From (2), one has that for almost all $r \epsilon \Omega$

$$\int_{\Omega} f(s)\tilde{g}(r, s)\mu(ds) = \int_{\Omega} f(s)g(s)\mu(ds)(r)$$

whenever $f \epsilon L^{\psi}$ is simple. Since simple functions are dense in L^{ψ}, it follows that for almost all $r \epsilon \Omega$. $t^*(h)(r) = \int_{\Omega} h(s)\tilde{g}(s, r)\mu(ds)$ for $h \epsilon L^{\psi}$. Arguments the same as those used in the necessity show that

$$t(f)(s) = \int_{\Omega} f(r)\tilde{g}(s, r)\mu(dr) \qquad \text{a. e.}$$

for all $f \epsilon L^{\Phi}$. The fact that t is of finite double norm follows immediately from (c). ∎

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On shrinking basic sequences in Banach spaces*

by

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Abstract. In § 1 we prove that a Banach space E with a basis $\{x_n\}$ contains a subspace with a separable conjugate space if and only if $\{x_n\}$ admits a shrinking block basic sequence. Hence, a Banach space E contains a subspace with a separable conjugate space if and only if E contains a shrinking basic sequence. In § 2 we prove that if E has a subspace with a separable conjugate space, then E^* (the conjugate of E) has a quotient space with a basis. In § 3 we prove that if E has a basis, then every shrinking basic sequence in E has a subsequence which can be extended to a basis of E. We also raise some related unsolved problems.

Introduction. A sequence $\{x_n\}$ in a Banach space E (we shall assume, without special mention, that dim $E = \infty$ and that the scalars are real or complex) is called a *basis* if E if for every $x \epsilon E$ there exists a unique sequence of scalars $\{a_n\}$ such that $x = \sum_{i=1}^{\infty} a_i x_i$. A sequence $\{z_n\} \subset E$ is said to be a *basic sequence* if $\{z_n\}$ is a basis of its closed linear span $[z_n]$. A sequence $\{z_n\} \subset E$ is called a *block basic sequence* with respect to a sequence $\{y_n\} \subset E$ if it is a basic sequence of the form $z_n = \sum_{i=m_{n-1}+1}^{m_n} \beta_i y_i \neq 0$ $(n = 1, 2, \ldots)$, where $\{m_n\}$ is an increasing sequence of positive integers and $m_0 = 0$; it is well known and easy to see that if $\{y_n\}$ is a basic sequence, then $\{z_n\}$ is necessarily a basic sequence. A basic sequence $\{z_n\} \subset E$ is called *shrinking*, if $\lim_{n} \|\chi|_{[z_n, z_{n+1}, \ldots]}\| = 0$ for all $\chi \epsilon [z_n]^*$. Say that a basic sequence $\{z_n\}$ *can be extended to a basis of E* if there exists a basis $\{x_n\}$ of E and a sequence of positive integers $\{k_n\}$ such that $z_n = x_{k_n} (n = 1, 2, \ldots)$.

In § 1 of the present paper we shall prove some results on the existence of shrinking basic sequences. Among other results, we shall prove that if E has a basis $\{x_n\}$, then E contains a subspace G having a separable

conjugate space if and only if $\{x_n\}$ admits a shrinking block basic sequence $\{z_n\}$. Furthermore, we shall show that a Banach space E contains a subspace having a separable conjugate space if and only if E contains a shrinking basic sequence. We shall also raise some related unsolved problems. In § 2, using the results of § 1, we shall make a contribution to the (unsolved) problem of the existence, for every Banach space E, of a quotient space with a basis, by proving that if E contains a subspace having a separable conjugate space (in particular, if E^* is separable), then E^* has a quotient space with a basis. Finally, in § 3 we shall make a contribution to the (unsolved) problem whether every basic sequence in a Banach space E with a basis can be extended to a basis of E, by proving, among other results, that if E has a basis, every shrinking basic sequence in E has a subsequence which can be extended to a basis of E.

§ 1. On the existence of shrinking basic sequences.

