papers; an accessible proof of the version stated here is given in Littman, McCarthy, and Rivière [5].

MULTIPLIER THEOREM 2.1. *Let M be a function of n variables which is of class C^n in all of R^n except possibly the coordinate planes. Suppose that $|x^\alpha D^\alpha M(x)|$ is bounded for each multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with each $\alpha_i = 0$ or 1. Then for $1 < p < \infty$, the operator T defined by $(T\varphi)^\wedge = M\hat{\varphi}$ is a continuous operator from L^p into L^p. M is called a multiplier.*

THEOREM 2.2. *Let $1 < p < \infty$ and $\gamma > 0$. Then $f \in L^p_{a,\gamma}$ if and only if $f \in L^p$ and $f = H^\gamma_a \varphi$ for some $\varphi \in L^p$. In this case, $\|f\|_{p,a,\gamma} \approx \|f\|_p + \|\Phi\|_p$.*

Proof. Suppose $f \in L^p$, $\varphi \in L^p$, and $f = H^\gamma_a \varphi$. Then $\varphi = H^{-\gamma}_a f$. We have

$$\begin{aligned}(J^{-\gamma}_a f)^\wedge &= (1+[x]^m)^{\gamma/m}\hat{f}\\ &= \big((1+[x]^m)^{\gamma/m}(1+[x]^\gamma)^{-1}\big)(1+[x]^\gamma)\hat{f}\\ &= (\mathrm{Tf})^\wedge + (TH^{-\gamma}_a f)^\wedge = (Tf)^\wedge + (T\varphi)^\wedge,\end{aligned}$$

where T is defined by $(Tg)^\wedge = (1+[x]^m)^{\gamma/m}(1+[x]^\gamma)^{-1}\hat{g}$. Appling (2.1), T maps L^p into L^p continuously. Thus $f = J^\gamma_a(Tf + T\varphi)$, and $(Tf + T\varphi) \in L^p$ with $\|Tf + T\varphi\|_p \leqslant c_p(\|f\|_p + \|\varphi\|_p)$.

On the other hand, suppose $f \in L^p_{a,\gamma}$. Then $f = J^\gamma_a \psi$ for some $\psi \in L^p$. By (2.1), $f \in L^p$ and $\|f\|_p \leqslant c_p\|\psi\|_p = c_p\|f\|_{p,a,\gamma}$. Also, $f = H^\gamma_a(S\psi)$, where S is defined by $(Sg)^\wedge = [x]^\gamma(1+[x]^m)^{-\gamma/m}\hat{g}$. Again applying (2.1), $S\psi \in L^p$ with $\|S\psi\|_p \leqslant c_p\|\psi\|_p = c_p\|f\|_{p,a\gamma}$.

Remark. If $a_1 = a_2 = \ldots = a_n = 1$, then the above theorem is known to be true for $p = 1$ or ∞. See [7]. The above theorem is adequate for our purposes, so no attempt has been made to extend it.

PROPOSITION 2.3. *Let $0 < \gamma < |a|$ and $\varphi \in L^1 \cap L^p$, where $1 < p < \infty$. Then $H^\gamma_a\ \varphi$ is the sum of an L^∞ function and an L^p function.*

Proof. Let $\zeta \in C^\infty$ with $\zeta(x) \equiv 1$ for $|x| \leqslant 1$, $\zeta(x) \equiv 0$ for $|x| \geqslant 2$, and $0 < \zeta(x) < 1$ for $1 < |x| < 2$. Then $(H^\gamma_a\varphi)^\wedge(x) = [x]^{-\gamma}\zeta(x)\hat{\varphi}(x) + [x]^{-\gamma}\big(1-\zeta(x)\big)\hat{\varphi}(x)$. Since $0 < \gamma < |a|$, $[x]^{-\gamma}$ is locally integrable; since $\varphi \in L^1$, $\hat{\varphi}$ is bounded. Thus $[x]^{-\gamma}\zeta(x)\hat{\varphi}(x)$ defines an L^1 function; consequently its inverse Fourier transform is in L^∞. Applying (2.1) to $[x]^{-\gamma}\big(1-\zeta(x)\big)$, we see that $[x]^{-\gamma}\big(1-\zeta(x)\big)\hat{\varphi}(x)$ is the Fourier transform of an L^p function.

