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Functional theory of geodesic fields and its applications to the calculus of variations of multiple integrals

by

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Abstract. The author, using the notion of a bundle of the Banach or Fréchet–Schwartz type, formulates an abstract theory of geodesic fields. These fields lead to some sufficient conditions for a minimum of a given function. The Fréchet–Schwartz bundle variant of the theory is applied to the calculus of variations for multiple integrals. In this way a unified approach to the problems with common and with movable boundary is obtained. For the problems with common boundary the classic Lepage geodesic fields appear to be very special cases of abstract fields. The abstract condition for a minimum turns out to be the old Weierstrass condition. For the problems with movable boundary a class of abstract fields is constructed on the basis of classic Lepage fields for common boundary problems and the sufficient condition obtained seems to be reasonable. Thus not only Carathéodory fields but all Lepage fields prove useful for problems with movable boundary.

1. Introduction. The notion of geodesic fields was introduced to the calculus of variations by Weierstrass already in the 19th century. At first a geodesic field was defined to be a congruence of extremal curves satisfying some additional conditions ensuring the existence of minimal curves in the field. Attempts to give a similar construction for the common boundary variational problems with multiple integrals resulted in two different expositions: one given by Carathéodory (1926), and the other by de Donder and H. Weyl (1936). It was in 1937 that Lepage showed the Carathéodory and the de Donder–Weyl geodesic fields only to be examples taken from a continual variety of all possible cases. Lepage's idea was developed by Boerner and obtained its final shape in 1953 in Paul Dedecker's paper [2]. Boerner, using the Carathéodory fields, formulated also a sufficient condition for an extremum for the problems with movable boundary. In 1967 A. Liesen gave a local construction of a broad class of Lepage geodesic fields, formulating at the same time Dedecker's results in the language of modern differential geometry.

In recent years systematic studies of non-linear analysis, started by Eells, Palais, and Smale, have resulted in a series of papers by Kijowski, Komorowski and Szczyrba ([5], [4], [3]), where a new approach to the calculus of variations has been presented. The notion of a manifold modelled on a locally convex space of the Fréchet–Schwartz type has become the basic notion of this approach.

In the present paper we attempt to formulate the theory of geodesic fields in this language. In the first part of the paper, starting from a functional definition of variational problems, we give an abstract definition of a geodesic field which allows us to prove abstract analogues of the two well-known theorems: one (formulated by Dedecker) stating that a surface embedded in a geodesic field is an extremal, and the second giving the famous Weierstrass sufficient condition for an extremum.

In the further two parts of the paper we study, as examples of abstract problems, in turn: the common boundary and the movable boundary problems with multiple integrals. The classic Lepage geodesic fields, which lead to a useful Weierstrass condition for an extremum, are proved to be very special cases of abstract fields.

There is an opinion, originating with Boerner, that among Lepage fields only the Carathéodory fields are useful in the investigation of problems with movable boundary. In the 3rd part of the present paper it is shown, however that all Lepage fields may be applied in this field. Namely, suppose we are given an extremal manifold of a problem with movable boundary embedded in a Lepage field for the respective problem with common boundary. The author constructs a geodesic field for the problem with movable boundary with the initial manifold embedded in it. A reasonable sufficient condition for an extremum to which the field just constructed leads comprises, besides the respective sufficient condition for the problem with common boundary, a condition obtained from the estimation of boundary expresions only. If the initial Lepage field is a Carathéodory field, then our sufficient condition obtained in the way described above admits a geometric interpretation. In the final part of the paper, following Boerner's idea, we give an exact formulation of this interpretation.

Thus our abstract theory of geodesic fields appears to lead to a unified approach to the integral problems with a common and with movable boundary.

Some results published in the present paper were announced earlier in the *General theory of geodesic fields* (Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys., 7 (1970), pp. 363–366).

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2. Abstract theory of geodesic fields. Let X be a manifold of the Fréchet–Schwartz (FS-manifold) type or of the Banach type of class C^1.

Let $f\colon X \to R^1$ be a C^1-morphism and Δ — a distribution of subspaces of the tangent bundle on X.

DEFINITION 1. We call $x \in X$ a *critical point of the variational problem* (f, Δ) if $T_x f \mid \Delta(x) = 0$.

Suppose we are given an additional structure: a manifold Y (of the same type as X) and C^1-morphisms $\Pi\colon Y \to X$, and $D\colon X \to Y$ (D being a section of Π, i. e., $\Pi \circ D = \mathrm{Id}_X$).

