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Extension of algebra norms and applications*

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Abstract. The main result of the paper is Lemma 1: Let $(B, \|\cdot\|')$ be a commutative normed algebra with identity and A be a subalgebra of B containing the identity. If $\|\cdot\|$ is an algebra norm on A equivalent to $\|\cdot\|'$ restricted to A, then $\|\cdot\|$ can be extended to an algebra norm on B equivalent to $\|\cdot\|'$. Two applications are given. We show first that the cortex of a commutative normed algebra depends only on the metric topology induced by the given algebra norm and not the particular norm. Secondly, we use Lemma 1 in proving conditions sufficient for the solvability of the equation $\exp x = a$ in complete normed extensions of the given algebra.

The main result of this paper, Lemma 1, grew out of the attempt to solve certain equations in normed extensions of commutative Banach algebras. Usually, we are able to construct a Banach algebra which contains an isomorphic copy of the original algebra and in which the given equation is solvable. The isomorphism is usually norm decreasing and in some situations, we can establish that it is bi-continuous but not norm preserving. As a consequence of Lemma 1, if we can establish a bi-continuous isomorphism, then the extension can be renormed so that the isomorphism is now norm preserving. Applications of this technique are given.

All algebras in the following will be commutative and complex, and have an identity element. If B, A are normed algebras, then we say B is a normed extension of A if there exists a norm preserving isomorphism of A into B that takes the identity of A onto the identity of B. When convenient, we view A as a subalgebra of B. If B is complete, we call it a complete normed extension of A.

§ 1. Extending norms from subalgebras. Let $(A, \|\cdot\|)$ be a commutative normed algebra with identity e. Let $X = \{x_\alpha\colon \alpha \in \mathfrak{A}\}$ be a family of commuting indeterminates over A and $T = \{t_\alpha\colon \alpha \in \mathfrak{A}\}$ a family of positive real numbers. By $A[X]$ we mean the algebra of polynomials in the x_α's, with coefficients in A. A typical element $p(x_{\alpha_1}, \ldots, x_{\alpha_k})$ of $A[X]$ is

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a finite sum of the form $\sum p_{\nu_1\ldots\nu_k} x_{a_1}^{\nu_1}\ldots x_{a_k}^{\nu_k}$, where $p_{\nu_1\ldots\nu_k}\in A$ and $x_{a_1},\ldots,x_{a_k}\in X$. The norm $\|\cdot\|_T$ on $A[X]$ is taken to be $\|p(x_{a_1},\ldots,x_{a_k})\|_T = \sum \|p_{\nu_1\ldots\nu_k}\| t_{a_1}^{\nu_1}\ldots t_{a_k}^{\nu_k}$. Then $(A[X], \|\cdot\|_T)$ is a normed algebra and a normed extension of A. For convenience, we sometimes write $p(x) = \sum p_\nu x^\nu$ instead of $p(x_{a_1},\ldots,x_{a_k}) = \sum p_{\nu_1\ldots\nu_k} x_{a_1}^{\nu_1}\ldots x_{a_k}^{\nu_k}$. For a polynomial $p(x)\in A[X]$, $p(0)$ will denote the "constant" term.

LEMMA 1. *Let $(B, \|\cdot\|')$ be a normed algebra and A a subalgebra of B containing the identity of B. Suppose $\|\cdot\|$ is a norm on A satisfying $a\|a\| \leqq \|a\|' \leqq \beta\|a\|$ for all $a\in A$, and $\beta \geqq \alpha > 0$. Then $\|\cdot\|$ can be extended to a norm on B (we simply denote the extension by $\|\cdot\|$) satisfying $(\alpha/\beta)\|b\| \leqq \|b\|' \leqq \beta\|b\|$ for all $b\in B$.*

Proof. First note that $\alpha \leqq 1 \leqq \beta$. Let $\mathfrak{A} = B\setminus A$ and set $t_b = \|b\|'$, $b\in\mathfrak{A}$. Let $T = \{t_b\colon b\in\mathfrak{A}\}$. There is a natural homomorphism ψ of $A[X]$ onto B satisfying

$$\psi\big(p(x_{b_1},\ldots,x_{b_k})\big) = p(b_1,\ldots,b_k),\quad b_1,\ldots,b_k\in\mathfrak{A},\quad p(x_{b_1},\ldots,x_{b_k})\in A[X].$$