Note in particular that the above hypotheses are satisfied for every p if $\varphi \in L^\infty_0$.

LEMMA 2.4. *Let $1 < p < \infty$, $b > 0$, $\gamma > 0$.* Then

$$\int_{-\infty}^{\infty}\Big(\int_0^\infty\Big|\int_{-1}^{1}F(r, t - r^b s)\,ds\Big|r^{-1-2\gamma}\,dr\Big)^{p/2}dt \leqslant C^p_p\int_{-\infty}^{\infty}\Big(\int_0^\infty|F(r,t)|^2 r^{-1-2\gamma}\,dr\Big)^{p/2}dt.$$

Proof. Let H be the Hilbert space of measurable functions f on $(0, \infty)$ for which $\int_0^\infty |f(r)|^2 r^{-1-2\gamma}\,dr < \infty$. Let us consider F to be an H-valued function of t. We define another H-valued function TF by

$$TF(r, t) = \int_{-1}^{1}F(r, t - r^b s)\,ds.$$

Then the desired conclusion is $\|TF\|_{L^p(H)} \leqslant C_p\|f\|_{L^p(H)}$. Suppose $F \in L^\infty_0(H)$; i. e., $\|F(\cdot, t)\|_H$ is essentially bounded and vanishes for a. e. t outside some compact interval. Computing Fourier transforms (in the sense of H-valued functions),

$$\begin{aligned}(Tf)^\wedge(r, \tau) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-it\tau}\,dt\int_{-1}^{1}F(r, t - r^b s)\,ds\\ &= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}ds\int_{-\infty}^{\infty}e^{-it\tau}F(r, t - r^b s)\,dt,\end{aligned}$$

the change in order of integration being justified because the latter integral converges absolutely in H.

$$(Tf)^{\wedge}(r,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-ir^b s\tau}\, ds \int_{-\infty}^{\infty} e^{-i(t-r^b s)\tau} F(r, t-r^b s)\, dt$$

$$= \int_{-1}^{1} e^{ir^b s\tau} \hat{F}(r,\tau)\, ds = \frac{2\sin r^b\tau}{r^b\tau}\hat{F}(r,\tau).$$

Since $\left|\frac{\sin r^b\tau}{r^b\tau}\right| \leqslant 1$ and $\left|\tau\frac{\partial}{\partial\tau}\frac{\sin r^b\tau}{r^b\tau}\right| \leqslant 2$, for each τ multiplication by $\frac{\sin r^b\tau}{r^b\tau}$ and by $\tau\frac{\partial}{\partial\tau}\frac{\sin r^b\tau}{r^b\tau}$ define bounded maps from H into H. Appealing to a Hilbert space version of the multiplier theorem. we have that T is a continuous map of $L^p(H)$ into $L^p(H)$ for $1 < p < \infty$,

As the authors point out in Benedek, Calderón, and Panzone [2] in Theorem 4 of their paper, such a multiplier theorem can be proved using their Theorem 2 and the method of proof of Hörmander's multiplier theorem [4].

3. A new norm for $L^p_{a,\gamma}$.

DEFINITION. $S_a^{(\gamma)}f(x) = \left(\int_0^\infty \left[\int_{I^n} |f(x-r^{(a)}y)-f(x)|\,dy\right]^2 r^{-1-2\gamma}\,dr\right)^{1/2}$ where $I = [-1, 1]$.

Note that for each x, $S_a^{(\gamma)}f(x)$ is a semi-norm. Another useful property is that if $f\in L^\infty$, then $S_a^{(\gamma)}(fg)(x) \leqslant \|f\|_\infty S_a^{(\gamma)}g(x) + |g(x)|\, S_a^{(\gamma)}f(x)$.