PROPOSITION 1. *Let $\{x_n\}$ be a basis of a Banach space E with a separable conjugate space E^*. Then $\{x_n\}$ has a shrinking block basic sequence.*

Proof. Let $\{g_n\}$ be a dense sequence in E^*. Then, since

$$(1) \quad \dim[x_{\frac{n(n+1)}{2}}, \ldots, x_{\frac{(n+1)(n+2)}{2}-1}] = n+1, \quad \operatorname{codim}_E[g_1, \ldots, g_n]_\perp \leqslant n,$$

there exist (see e. g. [20], Lemma 1), elements

$$(2) \quad z_n \in [x_{\frac{n(n+1)}{2}}, \ldots, x_{\frac{(n+1)(n+2)}{2}-1}] \cap [g_1, \ldots, g_n]_\perp \quad (n = 1, 2, \ldots)$$

such that $\|z_n\| = 1$ $(n = 1, 2, \ldots)$, where $G_\perp = \{z \in E \mid g(z) = 0 \text{ for all } g \in G\}$. Obviously, $\{z_n\}$ is a block basic sequence with respect to $\{x_n\}$. We claim that $\{z_n\}$ is shrinking. Indeed, let $\varepsilon > 0$ and $\psi \in [z_n]^*$ be arbitrary and let $g \in E^*$ be an extension of ψ. Since $\{g_n\}$ is dense in E^*, there exists an index $N = N(\varepsilon)$ such that $\|g - g_N\| < \varepsilon$. By (2), we have $g_N(z_n) = 0$ for $n \geqslant N$, hence $g_N|_{[z_n, z_{n+1}, \ldots]} = 0$ for $n \geqslant N$. Consequently,

$$\|\psi|_{[z_n, z_{n+1}, \ldots]}\| = \|g|_{[z_n, z_{n+1}, \ldots]}\|$$
$$= \|(g - g_N)|_{[z_n, z_{n+1}, \ldots]}\| \leqslant \|g - g_N\| < \varepsilon \quad (n \geqslant N),$$

and thus $\{z_n\}$ is shrinking, which completes the proof or Proposition 1.

The converse of Proposition 1 is not valid, as shown for example by $E = c_0 \times l^1$ and $x_{2n-1} = \{e_n, 0\}$, $x_{2n} = \{0, e_n\}$, where $e_n = \{\underbrace{0, \ldots, 0}_{n-1}, 1, 0, \ldots\}$

$(n = 1, 2, \ldots)$. However, we raise

PROBLEM 1. *Let $\{x_n\}$ be a basis of a Banach space E. If every block basic sequence with respect to $\{x_n\}$ (respectively, every subsequence of $\{x_n\}$) has a shrinking block basic sequence (respectively, a shrinking subsequence), then is E^* separable? Moreover, is $\{x_n\}$ a shrinking basis of E?*

From Proposition 1 we obtain the following result, in which the existence of a basis is no longer assumed:

PROPOSITION 2. *If E is a Banach space with a separable conjugate space E^*, then every basic sequence in E has a shrinking block basic sequence.*

Proof. Let $\{y_n\}$ be a basic sequence in E. Then, since E^* is separable, so is $[y_n]^*$, whence, by Proposition 1, $\{y_n\}$ has a shrinking block basic sequence.

PROBLEM 2. *Is the converse of Proposition 2 valid? That is, if E is a Banach space such that every basic sequence in E has a shrinking block basic sequence, is E^* separable?*

Proposition 1 above gives a sufficient condition in order that a basis $\{x_n\}$ of a Banach space E have a shrinking block basic sequence. A necessary and sufficient condition is given in

THEOREM 1. *A Banach space E with a basis $\{x_n\}$ contains a subspace G with a separable conjugate space G^* if and only if $\{x_n\}$ has a shrinking block basic sequence.*