DEFINITION 2. We say that (Π, D, g, s) is a *geodesic field* (g. f.) of the problem (f, Δ) if

(i) $g\colon Y \to \boldsymbol{R}^1$ is a $\hat{C}$-morphism, $g \circ D = f$,

(ii) $T_{D(x)} g$ gives zero when acting on Π-vertical vectors ($x \in X$),

(iii) s is a C^1-section of Π,

(iv) $T(g \circ s) \mid \Delta = 0$.

We say that (Π, D, g, s) is a geodesic field in the weak sense (w. g. f.) of (f, Δ) if it satisfies (i) and (ii), $s\colon X \to Y$, $\Pi \circ s = Id_X$, and

(v) if $t \to x_t$ is an integral C^1-curve of Δ, then $t \to s(x_t)$ is of class C^1 and the derivative of the mapping $t \to g \circ s(x_t)$ vanishes.

Remark 1. Obviously a g. f. is a w. g. f.

THEOREM 1. *Let (Π, D, g, s) be a g. f. and let x be embedded in the field (i.e. $s(x) = D(x)$). Then x is a critical point.*

Proof. Let $e \in \Delta(x)$. We must show that $T_x f(e) = 0$. Indeed, $T_x f(e) = T_x(g \circ D)(e) = T_{D(x)} g\big(T_x D(e)\big) = T_{s(x)} g\big(T_x s(e)\big) = T_x(g \circ s)(e) = 0$ as $T_x D(e) - T_x s(e)$ is a Π-vertical vector tangent to Y at $D(x)$.

DEFINITION 3. The Weierstrass function for a field (Π, D, g, s) is defined to be $E = f - g \circ s$. E is a mapping of X into R^1.

Now let us consider an integral C^1-curve of Δ: $]-1, 1[\ni t \to x_t \in X$. Let (Π, D, g, s) be a w. g. f. and let $D(x_0) = s(x_0)$. Then

$$f(x_t) - f(x_0) = f(x_t) - g \circ D(x_0) = f(x_t) - g \circ s(x_0)$$
$$= f(x_t) - g \circ s(x_t) + \int_0^t \frac{d\big(g \circ s(x_\delta)\big)}{d\delta}\, d\delta = f(x_t) - g \circ s(x_t).$$

Hence

(1) $$f(x_t) - f(x_0) = E(x_t).$$

THEOREM 2. *Let x_0 be embedded in a w. g. f. (Π, D, g, s) and*

(2) $$E(x) > 0 \quad \big(E(x) < 0\big) \quad \text{for } x \neq x_0.$$

Then f has at x_0 an essential minimum (maximum) in the set of all points connectable with x_0 by means of integral C^1-curves of Δ.

If we replace the condition (2) by

$$(2') \qquad E(x) \geqslant 0 \qquad (E(x) \leqslant 0)$$

and cancel the word "essential", the assertion given above will remain true.

Proof. Our statements are obvious in virtue of (1).

In the further part of the paper we study two examples of variational problems, for which the conditions for an extremum given by Theorem 2 simplify the question considerably and are of practical significance.

3. Classical integral variational problems with common boundary. Now we want to deal with variational problems (in the traditional sense of the term) determined by integral functionals. Let $X_c := \mathscr{P}_m(\mathscr{X})$ be a C^∞-manifold of compact oriented m-dimensional submanifolds with boundary of class C^∞ of an n-dimensional Hausdorff C^∞-manifold $\mathscr{X}$. $\mathscr{P}_m(\mathscr{X})$ is of the FS-type.

Remark 2. The construction of the differentiable structure on $\mathscr{P}_m(\mathscr{X})$ has been given in [2]. The submanifolds considered there were not required to be oriented. However, this additional condition does not change the construction.

From now on we assume our finite-dimensional manifolds and their morphisms to be of class C^∞.