Our goal is the following:

THEOREM 3.1. *Let $1 < p < \infty$ and $0 < \gamma < 1$. Then $f\in L^p_{a,\gamma}$ if and only if $f\in L^p$ and $S_a^{(\gamma)}f\in L^p$. Moreover, $\|f\|_{p,a,\gamma} \approx \|f\|_p + \|S_a^{(\gamma)}f\|_p$.*

LEMMA 3.2. *For $\varphi\in L_0^\infty$ and $f = H_a^\gamma\varphi$,*

$$\|S_a^{(\gamma)}f\|_p \leqslant C_p\|\varphi\|_p.$$

Proof. (Induction on n). When $n = 1$, $[x] = |x|$ and the result is contained in the proof of Theorem 2.3 of Strichartz [8]. Assume the lemma is valid in R^n. Let $\varphi\in L_0^\infty(R^{n+1})$, $f = H_A^{(\gamma)}\varphi$. We have

$$S_A^{(\gamma)}f(\tilde{x}) = \left(\int_0^\infty \left[\iint_{I^{n+1}} |f(x-r^{(a)}y,\, t-r^b s)-f(x,t)|\,dy\,ds\right]^2 r^{-1-2\gamma}\,dr\right)^{1/2}$$

$$\leqslant \left(\int_0^\infty \left[\iint_{I^{n+1}} |f(x-r^{(a)}y,\, t-r^b s)-f(x,\, t-r^b s)|\,dy\,ds\right]^2 r^{-1-2\gamma}\,dr\right)^{1/2}$$

$$+\left(\int_0^\infty \left[\iint_{I^{n+1}} |f(x,\, t-r^b s)-f(x,t)|\,dy\,ds\right]^2 r^{-1-2\gamma}\,dr\right)^{1/2}.$$

Putting

$$F(x,r,t) = \int_{I^n} |f(x-r^{(a)}y,\, t)-f(x,t)|\,dy$$

and

$$S^{(\gamma)}f(x,t) = \left(\int_0^\infty \left[\int_{-1}^{1} |f(x,\, t-r^b s)-f(x,t)|\,ds\right]^2 r^{-1-2\gamma}\,dr\right)^{1/2},$$

we have

$$S_A^{(\gamma)}f(x,t) \leqslant \left(\int_0^\infty \left|\int_{-1}^{1} F(x,r,\, t-r^b s)\,ds\right|^2 r^{-1-2\gamma}\,dr\right)^{1/2} + 2^n S^{(\gamma)}f(x,t).$$

By (2.4),

$$\int_{-\infty}^{\infty}\left(\int_0^\infty \left|\int_{-1}^{1} F(x,r,\, t-r^b s)\,ds\right|^2 r^{-1-2\gamma}\,dr\right)^{\frac{p}{2}} dt \leqslant C_p^p \int_{-\infty}^{\infty}\left(\int_0^\infty |F(x,r,t)|^2 r^{-1-2\gamma}\,dr\right)^{\frac{p}{2}} dt$$

$$= C_p^p \int_{-\infty}^{\infty} S_a^{(\gamma)}f(x,t)^p\,dt.$$

Hence $\|S_A^{(\gamma)}f\|_p \leqslant C_p\|S_a^{(\gamma)}f\|_p + 2^n\|S^{(\gamma)}f\|_p$. We have

$$\hat{f} = [\tilde{x}]^{-\gamma}\hat{\varphi} = [x]^{-\gamma}([x]^\gamma[\tilde{x}]^{-\gamma}\hat{\varphi}) = (H_a^\gamma\psi)^{\wedge},$$

where by (2.1) $\psi\in L^p$ with $\|\psi\|_p \leqslant C_p\|\varphi\|_p$. By the inductive hypothesis and a density argument,

$$\int S_a^{(\gamma)}f(x,t)^p\,dx \leqslant C_p^p \int |\psi(x,t)|^p\,dx$$

and hence $\|S_a^{(\gamma)}f\|_p \leqslant C_p\|\psi\|_p \leqslant C_p\|\varphi\|_p$.