Proof. Assume that E contains a subspace G which G^* separable. Then, by [2], Theorem 3 and Corollary 2, G contains a basic sequence $\{y_n\}$ equivalent to a block basic sequence $\{u_n\}$ with respect to $\{x_n\}$, i. e. there exists an isomorphism T of $[y_n]$ onto $[u_n]$, such that $T(y_n) = u_n$ $(n = 1, 2, \ldots)$. Since G^* is separable, so is $[y_n]^*$, whence, by Proposition 1, $\{y_n\}$ has a shrinking block basis sequence $\{z_n\}$, say $z_n = \sum\limits_{i=m_{n-1}+1}^{m_n} a_i y_i$ $(n = 1, 2, \ldots)$. Then, obviously, $T(z_n) = \sum\limits_{i=m_{n-1}+1}^{m_n} a_i T(y_i) = \sum\limits_{i=m_{n-1}+1}^{m_n} a_i u_i$ $(n = 1, 2, \ldots)$ is a shrinking block basic sequence with respect to $\{u_n\}$, whence also with respect to $\{x_n\}$.

Conversely, if $\{x_n\}$ has a shrinking block basic sequence, say $\{z_n\}$, then obviously $G = [z_n]$ is a subspace of E with G^* separable, which completes the proof of Theorem 1.

We recall that a Banach space X is said to be *somewhat reflexive* [5] if every infinite dimensional subspace of X contains an infinite dimensional reflexive subspace. A basic sequence $\{z_n\}$ in a Banach space E is called *boundedly complete* if the relation $\sup\limits_n \|\sum\limits_{i=1}^n a_i z_i\| < \infty$ implies that the series $\sum\limits_{i=1}^\infty a_i z_i$ converges.

COROLLARY 1. *If a Banach space E has a boundedly complete basis $\{x_n\}$ and if E^* is separable, then E is somewhat reflexive.*

Proof. Let G be an arbitrary infinite dimensional subspace of E. Then by [2], Theorem 3 and Corollary 2, G contains a basic sequence $\{z_n\}$ which is equivalent to a block basic sequence $\{y_n\}$ with respect to $\{x_n\}$. Since by our hypothesis $[y_n]^*$ is separable, from Proposition 1 it follows that $\{y_n\}$ has a shrinking block basic sequence $\{u_n\}$. Since $\{u_n\}$ is also a block basic sequence with respect to the boundedly complete basis $\{x_n\}$, the basic sequence $\{u_n\}$ is boundedly complete. Hence, by [7], Theorem 1, $[u_n]$ is a reflexive subspace of $[y_n]$. Since $\{z_n\} \sim \{y_n\}$ it follows that $[z_n]$, whence also G, contains a reflexive subspace, too, which completes the proof.

We recall that a Banach space E is called [3] quasi-reflexive, if dim $E^{**}/\pi(E) < \infty$, where π denotes the canonical embedding of E into E^{**}. Since every quasi-reflexive Banach space with a basis satisfies the conditions of Corollary 1 (by [4]), it follows that *every quasi-reflexive Banach space with a basis is somewhat reflexive*, a result obtained with different proofs in [5] and [21].

THEOREM 2. *A Banach space E contains an infinite dimensional subspace having a separable conjugate space if and only if E contains a shrinking basic sequence.*

Proof. Assume that E contains a subspace G with dim $G = \infty$, such that G^* is separable. By virtue of [1], p. 238 (for a proof see, for example, [2]), G contains a basic sequence $\{x_n\}$. Since G^* is separable, $\{x_n\}$ has a shrinking block basic sequence, by Proposition 2.

Conversely, assume that E contains a shrinking basic sequence $\{x_n\}$. Then $[x_n]^*$ is separable, which completes the proof of Theorem 2.

Let us now raise some related unsolved problems.

PROBLEM 3. *If E has a basis and E^* is separable, does E have a shrinking basis?*

By Proposition 1 and [22] it is known that if E has a basis and E^* is separable, then E has a shrinking basic sequence $\{y_n\}$ which can be extended to a basis of E. However, we do not know whether $\{y_n\}$ can be extended to a shrinking basis of E.