LEMMA 1. *A morphism ι mapping a compact manifold with boundary Ω into $\mathscr{X}$ is a diffeomorphism of Ω onto a submanifold with boundary of $\mathscr{X}$ if and only if it is an injective immersion. Then $\iota(\partial\Omega) = \partial(\iota(\Omega))$.*

Proof. Let $x_0 \in \Omega$. There exist an open neighbourhood U of x_0 and an open neighbourhood V of $\iota(x_0)$ such that $\iota|U$ is a diffeomorphism of U onto a submanifold with boundary of V and $\iota|U$ generates a bijection of $\partial\Omega \cap U$ onto $\partial(\iota(\Omega))$ (a conclusion from the inverse function theorem). Since $\iota(\Omega\setminus U)$ is compact, there exists an open subset V' of V with $\iota(x_0) \in V'$ and $V' \cap \iota(\Omega\setminus U) = \emptyset$ as ι is injective. Thus $\iota|\iota^{-1}(V')$ is a diffeomorphism of $\iota^{-1}(V')$ $(\subset U)$ onto a submanifold with boundary of V'. $\iota(\Omega)$ is then a compact submanifold with boundary of $\mathscr{X}$.

Let $\mathscr{Y}$ be a finite-dimensional Hausdorff manifold and $\pi: \mathscr{Y} \to \mathscr{X}$ — a morphism.

LEMMA 2. *The set Y_c of points $\Omega \in \mathscr{P}_m(\mathscr{Y})$ for which $\pi\circ\iota_\Omega$ satisfies the assumption of Lemma 1 is open in $\mathscr{P}_m(\mathscr{Y})$ (ι_Ω being the canonical injection of Ω into $\mathscr{Y}$).*

Proof. Let $\mathscr{E}_\Omega$ be the space of all diffeomorphisms of Ω into $\mathscr{Y}$ (i.e., onto submanifolds with boundary of $\mathscr{Y}$) equipped with the topology of uniform convergence of all derivatives. We divide the proof into two parts: first we prove that

(a) there exists a neighbourhood $\mathcal{O}$ of ι_Ω in $\mathscr{E}_\Omega$ such that if $\varkappa \in \mathcal{O}$ then $\pi\circ\varkappa$ is an immersion; then we prove that

(b) there exists a neighbourhood $\mathcal{O}'$ of ι_Ω in $\mathscr{E}_\Omega (\mathcal{O}' \subset \mathcal{O})$ such that if $\varkappa \in \mathcal{O}'$ then $\pi\circ\varkappa$ is injective.

The set

$$\{\Omega' \in \mathscr{P}_m(\mathscr{Y}): \text{there exists } \varkappa \in \mathcal{O}' \text{ with } \Omega' = \varkappa(\Omega)\}$$

will possess the required properties.

Ad (a). Let (U_i) be a finite covering of $(\pi\circ\iota_\Omega)(\Omega)$ by domains of charts of $\mathscr{X}$. Let us consider three finite coverings of $\iota_\Omega(\Omega)$ open in $\mathscr{Y}$: $(\pi^{-1}(U_i))$, (V_j'), (V_j) — the second finer than the first, V_j' — being a relatively compact domains of a chart of $\mathscr{Y}$, and compact $\overline{V}_j$ being a subset of V_j' for every j ($\mathscr{Y}$, as a locally compact manifold is normal).

Moreover, let $(\iota_\Omega^{-1}(V_j))$, (W_l'), (W_l) be three finite coverings of Ω open in Ω, the second finer than the first, W_l' being a domain of a chart of Ω and compact $\overline{W}_l$ being a subset of W_l' for every l (Ω, as a compact space, is also normal).

For every l there exists a continuous function ($\geqslant 0$) of image points and first derivatives of $\pi\circ\varkappa|W_l'$ which is positive if and only if $\pi\circ\varkappa|W_l$ is an immersion (e.g., the sum of moduli of decomposition coefficients of the image by $\overset{m}{\wedge} T(\pi\circ\varkappa)$ of an m-vector field tangent to Ω in the base generated by the chart of $\mathscr{X}$ defined on U_i). This function takes in $\overline{W}_l$ a minimum value >0 when calculated for $\pi\circ\iota_\Omega$. Now we can take a neighbourhood of ι_Ω in $\mathscr{E}_\Omega$ such that, for $\varkappa$ from this set and for every l, image points and first derivatives of $\pi\circ\varkappa|\overline{W}_l$ and $\pi\circ\iota_\Omega|\overline{W}_l$ (taken in the coordinates) differ uniformly sufficiently little (these differences, by the mean value theorem, can be estimated from above by the upper bound on $\overline{W}_l$ of the differences of image points and first derivatives of $\varkappa$ and ι_Ω and the upper bounds of first and second-order derivatives of π on the corresponding $\overline{V}_j$). Thus our test function will remain positive when applied to $\pi\circ\varkappa$ if $\varkappa$ is taken from the selected neighbourhood of ι_Ω just selected (since it is uniformly continuous on a compact neighbourhood of the compact set of its arguments taken when $\iota_\Omega|\overline{W}_l$ is considered).