For the other term, we have

$$S^{(\gamma)}f(x,t) = \left(\int_0^\infty \left[\int_{-1}^{1} |f(x,\, t-r^b s)-f(x,t)|\,ds\right]^2 r^{-1-2\gamma}\,dr\right)^{1/2}$$

$$= b^{-1/2}\left(\int_0^\infty \left[\int_{-1}^{1} |f(x,\, t-rs)-f(x,t)|\,ds\right]^2 r^{-1-2\gamma/b}\,dr\right)^{1/2}.$$

This time we apply the multiplier theorem to $|t|^{\gamma/b}[\tilde{x}]^{-\gamma}$ and use a similar argument to show $\|S^{(\gamma)}f\|_p \leqslant C_p\|\varphi\|_p$.

LEMMA 3.3. *Let $1 < p < \infty$, $0 < \gamma < 1$, $\varphi\in L_0^\infty$, and $f = H_a^\gamma\varphi$. Then $\|\varphi\|_p \leqslant C_p\|S_a^{(\gamma)}f\|_p$.*

Proof. The technique here is essentially that used in Strichartz [8], except that instead of applying Theorem 4 of Benedek, Calderón, and Panzone [2] we use some of the ideas present in its proof.

Let

$$T_a^{(\gamma)}f(x) = \Big(\int_0^\infty \Big|\int_{I^n} [f(x-r^{(a)}y)-f(x)]\,dy\Big|^2 r^{-1-2\gamma}\,dr\Big)^{1/2}.$$

Clearly $T_a^{(\gamma)}f \leqslant S_a^{(\gamma)}f$; we shall show $\|\varphi\|_p \leqslant C_p\|T_a^{(\gamma)}\|_p$. Let H be the Hilbert space defined in (2.4). Define $B\varphi(x) = \int_{I^n} [H_a^\gamma\varphi(x-r^{(a)}y)-H_a^\gamma\varphi(x)]\,dy$. By 3.2, $B\varphi \in L^p(H)$ with $\|B\varphi\|_{L^p(H)} = \|T_a^{(\gamma)}f\|_p \leqslant \|S_a^{(\gamma)}f\|_p \leqslant C_p\|\varphi\|_p$. We have

$$\begin{aligned}(B\varphi)^\wedge(x) &= \int_{I^n} [H_a^\gamma\varphi(\cdot - r^{(a)}y) - H_a^\gamma\varphi]^\wedge(x)\,dy\\ &= \int_{I^n} (H_a^\gamma\varphi)^\wedge(x)[\exp\{-i<r^{(a)}y, x>\}-1]\,dy\\ &= \hat\varphi(x)[x]^{-\gamma}\Big(\prod_{i=1}^n\int_{-1}^1 \exp\{-ir^{a_i}y_ix_i\}\,dy_i - 2^n\Big)\\ &= 2^n\varphi(x)[x]^{-\gamma}\Big(\prod_{i=1}^n \frac{\sin r^{a_i}x_i}{r^{a_i}x_i}-1\Big) = \hat\varphi(x)k(x, r),\end{aligned}$$

$k(\cdot, r)$ is an H-valued function; $\|k(x,\cdot)\|_H$ will be bounded subsequently. B^*, the adjoint of B, maps $L^p(H)$ into L^p continuously with norm bounded by C_p', where $\|B\varphi\|_p \leqslant C_p\|\varphi\|_p$, all $\varphi \in L_0^\infty$, $1<p<\infty$. Moreover, for $\varphi\in L_0^\infty(H)$,

$$(B^*\varphi)^\wedge(x) = (\hat\varphi(x), k(x, \cdot)),$$

where the inner product is taken in H. Hence for $\varphi \in L_0^\infty$,

$$(B^*B\varphi)^\wedge(x) = (\hat\varphi(x)k(x,\cdot), k(x,\cdot)) = \hat\varphi(x)\|k(x,\cdot)\|_H^2.$$

Since $\|B^*B\varphi\|_p \leqslant C_p'\|B\varphi\|_{L^p(H)}$, if we can apply (2.1) to $\|k(x,\cdot)\|_H^{-2}$ the proof is complete.