We first show that when $A[X]$ is given the norm $\|\cdot\|_T'$, ψ becomes a norm decreasing mapping. Let $p(x) = \sum p_{\nu_1\ldots\nu_k} x_{b_1}^{\nu_1}\ldots x_{b_k}^{\nu_k}\in A[X]$. Then

$$(*)\qquad \Big\|\sum p_{\nu_1\ldots\nu_k} b_1^{\nu_1}\ldots b_k^{\nu_k}\Big\|' \leqq \sum \|p_{\nu_1\ldots\nu_k}\|'(\|b_1\|')^{\nu_1}\ldots(\|b_k\|')^{\nu_k} = \sum \|p_{\nu_1\ldots\nu_k}\|' t_{b_1}^{\nu_1}\ldots t_{b_k}^{\nu_k} = \|p(x)\|_T'.$$

Thus, ψ is norm decreasing. Now, set $J = \psi^{-1}(0)$, and $\|b\|_Q' = \inf_{j(x)\in J}\|p(x)+j(x)\|_T'$, where $b = \psi\big(p(x)\big)$. Then $\|b\|_Q' = \|b\|'$ for all $b\in B$. For if $a\in A$, then $\|a\|' \leqq \|a\|_Q' \leqq \|a\|'$, and if $b\in\mathfrak{A}$, then $\|b\|' \leqq \|b\|_Q' \leqq \|x_b\|_T' = t_b = \|b\|'$.

Now, set $\|\sum_\nu a_\nu x^\nu\|_T'' = \|a_0\| + (\beta/\alpha)\|\sum_{\nu\neq 0} a_\nu x^\nu\|_T$. Then $\|\cdot\|_T''$ is an algebra norm on $A[X]$ since $(\beta/\alpha)\geqslant 1$. For $b\in B$, let $\|b\|_Q = \inf_{j(x)\in J}\|p(x)+j(x)\|_T''$, where $\psi\big(p(x)\big) = b$. Then $\|b\|' = \|b\|_Q' \leqq \beta\|b\|_Q$ for all $b\in B$ since

$$\Big\|\sum_\nu a_\nu x^\nu\Big\|_T' = \|a_0\|' + \Big\|\sum_{\nu\neq 0} a_\nu x^\nu\Big\|_T' \leqq \beta\|a_0\|' + \beta\Big\|\sum_{\nu\neq 0} a_\nu x^\nu\Big\|_T \leqq \beta\left(\|a_0\| + \frac{\beta}{\alpha}\Big\|\sum_{\nu\neq 0} a_\nu x^\nu\Big\|_T\right) = \beta\Big\|\sum_\nu a_\nu x^\nu\Big\|_T''.$$

This inequality shows that J is closed with respect to $\|\cdot\|_T''$. On the other hand, for $b\in\mathfrak{A}$, $\|b\|_Q = \inf_{j(x)\in J}\|x_b + j(x)\|_T'' \leqq \|x_b\|_T'' = \frac{\beta}{\alpha}t_b = \frac{\beta}{\alpha}\|b\|'$. Therefore, $\frac{\alpha}{\beta}\|b\|_Q \leqq \|b\|_Q'$ for $b\in\mathfrak{A}$. The proof that $\|\cdot\|_Q$ and $\|\cdot\|_Q'$ are equivalent on B and satisfy $\frac{\alpha}{\beta}\|b\|_Q \leqq \|b\|_Q' \leqq \beta\|b\|_Q$ for all $b\in B$ will be complete when we show that $\|a\|_Q = \|a\|$ for all $a\in A$.

To that end, let $a\in A$ and $j(x)\in J$. Then

$$\|a+j(x)\|_T'' = \|a+j(0)\| + \frac{\beta}{\alpha}\|j(x)-j(0)\|_T \geqq \|a\| - \|j(0)\| + \frac{1}{\alpha}\|j(x)-j(0)\|_T' \geqq \|a\| - \|j(0)\| + \frac{1}{\alpha}\|j(0)\|' \geqq \|a\| - \|j(0)\| + \|j(0)\| = \|a\|.$$

The second inequality holds since $\|j(x)-j(0)\|_T' \geqq \|j(0)\|'$ holds for all $j(x)\in J$ (see $(*)$ above). Therefore $\|a\| \leqq \|a\|_Q$. But $\|a\|_Q \leqq \|a\|$ so that $\|a\| = \|a\|_Q$ for all $a\in A$. This completes the proof of the lemma.