We recall that a basis $\{f_n\}$ of a conjugate Banach space E^* is said to be a *retro-basis* of E^*, if there exists a basis $\{x_n\}$ of E such that $f_i(x_j) = \delta_{ij}$ $(i, j = 1, 2, \ldots)$. If the answer to Problem 3 were affirmative, say $\{x_n\}$ would be a shrinking basis of E, then the sequence $\{f_n\} \subset E^*$ with $f_i(x_j) = \delta_{ij}$ $(i, j = 1, 2, \ldots)$ would be a retro-basis of E^*. This would solve in the affirmative the following two problems ([18], Ch. II, § 4, Problems 2.3' and 2.4'), the second of which is obviously equivalent to Problem 3:

(1) If E has a basis and E^* is separable, does E^* have a basis?

(2) If E has a basis and E^* is separable, does E^* have a retro-basis?

Since in the above case $\{f_n\}$ is also a boundedly complete basis of E^* (by [17]), the following question arises naturally:

PROBLEM 4. *Let E^* have a basis.* (a) *Does E^* have a boundedly complete basis?* (b) *Does E^* have a boundedly complete basic sequence?* ([19], Ch. III, § 2, Problem 3.8). (c) *Does E^* have a boundedly complete block basic sequence?* (d) *Is E^* isomorphic to the conjugate space of a Banach space with a basis?*

If the answer to Problem (4a) were affirmative, then by [8], Theorem 10, the answer to Problem (4d) would be affirmative, too. Let us also mention the following related problem of S. Karlin [8]: If E^* has a basis, does E have a basis? Again, if the answer to Karlin's problem were affirmative, then, obviously, the answer to Problem (4d) would be affirmative.

PROBLEM 5. *Let E^* be separable.* (a) *Does E^* have a boundedly complete basic sequence?* ([19], Ch. III, § 2, Problem 3. 8). (b) *Does every subspace Γ of E^* have a boundedly complete basic sequence?*

As has observed W. J. Davis, the answer to problem 5(a) would be obviously affirmative if E would have a quotient space E/G with a shrinking basis.

In order to give another approach to these problems let us observe that the technique of the proof of Proposition 1 gives the following result, which is also useful for other applications:

PROPOSITION 3. *Let E be a Banach space with a basis $\{x_n\}$. Then every sequence $\{y_n\} \subset E$ with dim $[y_n] = \infty$ has a block basic sequence which is equivalent to a block basic sequence with respect to $\{x_n\}$.*

Proof. We may assume (omitting, if necessary, a suitable subsequence of $\{y_n\}$), that dim $[y_{\frac{n(n+1)}{2}}, \ldots, y_{\frac{(n+1)(n+2)}{2}-1}] = n+1$ $(n = 1, 2, \ldots)$.

Then, as in the proof of Proposition 1, there exist elements

$$(3) \quad z_n \in [y_{\frac{n(n+1)}{2}}, \ldots, y_{\frac{(n+1)(n+2)}{2}-1}] \cap [f_1, \ldots, f_n]_\perp \quad (n = 1, 2, \ldots)$$

such that $\|z_n\| = 1$ $(n = 1, 2, \ldots)$, where $\{f_n\} \subset E^*$, $f_i(x_j) = \delta_{ij}$. Hence, by [2], Theorem 3, $\{z_n\}$ has a basic subsequence (which is obviously a block basic sequence with respect to $\{y_n\}$) equivalent to a block basic sequence with respect to $\{x_n\}$, which completes the proof.

COROLLARY 2. *Let E be a Banach space with a basis $\{x_n\}$.*

(a) *If $\{x_n\}$ is an unconditional basis of E, then every sequence $\{y_n\} \subset E$ with dim $[y_n] = \infty$ has an unconditional block basic sequence.*

(b) *If $\{x_n\}$ is a boundedly complete basis of E, then every sequence $\{y_n\} \subset E$ with dim $[y_n] = \infty$ has a boundedly complete block basic sequen e.*

By Corollary 2(b) the answer to Problems 5 and 4(b), (c) would be affirmative, if the answer to the following problem, suggested by W. J. Davis, is affirmative.