Ad (b). The covering (U_i) in the proof of (a) can be chosen so that in the local coordinates for every l a common subspace supplementary to $T_{\pi\circ\iota_\Omega(x)}(T_x\Omega)$ for every $x \in W_l'$ can be selected. Now the inverse function theorem states that, for every $x \in \Omega$, $\pi\circ\varkappa (\varkappa \in \mathscr{E}_\Omega)$ is a diffeomorphism on a neighbourhood of x common for $\varkappa$ such that $\pi\circ\varkappa$ and its derivatives to the second order inclusively do not differ largely from $\pi\circ\iota_\Omega$ and its corresponding derivatives (uniformly on every $\overline{W}_l$), which is guaranteed, in virtue of the mean value theorem, by the analogous requirements for $\varkappa$ and ι_Ω(π is of class C^∞ and so its derivatives are bounded on every $\overline{V_j}$).

Now suppose that for every neighbourhood of ι_Ω in $\mathscr{E}_\Omega$ subordinated to the one just selected there exist its element $\varkappa$ and $x_1^\varkappa \neq x_2^\varkappa$ (points in Ω) such that $\pi \circ \varkappa(x_1^\varkappa) = \pi \circ \varkappa(x_2^\varkappa)$. Let $a \to (x_1^{\varkappa_a}, x_2^{\varkappa_a}) \in \Omega \times \Omega$ be a sequence convergent to (x_1, x_2) (Ω is compact) with $\varkappa_a \underset{a}{\to} \iota_\Omega$ in $\mathscr{E}_\Omega$. As π (a morphism) is of class C^1, $\pi \circ \varkappa_a \underset{a}{\rightrightarrows} \pi \circ \iota_\Omega$ uniformly (by the mean value theorem). Thus

$$\pi \circ \varkappa_a(x_1^{\varkappa_a}) \underset{a}{\to} \pi \circ \iota_\Omega(x_1), \qquad \pi \circ \varkappa_a(x_2^{\varkappa_a}) \underset{a}{\to} \pi \circ \iota_\Omega(x_2),$$

i.e., $\pi \circ \iota_\Omega(x_1) = \pi \circ \iota_\Omega(x_2)$. But $\pi \circ \iota_\Omega$ is an injection. Hence $x_1 = x_2 =: x$. We have shown, however, that there is a neighbourhood of x in Ω $\mathcal{O}(x)$ such that, for every $\varkappa$ sufficiently close to ι_Ω, $\varkappa | \mathcal{O}(x)$ is a diffeomorphism. Thus we have got a contradiction.

Let us suppose that Y_c is not void. We define on the open subset Y_c of $\mathscr{P}_m(\mathscr{Y})$ a mapping Π_c that assigns to a submanifold its diffeomorphic image in $\mathscr{X}$ under $\pi \circ \iota_\Omega$ (the orientation being that shifted by $\pi \circ \iota_\Omega$), i. e. a point of $X_c = \mathscr{P}_m(\mathscr{X})$.

Lemma 3. *Π_c is continuous.*

Proof. Lemma 3 is an immediate conclusion from the fact that (see the proof of Lemma 2) on every $\overline{W_l}$ derivatives of the k-th order of $(\pi \circ \varkappa_1 - \pi \circ \varkappa_2)$ (in the local coordinates) can be estimated from above by derivatives up to the k-th order of $(\varkappa_1 - \varkappa_2)$ $(\varkappa_1, \varkappa_2 \in \mathscr{E}_\Omega)$ and the upper bounds on the corresponding $\overline{V_j}$ of derivatives up to the $(k+1)$-th order of π (the mean value theorem) if $\varkappa_2$ lies in a sufficiently small neighbourhood of $\varkappa_1$.

Y_c, as an open subset of $\mathscr{P}_m(\mathscr{Y})$, is an FS-manifold of class C^∞.

Lemma 4. *Π_c: $Y_c \to X_c$ is a C^∞-morphism.*

Proof. The proof is fully analogous to that of Theorem 2 point (b) in [3] (one needs only to replace the morphism Φ_1 appearing there by $\pi \circ \Phi_1$).