COROLLARY 1. *If $\beta = 1$, then $\|\cdot\|$ can be extended without change in its bounds relative to $\|\cdot\|'$.*

If we only assume $a\|\cdot\| \leqq \|\cdot\|'$ on A, then it need not be true that $\|\cdot\|$ can be extended to all of B. Let Δ denote the unit disk in the field of complex numbers. For B, take $C(\Delta)$ and for A, take $\mathrm{Hol}(\Delta) = \{f\in C(\Delta)\colon f \text{ analytic on } \mathrm{int}(\Delta)\}$. Set $\|f\|_\infty = \sup_{\omega\in\Delta}|f(\omega)|$ for $f\in C(\Delta)$ and $\|f\| = \sup_{0\leqslant\arg\omega\leqslant\pi}|f(\omega)|$ for $f\in\mathrm{Hol}(\Delta)$. Then $\|f\|_\infty \geqq \|f\|$ for $f\in\mathrm{Hol}(\Delta)$ but $\|\cdot\|$ cannot be extended to $C(\Delta)$ since any extension would automatically majorize $\|\cdot\|_\infty$ on $C(\Delta)$ by a result of Kaplansky's [2], evidently a contradiction.

§ 2. The invariance of the cortex of a normed algebra. If $(A, \|\cdot\|)$ is a normed algebra, then $\Gamma(A, \|\cdot\|)$ will denote the set of $\|\cdot\|$-continuous non-trivial complex homomorphisms that extend continuously as complex homomorphisms to every normed extension of A. R. Arens, in [1], calls $\Gamma(A, \|\cdot\|)$ the cortex of A.

THEOREM 1. *If $\|\cdot\|$ and $\|\cdot\|'$ are equivalent algebra norms on A, then $\Gamma(A, \|\cdot\|) = \Gamma(A, \|\cdot\|')$.*

Proof. If B is a normed extension of A with respect to $\|\cdot\|$, and if $\varphi\in\Gamma(A, \|\cdot\|')$, then $\varphi\in\Gamma(A, \|\cdot\|)$ since by Lemma 1, B can be renormed with an equivalent norm extending $\|\cdot\|'$. Hence $\Gamma(A, \|\cdot\|') \subseteq \Gamma(A, \|\cdot\|)$. Since the argument is symmetric, $\Gamma(A, \|\cdot\|') = \Gamma(A, \|\cdot\|)$.

COROLLARY 2: *If $(A, \|\cdot\|)$ is a semi-simple Banach algebra, then the cortex $\Gamma(A, \|\cdot\|)$ depends only on the complete norm topology induced by $\|\cdot\|$.*

The corollary follows immediately from the theorem since all complete algebra norms on semi-simple algebras are equivalent [4].

§ 3. Simple logarithmic extensions. In a 1944 paper [3], E. Lorch raised the question of solving for logarithms in complete normed extensions of Banach algebras. Specifically, if A is a Banach algebra and $a \in A$, when is $\exp x = a$ solvable in some complete normed extension? For a uniform algebra A, it is easily shown that $\exp x = a$ is solvable in such extension if and only if a is not a topological divisor of zero in A. For arbitrary Banach algebras, necessary and sufficient conditions are given in [1]. However, these conditions are difficult to use in specific cases because the norms are not well enough understood. In the remainder of this section, we give conditions on A and $a \in A$ which enable us to solve the equation $\exp x = a$.

Since A will be assumed to be semi-simple in the next theorem, we will view the elements of A as continuous functions on the carrier space Φ_A.

THEOREM 2. *Let A be a semi-simple Banach algebra and $a \in A$ be a unit. If A contains all its local belonging functions and if $\exp x = a$ has local solutions at each point of Φ_A, then $\exp x = a$ has a solution in some complete normed semi-simple extension of A.*

Proof. By $A(x; t)$, $t > 0$, we denote the algebra of formal power series $\sum_{\nu=0}^{+\infty} a_\nu x^\nu$ in the commuting indeterminate x over A for which $\sum_{\nu=0}^{+\infty} \|a_\nu\| t^\nu < +\infty$, where $\|\cdot\|$ is the norm on A. We set $\|\sum_{\nu=0}^{+\infty} a_\nu x^\nu\|_t = \sum_{\nu=0}^{+\infty} \|a_\nu\| t^\nu$ for $\sum_{\nu=0}^{+\infty} a_\nu x^\nu \in A(x; t)$. Then $(A(x; t), \|\cdot\|_t)$ is a Banach algebra. The carrier space of $A(x; t)$ is identifiable with $\Phi_A \times \Delta_t$, $\Delta_t = \{\lambda \in C: |\lambda| \leqq t\}$, and $(\varphi, \lambda) \sum_{\nu=0}^{+\infty} a_\nu x^\nu = \sum_{\nu=0}^{+\infty} a_\nu(\varphi)\lambda^i = \varphi(\sum_{\nu=0}^{+\infty} a_\nu \lambda^\nu)$.