PROBLEM 6. *If E^* is separable or if E^* has a basis, can E^* be embedded into a Banach space with a boundedly complete basis?*

Let us also mention the following related question of A. Pełczyński ([13], Problem 1): If E^* is separable, can E^* be isomorphically embedded into a conjugate space with a basis?

COROLLARY 2′. *Let E be a Banach space. Then every sequence $\{y_n\} \subset E$ with $\dim[y_n] = \infty$ has a block basic sequence.*

Proof. It is sufficient to embed $[y_n]$ into $C([0, 1])$ and to apply Proposition 3.

COROLLARY 3. *Let E be a Banach space with a basis.*

(a) *If E contains an unconditional basic sequence, then every basis of E has an unconditional block basic sequence.*

(b) *If E contains a boundedly complete basic sequence, then every basis of E has a boundedly complete block basic sequence.*

Indeed, this follows from Proposition 3 and from the fact that every block basic sequence with respect to an unconditional (boundedly complete) basic sequence is unconditional (respectively, boundedly complete).

From Corollary (3 b) it follows that Problems (4 b) and (4 c) above are equivalent. On the other hand, it is an open problem raised by C. Bessaga and A. Pełczyński [2], whether every Banach space E satisfies the assumption of Corollary (3 a), i. e. *whether every Banach space contains an unconditional basic sequence*. By virtue of Corollary (3 a) and of the fact that every Banach space contains a subspace with a basis, this problem becomes now *equivalent* to the following:

PROBLEM 7. *Let E be a Banach space with a basis $\{x_n\}$. Does $\{x_n\}$ have an unconditional block basic sequence?*

By a well known theorem of B. J. Pettis [16], a separable Banach space E is reflexive if and only if E^* is separable and E is weakly complete. This, together with Problem 3 above and with [7], Theorem 1, suggests the question, whether every weakly complete Banach space with a basis has a boundedly complete basis. The answer is negative, since there exists a subspace of l^1 which has a basis but has no boundedly complete basis [9]. However, since this space obviously has a boundedly complete basic sequence, it is natural to ask

PROBLEM 8. (a) *Does every weakly complete Banach space contain a boundedly complete basic sequence?* (b) *Does every basis of a weakly complete Banach space have a boundedly complete block basic sequence?*

While Theorem 2 above characterizes Banach spaces which contain at least one shrinking basic sequence, it is known that *all* basic sequences in a Banach space E are shrinking if and only if E is reflexive ([17], [11]). It is natural to raise the problem, what intermediate situations occur. We recall that a basis $\{x_n\}$ of E is called [20] k-shrinking, where $k \geqslant 0$ is an integer, if $\operatorname{codim}_{E^*}[f_j] = \dim E^*/[f_j] = k$, where $\{f_i\} \subset E^*$, $f_i(x_j) = \delta_{ij}$ $(i, j = 1, 2, \ldots)$. For $k = 0$ the 0-shrinking bases are nothing else than the shrinking bases [7]. Combining the results of [12], [20] and [4] (see also [21]), we obtain the following result: Let $n \geqslant 0$ be an integer. All basic sequences in E are k-shrinking, where $k \leqslant n$, if and only if E is quasi-reflexive of order $\leqslant n$ (this result is intermediate between the above two results, since if all basic sequences in E are k-shrinking, with $k \leqslant n$, then E has also a shrinking basic sequence [21], [4]).

Let us introduce the following generalization of k-shrinking bases:

DEFINITION 1. We shall say that a basis $\{x_n\}$ of a Banach space E is *quasi-shrinking* if the subspace $[f_n]$ of E^*, where $f_i(x_j) = \delta_{ij}$ $(i, j = 1, 2, \ldots)$, is complemented in E^*.