From now on $\mathscr{Y}$ will denote the manifold $J_m(\mathscr{X})$ of oriented m-dimensional elements tangent to $\mathscr{X}$ (i. e., the Grassmann manifold — see [2]). π will be the natural projection in the manifold of tangent elements.

As D_c we take the mapping of X_c into Y_c which assigns to every $\Omega \in X_c$ the set of m-dimensional oriented elements tangent to Ω — which is an m-dimensional compact submanifold with boundary of $\mathscr{Y}$, i. e., an element of Y_c.

Lemma 5. *D_c (being obviously a section of Π_c) is continuous.*

Proof. We cover $D(\Omega)$ by domains of charts of $J_m(\mathscr{X})$ generated by charts of $\mathscr{X}$ (see [1]) and choose a finite covering (U_i). Let $X_\varkappa$ $(\varkappa \in \mathscr{E}_\Omega)$

be the morphism from Ω into $J_m(\mathscr{X})$ which assigns to every point x of Ω the m-dimensional oriented element tangent at $\varkappa(x)$ to $\varkappa(\Omega)$. Let $\big(X_{\iota_\Omega}^{-1}(U_i)\big)$, (W_l'), (W_l) be three finite coverings, the second finer than the first, W_l' being a domain of a chart of Ω and a compact $\overline{W_l} \subset W_l'$ for every l. The differences on $\overline{W_l}$ of $X_\varkappa$ and X_{ι_Ω} and their derivatives up to the k-th order can be estimated from above by the upper bounds on $\overline{W_l}$ of the differences of $\varkappa$ and ι_Ω and their derivatives up to the $(k+1)$-th order (in the local coordinates).

Lemma 6. *D_c is a C^∞-morphism.*

Proof. We shall consider on X_c and Y_c atlases of the type constructed in [3] with charts described by sixes $(N, \Phi, H, \Psi, C, \xi)$, where N is a vector bundle over some extension $\tilde{\Omega}$ of a submanifold $\Omega \in X_c$ (Y_c), i. e., over some m-dimensional submanifold without boundary containing Ω, Φ is a diffeomorphism of N onto an open neighbourhood of $\tilde{\Omega}$ in $\mathscr{X}(\mathscr{Y})$, H is a vector bundle over $\partial\Omega$ diffeomorphic (through Ψ) to a neighbourhood of $\partial\Omega$ in $\tilde{\Omega}$, C is a linear connection in N, and ξ is a real function possessing some additional properties (see [3]).

Let $(N_1, \Phi_1, H_1, \Psi_1, C_1, \xi_1)$ determine a chart $\varkappa_1$ at $\Omega \in X_c$. Let $N_2 := \mathrm{Hom}\big(T(\tilde{\Omega}), N_1\big) \oplus N_1$. N_2 can be treated as a bundle over $X_{\iota_{\tilde{\Omega}}}(\tilde{\Omega})$ (for the definition of $X_{\iota_{\tilde{\Omega}}}$ see the proof of Lemma 5) as $X_{\iota_{\tilde{\Omega}}}$ is a diffeomorphism of $\tilde{\Omega}$ into $J_m(\mathscr{X})$. Let us define a diffeomorphism Φ_2 of N_2 onto an open neighbourhood of $X_{\iota_{\tilde{\Omega}}}(\tilde{\Omega})$ in $J_m(\mathscr{X})$ $\big(X_{\iota_{\tilde{\Omega}}}(\tilde{\Omega})$ is an extension of $D_c(\Omega)\big)$. We assign to a point (l, u) $\big(l \in \mathrm{Hom}\big(T(\tilde{\Omega}), N_1\big), u \in N_1\big)$ from the fibre over $p \in \tilde{\Omega}$ a tangent element in the following way: let (e_i) be a base for $T_p(\tilde{\Omega})$; we take an element generated by a simple non-zero m-vector $\bigwedge_{i=1}^{m} \big(\tilde{e}_i^u + l(e_i)\big)$ tangent to N_1 at u $\big(\tilde{e}_i^u$ is a C_1-horizontal lifting of e_i to the point $u\big)$.

Let H_2 and Ψ_2 be the same as H_1 and Ψ_1 provided we identify points diffeomorphic through $X_{\iota_{\tilde{\Omega}}}$. Let $\xi_2 = \xi_1 =: \xi$. C_2 will be constructed in due course.