$$\|k(x,\cdot)\|_H^2 = 2^{2n}[x]^{-2\gamma}\int_0^\infty \Big|\prod_{i=1}^n \frac{\sin r^{a_i}x_i}{r^{a_i}x_i}-1\Big|^2 r^{-1-2\gamma}\,dr.$$

Since $\Big|\prod_{i=1}^n \frac{\sin r^{a_i}x_i}{r^{a_i}x_i}-1\Big|$ is bounded by 2 and is $O(r^2)$ as $r\to 0$ for each x, the integral converges. For $\lambda>0$,

$$\|k(\lambda^{(a)}x,\cdot)\|_H^2 = 2^{2n}[\lambda^{(a)}x]^{-2\gamma}\int_0^\infty\Big|\prod_{i=1}^n\frac{\sin r^{a_i}\lambda^{a_i}x_i}{r^{a_i}\lambda^{a_i}x_i}-1\Big|^2 r^{-1-2\gamma}\,dr = \|k(x,\cdot)\|_H^2,$$

as can be seen by substituting $r = \lambda^{-1}r'$.

But $\|k(x,\cdot)\|_H^2$ is bounded away from 0 on $\{x: [x]=1\}$; hence it is also bounded away from 0 in $R^n \sim \{0\}$. Consequently, it suffices to bound $x^\alpha D^\alpha\|k(x,\cdot)\|_H^2$ for x not lying in a coordinate plane, and α of the form $(\alpha_1, \ldots, \alpha_n)$ with each $\alpha_i = 0$ or 1.

An elementary calculation shows that $|x^\alpha D^\alpha[x]^{-2\gamma}| \leqslant C[x]^{-2\gamma}$ for all such α; thus by Leibnitz's rule it suffices to show

$$\Big|x^\alpha D^\alpha\int_0^\infty\Big|\prod_{i=1}^n\frac{\sin r^{a_i}x_i}{r^{a_i}x_i}-1\Big|^2 r^{-1-2\gamma}\,dr\Big| \leqslant C[x]^{2\gamma}.$$

Differentiating under the integral sign, we see that

$$x^\alpha D^\alpha\Big|\prod_{i=1}^n\frac{\sin r^{a_i}x_i}{r^{a_i}x_i}-1\Big|^2$$

s a sum of products of terms of the form

$$\prod_{i=1}^n\frac{\sin r^{a_i}x_i}{r^{a_i}x_i}-1, \tag{1}$$

$$\frac{\sin r^{a_i}x_i}{r^{a_i}x_i}, \tag{2}$$

or

$$x_iD_{x_i}\frac{\sin r^{a_i}x_i}{r^{a_i}x_i} = \cos r^{a_i}x_i - \frac{\sin r^{a_i}x_i}{r^{a_i}x_i} \tag{3}$$

due to the nature of α; the number of terms of the third type in each product is the order of α.

As pointed out previously, the term of the first type is bounded by 2 and is $O(r^2)$ as $r\to 0$. Terms of the second type are bounded by 1. Noting that $\cos t = 1+O(t^2)$ and $\frac{\sin t}{t} = 1+O(t^2)$, we see that terms of the third type are $O(r^2)$ as $r\to 0$, uniformly for bounded x; moreover, such terms are bounded by 2 for large r.

Thus $\int_0^\infty D^\alpha\Big|\prod_{i=1}^n\frac{\sin r^{a_i}x_i}{r^{a_i}x_i}-1\Big|^2 r^{-1-2\gamma}\,dr$ converges uniformly in a neighborhood of each x not in a coordinate plane; consequently differentiation under the integral sign is valid. It follows also that $x^\alpha D^\alpha\|k(x,\cdot)\|_H^2$ is continuous on $R^n\sim\{0\}$.