Obviously, every k-shrinking basis $(k \geqslant 0)$ is quasi-shrinking. Thus, for instance, the unit vector basis of c_0 is quasi-shrinking. On the other hand, c_0 also has non-quasi-shrinking bases, e. g. such is "the Lindenstrauss basis" [6] of c_0. Let us also observe that every basis of a quasi-reflexive Banach space E with a basis is quasi-shrinking (and hence so is every basic sequence in any quasi-reflexive space) [20]. On the other hand, every basis of l^1 is non-quasi-shrinking, since no separable subspace of m is complemented in m. It is also easy to see that every (infinite) basic sequence in l^1 is non-quasi-shrinking.

PROBLEM 9. *Characterize those Banach spaces with bases in which* (a) *there exists a quasi-shrinking basis*; (b) *all bases are quasi-shrinking. Characterize those Banach spaces E in which* (a) *or* (b) *holds, with "basis" replaced by "basic sequence" (we do not assume that E has a basis).* (c)*Characterize those spaces E with a basis $\{x_n\}$ in which* (a) *or* (b) *holds, with "basis" replaced by "block basic sequence with respect to $\{x_n\}$".*

DEFINITION 2. We shall say that a basis $\{x_n\}$ of a Banach space E is ∞-*shrinking*, if it is quasi-shrinking and $\operatorname{codim}_{E^*}[f_n] = \infty$, where $\{f_n\} \subset E^*$ $f_i(x_j) = \delta_{ij}$ $(i, j = 1, 2, \ldots)$.

Thus, a quasi-shrinking basis is either k-shrinking for some k, $0 \leqslant k < \infty$, or ∞-shrinking.

An example of an ∞-shrinking basis of c_0 can be obtained as follows: Let $E = (E_1 \times E_2 \times \ldots)_{c_0} \equiv c_0$, where $E_n = c_0$ $(n = 1, 2, \ldots)$. Then the basis $\{x_j\} = \{x_j^{(1)}\} \times \{x_j^{(2)}\} \times \ldots$ of E, where $x_j^{(n)} = \sum_{k=1}^{j} e_k = \{\underbrace{1, \ldots, 1}_{j}, 0, 0, \ldots\}$ $(j, n = 1, 2, \ldots; e_k =$ the k-th unit vector in $c_0)$, is an ∞-shrinking

basis of $E \equiv c_0$. Indeed, $\{x_j^{(n)}\}$ is a 1-shrinking basis of $E_n = c_0$ [17], whence $\{x_j\}$ is a basis of $E \equiv c_0$ [15], with $\mathrm{codim}_{E^*}[f_j] = \infty$, where $f_i(x_j) = \delta_{ij}$ $(i, j = 1, 2, \ldots)$. Finally, since there exist uniformly bounded projections of E_n^* onto $[f_j^{(n)}]$, where $f_i^{(n)}(x_j^{(n)}) = \delta_{ij}$ $(i, j, n = 1, 2, \ldots)$, the subspace $[f_j] = [f_j^{(1)}] \times [f_j^{(2)}] \times \ldots$ is complemented in $E^* \equiv (E_1^* \times \times E_2^* \times \ldots)_{l^1}$ [10], and hence $\{x_j\}$ is an ∞-shrinking basis of $E \equiv c_0$. On the other hand, every basis (and every basic sequence) in a quasi-reflexive Banach space is non-∞-shrinking [20]. One can ask the questions of Problem 9 with "quasi-shrinking" replaced by "∞-shrinking".

§ 2. Existence of quotient spaces with bases.

The following problem has been raised by A. Pełczyński [13]: Does every Banach space have a quotient space with a basis? Using Theorem 2 of § 1, we shall prove now that for a large class of conjugate Banach spaces, containing all separable conjugate spaces, the answer is affirmative.