Now it remains to show (we trivialise H_1) that the mapping

$$\Gamma(\Omega, N_1) \times C^\infty(\partial\Omega) \supset V \overset{\varkappa_1^{-1}}{\to} X_c \overset{D_c}{\to} Y_c \overset{\varkappa_2}{\to} \Gamma(\Omega, N_2) \times C^\infty(\partial\Omega)$$

is of class C^∞, $\varkappa_1$ and $\varkappa_2$ being the charts under consideration for X_c and Y_c respectively (see the proof of Theorem 2 in [2]).

Let $(\varkappa_2 \circ D_c \circ \varkappa_1^{-1})(v, \varphi) =: (\theta, \nu)$. One can easily see that $\nu = \varphi$. So we have to show that the mapping $(v, \varphi) \to \theta(v, \varphi)$ is of class C^∞.

We choose the connection C_2 in $\mathrm{Hom}(T(\tilde{\Omega}), N_1) \oplus N_1$ as follows: points of N_1 are to be displaced parallelwise according to C_1, and the displacement in $\mathrm{Hom}(T(\tilde{\Omega}), N_1)$ is to be generated by C_1 and a connection in $T(\tilde{\Omega})$ for which the parallel displacement along the curves $t \to (p, t) \in \partial\Omega \times]-r, r[\cong H_1$ agrees with the trivial structure of $T(H_1) \cong T(\partial\Omega) \times]-r, r[\times R^1$. We have

$$\Gamma(\Omega, N_2) \cong \Gamma\big(\Omega, \mathrm{Hom}(T(\tilde{\Omega}), N_1)\big) \times \Gamma(\Omega, N_1)$$

(the topologies being those of uniform convergence of all derivatives). Furthermore, the superposition

$$\Gamma(\Omega, N_1) \times C^\infty(\partial\Omega) \xrightarrow{\varkappa_2 \circ D_C \circ \varkappa_1^{-1}} \Gamma(\Omega, N_2) \times C^\infty(\partial\Omega) \to \Gamma(\Omega, N_2) \to \Gamma(\Omega, N_1)$$

— is just the projection on the first factor and thus is of class C^∞. Now we need only to prove that the mapping

$$(v, \varphi) \to \mathrm{pr}_1 \circ \theta(v, \varphi) \in \Gamma\big(\Omega, \mathrm{Hom}(T(\tilde{\Omega}), N_1)\big)$$

is of class C^∞.

Let us cover $\partial\Omega$ by a finite number of chart domains $\mathcal{O}_i$. The points of N_1 lying over H_1 can be represented locally as triples $(p', t, a) \in R^{m-1} \times R^1 \times R^{n-m}$, the trivial structure being determined by the paralled displacement along the fibres of H_1. Now let $(\mathcal{O}'_j)$ be a finite covering of $\Omega \setminus H_1$ by chart domains of Ω such that the intersection of $\mathcal{O}'_j$ and H_1 does not contain points with $t \leqslant \frac{3}{4} r$ for any j and that $\pi_N^{-1}(\mathcal{O}'_j)$ is trivial. Restricting the consideration to functions φ with $|\varphi| < r/4$ we see that at points of $\mathcal{O}'_j$ the displacements along fibres of H_1 will not intervene when $\varkappa$ is being computed. The family $(\mathcal{O}_i \times [0, r[\, , \mathcal{O}'_j)$ is a finite covering of Ω. Let (f_i, f_j) be a subordinated partition of unity (of class C^∞). It can easily be verified that $l(v, \varphi) := \mathrm{pr}_1 \circ \theta(v, \varphi)$ is linear in v. Hence

$$l\Big(\sum_i f_i v + \sum_j f_j v, \varphi\Big) = \sum_i l(f_i v, \varphi) + \sum_j l(f_j v, \varphi).$$