Now, let $U_1, \ldots, U_n$ be open subsets of Φ_A, and let $f_1, \ldots, f_n \in A$ such that $\bigcup_{\nu=1}^{n} U_\nu = \Phi_A$ and $\exp(f_\nu | U_\nu) = a | U_\nu$, $\nu = 1, \ldots, n$. Let $t = \max(\|f_1\|, \ldots \ldots, \|f_n\|)$. If $\varphi \in \Phi_A$, then $\varphi \in U_\nu$ for some ν; hence, $|f_\nu(\varphi)| \leqq \|f_\nu\| \leqq t$ and therefore $(\varphi, f_\nu(\varphi)) \in H(J)$, where $J = K(H(\exp x - a)) \subset A(x; t)$. Since $\Phi_{A(x;t)/J} = H(J)$ and each $\varphi \in \Phi_A$ has an extension in $H(J)$, the semi-simplicity implies that $a \to a + J$, $a \in A$, is an isomorphism. We shall show that $\{a + J: a \in A\}$ is closed in the quotient topology on $A(x; t)/J$. To this end, suppose that $a_n + J \to f(x) + J$, $f(x) = \sum_{k=0}^{+\infty} a_k x^k$; that is, suppose there exists a sequence $\{g_n(x)\} \subset J$ such that

$$a_n + g_n(x) \to f(x)$$

in the norm on $A(x; t)$. Since $\Phi_{A(x;t)} = \Phi_A \times \Delta_t$, $H(J) = \{(\varphi, \lambda) \in \Phi_{A(x;t)}: \exp\lambda = \varphi(a)\}$. Hence, if (φ, λ), $(\varphi, \lambda') \in H(J)$, then $\varphi(f(\lambda)) = \varphi(f(\lambda'))$ since $\{\varphi(a_n)\}$ converges to $\varphi(f(\lambda))$ as well as to $\varphi(f(\lambda'))$. Then g defined by $g(\varphi) = \sum_{k=0}^{+\infty} \varphi(a_k)\lambda^k$ for any $(\varphi, \lambda) \in H(J)$ is a continuous function on Φ_A. Now, set $g_\nu = \sum_{k=0}^{+\infty} a_k f_\nu^k$, $\nu = 1, \ldots, n$. Clearly $\sum_{k=0}^{+\infty} a_k f_\nu^k$ converges in A since $\|f_\nu\| \leqq t$ for all ν and $\sum_{k=0}^{+\infty} a_k x^k \in A(x; t)$. If $\varphi \in U_\nu$, then $g_\nu(\varphi) = g(\varphi)$. Thus, g belongs locally to A and consequently $g \in A$. Now, $g - f(x) \in J$ since for each $(\varphi, \lambda) \in H(J)$, $g(\varphi) = \varphi(f(\lambda))$. Thus, the image of A in $A(x; t)/J$ is $\|\cdot\|_Q$-closed and there exists $s > 0$ such that $s\|a\| \leqq \|a + J\|_Q \leqq \|a\|$, $a \in A$. By Lemma 1, $A(x; t)/J$ can be renormed with an equivalent norm so that $a \to a + J$, $a \in A$, is an isometry. Since $A(x; t)/J$ is complete under $\|\cdot\|_Q$, it will be complete under the new norm. Finally, note that $A(x; t)/J$ is semi-simple since $J = K(H(J))$.

COROLLARY 3. *Let A be a regular semi-simple Banach algebra. Then $\exp x = a$ is solvable in some complete normed semi-simple extension of A if and only if a is a unit in A.*

Proof. The "if" part follows immediately from the theorem, since for a unit a in a semi-simple regular Banach algebra, $\exp x = a$ can be solved locally in A and such an algebra contains all its local belonging functions. To prove the "only if" part, note first that if $\exp x = a$ is solvable in a complete normed extension B of A, then a is a unit in B and hence a unit in A since each $\varphi \in \Phi_A$ extends to a complex homomorphism on B (see page 175, [4]).

Our final application of Lemma 1 is

THEOREM 3: *Let A and B be Banach algebras, with B an extension of A. If the given isomorphism is bi-continuous and if $a \in A$ has the property that $\exp x = a$ is solvable in a complete normed extension of B, then $\exp x = a$ is solvable in a complete normed extension of A.*

References

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