THEOREM 3. *If a Banach space E contains a subspace having a separable conjugate space (in particular, if E^* is separable), then E^* has a $\sigma(E^*, E)$-closed subspace Γ such that the quotient space E^*/Γ has a boundedly complete basis.*

Proof. By Theorem 2, E contains a shrinking basic sequence $\{x_n\}$. Then, by [17], the sequence $\{\varphi_i\} \subset [x_n]^*$ with $\varphi_i(x_j) = \delta_{ij}$ is a boundedly complete basis of $[\varphi_n] = [x_n]^*$. Hence, since $[x_n]^* \equiv E^*/_{[x_n]^\perp}$, the subspace $\Gamma = [x_n]^\perp$ of E^* has the required properties, which completes the proof.

COROLLARY 4. *If a Banach space E contains a subspace having a separable conjugate space (in particular, if E^* is separable), then there exists a continuous linear mapping of E^* onto the conjugate space X^* of a Banach space X having a basis.*

Proof. By Theorem 3, E^* has a $\sigma(E^*, E)$-closed subspace Γ such that E^*/Γ has a boundedly complete basis $\{\varphi_n\}$. Then, by [8], E^*/Γ is isomorphic to the conjugate space X^* of a Banach space X having a basis and there exists a continuous linear mapping of E^* onto E^*/Γ (namely, the canonical mapping).

PROBLEM 10. *Let E be either* (a) *a Banach space such that E^* is separable or* (b) *a subspace of a separable conjugate Banach space X^*. Does E have a quotient space with a basis?*

§ 3. Extension of basic sequences to bases.

A. Pełczyński [14] has raised the problem whether every basic sequence in a Banach space E with a basis $\{x_n\}$ can be extended to a basis of E (see the Introduction) and has conjectured that the answer is negative. Recently M. Zippin [22] has given an important class of basic sequences which can be extended to a basis of E, namely, he has proved that every block basic sequence with respect to $\{x_n\}$ has this property. In the present section we shall

prove that for a large class of sequences, containing, among others, all shrinking basic sequences, there exist subsequences which can be extended to a basis of E.

THEOREM 3. *Let $\{x_n\}$ be a basis of a Banach space E, with biorthogonal functionals $\{f_n\}$ and let $\{y_n\}$ be a sequence in E such that $\inf \|y_n\| > 0$, $\lim\limits_{n \to \infty} f_j(y_n) = 0$ $(j = 1, 2, \ldots)$. Then $\{y_n\}$ has a subsequence $\{y_{p_n}\}$ which can be extended to a basis of E.*

Proof. Let $0 < \varepsilon_n < \delta \leqslant \|y_n\|$ $(n = 1, 2, \ldots)$ be such that

$$(4) \qquad \frac{72 K (K+1)}{\delta} \sum_{n=1}^{\infty} \varepsilon_n < 1,$$

where K is the constant of the basis $\{x_n\}$ (i. e. the infimum of all M such that $\|\sum_{i=1}^{n} a_i x_i\| \leqslant M \|\sum_{i=1}^{n+m} a_i x_i\|$ for all scalars $a_1, \ldots, a_{n+m}$ and all n, m). By [2], Theorem 3 and its proof, there exist a subsequence $\{y_{p_n}\}$ of $\{y_n\}$ and a block basic sequence $\{z_n\}$ with respect to $\{x_n\}$ such that

$$\|y_{p_n} - z_n\| \leqslant \frac{\varepsilon_n}{2} \qquad (n = 1, 2, \ldots).$$

We shall show that $\{y_{p_n}\}$ can be extended to a basis of E, which will complete the proof. Since $\varepsilon_n < \delta \leqslant \|y_n\|$ $(n = 1, 2, \ldots)$, we have

$$\|z_n\| \geqslant |\,\|y_{p_n}\| - \|y_{p_n} - z_n\|\,| \geqslant \frac{\delta}{2} \qquad (n = 1, 2, \ldots).$$