We define $l_i(v, \varphi) := l(f_i v, \varphi)$, $l_j(v, \varphi) := l(f_j v, \varphi)$. We shall prove l_i and l_j to be of class C^∞. Let h_i be a smooth function on $\partial\Omega$ equal to 1 on the compact subset $\pi_{H_1}(\mathrm{supp} f_i)$ of $\mathcal{O}_i$ with a compact $\mathrm{supp}\, h_i \subset \mathcal{O}_i$. The mappings

$$\Gamma(\Omega, N_1) \ni v \to \tilde{v}_i := \big(f_i v, T(f_i v)\big) \in C^\infty(\Omega, R^{n-m+m(n-m)})$$

and the ones similarly defined for every j and

$$C^\infty(\partial\Omega) \ni \varphi \to \tilde{\varphi}_i \in C^\infty(\Omega, R^{1+(m-1)}),$$
$$\tilde{\varphi}_i(p', t) := \big(h_i(p'_i)\varphi(p')\xi(t), h_i(p')T_{p'}(\varphi)\xi(t)\big)$$

for $(p', t) \in \mathcal{O}_i \times [0, r[$ (and 0 elsewhere) are linear mappings continuous in the topologies of uniform convergence of all derivatives, and hence of class C^∞. One can easily notice that l_j does not depend on φ and l_i depends really on $\tilde{\varphi}_i$ only. Furthermore, $l_i(v, \varphi)$ is equal to zero beyond the compact $\mathrm{supp} f_i \subset \mathcal{O}_i \times [0, r[$ and so is $l_j(v, \varphi)$ beyond the compact $\mathrm{supp} f_j \subset \mathcal{O}'_j$. Thus l_i maps into a closed subspace of $\Gamma\big(\Omega, \mathrm{Hom}(T(\tilde{\Omega}), N_1)\big)$ consisting of sections vanishing beyond $\mathrm{supp} f_i$ and analogically for every l_j. But this subspace is isomorphic to the closed subspace of $C^\infty(\Omega, R^{m(n-m)})$ consisting of mappings equal to zero beyond $\mathrm{supp} f_i$ and analogically for every l_j. It can easily be verified that l_i, treated as a mapping into $C^\infty(\Omega, R^{m(n-m)})$, possesses, as a result of the choice of C_2, the form

$$l_i(v, \varphi)(p', t) = \delta^i\big(p', t, \tilde{\varphi}_i(p', t), \tilde{v}_i(p', t)\big),$$

δ^i being a smooth mapping vanishing for $(p', t) \notin \mathcal{O}_i \times [0, r[$. It is even more evident that

$$l_j(v, \varphi)(p) = \delta^j\big(p, \tilde{v}_j(p)\big),$$

δ^j being a smooth mapping equal to zero beyond $\mathcal{O}'_j$. Such mappings are of class C^∞ (see [3]). Thus l, as a sum of C^∞-morphisms, is of class C^∞, which we were to show.

We define f_c as follows: let L be a Lagrange function (i. e., a positively homogeneous function on the manifold of non-zero simple m-vectors tangent to $\mathcal{X}$ — see [1]); on every m-dimensional submanifold, L determines in a natural way a tensor density $\mathcal{L}$; f_c will assign to $\Omega \in X_c$ the integral of $\mathcal{L}$ over Ω.

LEMMA 7. *f_c is of class C^1.*

Proof. Let us cover $D(\Omega)(\Omega \in X_c)$ by means of a finite number of relatively compact domains U_i of charts of $J_m(\mathcal{X})$ generated by charts of $\mathcal{X}$ $(x_k, \xi^a_l)^{k=1,\ldots,n}_{\substack{l=m+1,\ldots,n\\ a=1,\ldots,m}}$ being the corresponding coordinates). Let (g_i) be a subordinated partition of unity of class C^∞. On some neighbourhood of Ω, f_c is a superposition of D_c and a function which assigns to a submanifold of $J_m(\mathcal{X})$ the sum of the integrals over Ω of forms $g_i(x, \xi)L_i(x, \xi)\,dx^1 \wedge \ldots \wedge dx^m$, where L_i is the local Lagrange function for U_i (see [1]). D_c is of class C^∞, and a function assigning to submanifolds the integrals of smooth forms over them is of class C^1 (see [2]). Thus f_c is of class C^1.