Evaluating terms of the three types at $\lambda^{(a)}x$ instead of x and then substituting $r=\lambda^{-1}r'$, it follows readily that

$$|x^\alpha D^\alpha\|k(x,\cdot)\|_H^2| \leqslant C[x]^{2\gamma}.$$

Proof of Theorem 3.1. Here the details are identical to the proof of the corresponding theorem in [1]; the method is due to Strichartz [8].

Let $f\in L^p_{a,\gamma}$. By (2.2), $f = H^\gamma_a\varphi$ for some $\varphi\in L^p$ and $\|f\|_{p,a,\gamma}\approx\|f\|_p+\|\varphi\|_p$. By (3.2), (3.3), and a density argument, we have $\|\varphi\|_p\approx\|S^{(\gamma)}_a f\|_p$.

On the other hand, suppose both f and $S^{(\gamma)}_a f$ are in L^p. Let g_n be a sequence of functions in C^∞_0 such that

(1) $g_n\geqslant 0$,

(2) $\|g_n\|_1 = 1$,

(3) $\|F*g_n-F\|_p\to 0$ for each $F\in L^p$.

A routine argument shows that both g_n and $f*g_n$ are in $L^p_{a,\gamma}$. By what we have shown,

$$\|f*g_n\|_{p,a,\gamma}\leqslant C\big(\|f*g_n\|_p+\|S^{(\gamma)}_a(f*g_n)\|_p\big).$$

But $\|f*g_n\|_p\leqslant\|f\|_p$, and since $g_n\geqslant 0$ Minkowski's inequality gives $S^{(\gamma)}_a(f*g_n)\leqslant g_n*S^{(\gamma)}_a f$ and thus $\|S^{(\gamma)}_a(f*g_n)\|_p\leqslant\|g_n S^{(\gamma)}_a f\|_p\leqslant\|S^{(\gamma)}_a f\|_p$. Hence $\{f*g_n\}$ forms a bounded sequence in $L^p_{a,\gamma}$. It is shown in [3] that $L^p_{a,\gamma}$ is reflexive; consequently, some subsequence $f*g_{n_k}$ is weakly convergent in $L^p_{a,\gamma}$. Since $f*g_{n_k}\to f$ in L^p, it follows that $f\in L^p_{a,\gamma}$.

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Convolution of functions in Lorentz spaces

by

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Abstract. The well-known results of M. Rajagopalan, W. Żelazko, and N. Rickert concerning the *L^p-conjecture* are extended to the Lorentz spaces $L(p,q)(G)$ defined on (non-compact) locally compact groups G. Related results for compact groups are also given. The theorems presented here are complementary to, and motivated by, earlier results on convolution of functions in Lorentz spaces by R. O'Neil and the author.

1. Introduction. In O'Neil [2] and Yap [8] convolution of functions in various Lorentz spaces $L(p,q)(X,\mu)$ (for definition, see (2.1) *infra*) were considered and interesting and useful results were obtained. For example, if G is a *unimodular* locally compact group with Haar measure λ and p_1, p_2 are real numbers such that $1/p_1+1/p_2\geqslant 1$, then for f in $L(p_1,q_1)(G,\lambda)$ and g in $L(p_2,q_2)(G,\lambda)$, the usual convolution product $f*g$ is always defined (under some mild restriction on q_1 and q_2) and it has further pleasant properties [2], [8]. In this note we consider the case $1/p_1+1/p_2<1$ and thus answer some natural questions left open in the earlier papers. In addition to this we also consider the Banach algebras $L(p,q)(G,\lambda)$ when G is a compact group and $p>1$. Our results are generalizations of theorems in [1], [3], [4], [7], [9].

2. Definitions and preliminary results. In this section we review the basic definitions, and give some preliminary results which are needed in the sequel.

DEFINITIONS 2.1. Let f be a measurable function defined on a measure space (X,μ). For $y\geqslant 0$, we define $m(f,y) = \mu\{x\in X: |f(x)|>y\}$, and let f^* be the function on $[0,\infty)$ which is inverse to $m(f,\cdot)$ and right-continuous. For $x>0$, we define

$$f^{**}(x) = x^{-1}\int\limits_0^x f^*(t)\,dt.$$