By [22], the block basic sequence $\{z_n\}$ can be extended to a basis $\{u_n\}$ of E, having basis constant no larger than $18 K(K+1)$, say $u_{m_n} = z_n$ $(n = 1, 2, \ldots)$. Let $\{g_n\} \subset E^*$ be the sequence biorthogonal to $\{u_n\}$. Then for every $x \in E$ we have

$$|g_{m_n}(x)| = \frac{\|g_{m_n}(x) z_n\|}{\|z_n\|} \leqslant \frac{2 \cdot 18 K (K+1)}{\delta} \|x\|,$$

whence $\|g_{m_n}\| \leqslant \dfrac{72 K (K+1)}{\delta}$ $(n = 1, 2, \ldots)$. Put

$$v_{m_n} = y_{p_n} \qquad (n = 1, 2, \ldots),$$
$$v_j = u_j \qquad (j \notin \{m_n\}).$$

Then, taking into account (4),

$$\sum_{j=1}^{\infty} \|g_j\| \, \|v_j - u_j\| = \sum_{n=1}^{\infty} \|g_{m_n}\| \, \|y_{p_n} - z_n\| \leqslant \frac{72 K (K+1)}{\delta} \sum_{n=1}^{\infty} \varepsilon_n < 1,$$

whence (see e. g. [2], Theorem 1) $\{v_n\}$ is a basis of E, which completes the proof.

COROLLARY 5. *Let E be a Banach space with a basis. Then every infinite dimensional subspace G of E contains a basic sequence which can be extended to a basis of E.*

Proof. By [2], proof of Corollary 2, G contains a sequence $\{y_n\}$ satisfying the conditions of Theorem 3.

For its special interest, let us state separately

COROLLARY 6. *Let E be a Banach space with a basis and let $\{y_n\}$ be a sequence in E such that $\inf_n \|y_n\| > 0$, $y_n \to 0$ weakly (e. g., in particular, a shrinking basic sequence). Then $\{y_n\}$ has a subsequence which can be extended to a basis of E.*

Proof. Obviously, $\{y_n\}$ satisfies the conditions of Theorem 3.

COROLLARY 7. *Let E be a Banach space with a basis $\{x_n\}$. Then every sequence $\{y_n\} \subset E$ with $\dim[y_n] = \infty$ has a block basic sequence which can be extended to a basis of E.*

Proof. As in the proof of § 1, Proposition 3, there exists a sequence $\{z_n\} \subset E$ satisfying (3), where $\{f_n\} \subset E^*$, $f_i(x_j) = \delta_{ij}$. Hence, by Theorem 3, $\{z_n\}$ has a basic subsequence (which is obviously a block basic sequence with respect to $\{y_n\}$) which can be extended to a basis of E. This completes the proof.

Finally, let us give the following sharpening of Proposition 1 and Corollary 7:

THEOREM 4. *Let E be a Banach space with a basis $\{x_n\}$ and with a separable conjugate space E^*. Then every sequence $\{y_n\} \subset E$ with $\dim[y_n] = \infty$ has a block basic sequence which is shrinking and which can be extended to a basis of E.*

Proof. By Proposition 3, $\{y_n\}$ has a block basic sequence $\{z_n\}$. Since E^* is separable, by virtue of Proposition 2 $\{z_n\}$ has a shrinking block basic sequence $\{u_n\}$. Then $\left\{\dfrac{u_n}{\|u_n\|}\right\}$ is a shrinking block basic sequence with respect to $\{z_n\}$, whence also with respect to $\{y_n\}$. By Corollary 6, $\left\{\dfrac{u_n}{\|u_n\|}\right\}$ has a subsequence $\{v_n\}$ (which is obviously a block basic sequence with respect to $\{y_n\}$), which can be extended to a basis of E. This completes the proof of Theorem 4.

Remark. In the results of this section, if the initial basis $\{x_n\}$ of E is k-shrinking (k-boundedly complete), then one may choose the new (extended) basis of E to be k-shrinking (respectively, k-boundedly complete). Indeed, as observed in [4], Remark 1, in the Zippin extension of block basic sequences to bases the orders of shrinkingness and of boundedly completeness of the new (extended) basis are the same as the corre-

sponding orders of the initial basis $\{x_n\}$ and, obviously, the property of being a k-shrinking (k-boundedly complete) basis is conserved by isomorphisms.

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