We recall that (after [2])

$$T_\Omega X_c = \Gamma\big(\Omega, T(\mathcal{X})\big) / \Gamma\big(\Omega, T(\Omega)\big),$$

where $\Gamma\big(\Omega, T(\mathcal{X})\big)$ is the space of smooth vector fields on Ω and $\Gamma\big(\Omega, T(\Omega)\big)$

is the space of smooth vector fields tangent on $\operatorname{int}\Omega$ to Ω and on $\partial\Omega$ to $\partial\Omega$, the topologies being those of uniform convergence of all derivatives. Let

$$\Delta_c(\Omega) := \{[u] \in T_\Omega X_c\colon\ u|\partial\Omega = 0\}.$$

The classical theory of geodesic fields studies fields $(\Pi_c, D_c, g_\vartheta, s_\sigma)$ of the problem (f_c, Δ_c), g_ϑ assigning to every submanifold ($\in Y_c$) the integral over it of a π-horizontal m-form ϑ on $J_m(\mathscr{X})$ and $s_\sigma\colon X_c \to Y_c$ being generated by a section σ of $J_m(\mathscr{X})$ over $\mathscr{X}$.

By [2] g_ϑ is of class C^1 and

$$T_\Omega g_\vartheta([u]) = \int_\Omega u \lrcorner\, d\vartheta - \int_{\partial\Omega} u \lrcorner\, \vartheta, \tag{3}$$

where $[u] \in T_\Omega Y_c = \Gamma(\Omega, T(\mathscr{Y}))/\Gamma(\Omega, T(\Omega))$.

LEMMA 8. *($[u]$ is Π_c-vertical) $\Leftrightarrow$ (there exists a π-vertical vector field u' on Ω such that $[u'] = [u]$).*

Proof. By [2] there exists a homotopy $\Omega \times]-r, r[\ni (p, t) \to h(p, t) \in J_m(\mathscr{X})$, where $h(p, 0) \equiv p$, $h(\cdot, t)$ is a diffeomorphism onto a submanifold with the boundary of $J_m(\mathscr{X})$ and $u = \left.\frac{\partial h}{\partial t}\right|_{t=0}$. Hence

$$T_\Omega \Pi_c([u]) = \left[\left.\frac{\partial \pi \circ h(p, t)}{\partial t}\right|_{t=0}\right].$$

Thus $[u]$ is π_c-vertical if and only if $T\pi \circ u \circ \pi|_\Omega^{-1} \in \Gamma(\Pi_c(\Omega), T(\Pi_c(\Omega)))$. But π is a diffeomorphism of Ω onto $\Pi_c(\Omega)$. Thus there exists $u'' \in \Gamma(\Omega, T(\Omega))$ such that $T\pi \circ u'' = T\pi \circ u$. We set $u' := u - u''$.

From the definition of g_ϑ and Lemma 8 we get

COROLLARY 1. *$g_\vartheta \circ D_c = f_c$ and $T_{D(\Omega)} g_\vartheta$ gives zero when acting on Π_c-vertical vectors for every $\Omega \in X_c$ if and only if*

I $\bar{X} \lrcorner\, \vartheta(X) = L(\bar{X})$,

II $\langle \bar{X} | \bar{Y}_X \lrcorner\, d\vartheta(X)\rangle = 0$ *for every simple m-vector $\bar{X}$ tangent to $\mathscr{X}$, where X is the tangent element generated by $\bar{X}$, and for every π-vertical vector $\bar{Y}_X$ tangent at X to $J_m(\mathscr{X})$.*

On $\overset{m}{\wedge} T^*_-(J_m(\mathscr{X}))$ (i.e., on the bundle of horizontal m-covectors tangent to $J_m(\mathscr{X})$) there exists a canonical π_2-horizontal m-form $\tilde{\theta}$ (π_1, π_2 denote the projections of $\overset{m}{\wedge} T^*_-(J_m(\mathscr{X}))$ onto $J_m(\mathscr{X})$ and $\mathscr{X}$ respectively). We immediately get

LEMMA 9. *ϑ satisfies I (see Corollary 1) if and only if it is section of*

$$\mathscr{E}' := \{\omega_X \in \overset{m}{\wedge} T^*_-(J_m(\mathscr{X}))\colon \langle \bar{X} | \tilde{\theta}(\omega_X)\rangle = L(X)\}$$

($\bar{X}$ is a simple m-vector generating the tangent element X).

$\mathscr{E}'$ is a submanifold of $\overset{m}{\wedge} T^*_-(J_m(\mathscr{X}))$ of codimension 1.

Proof. The first assertion is obvious. Showing $\mathscr{E}' \cap \pi_1^{-1}(U_i)$ to be a submanifold of $\pi_1^{-1}(U_i)$ for some covering (U_i) of $J_m(\mathscr{X})$, we shall complete the proof.

We take U_i sufficiently small, so that there exists a section S of $\overset{m}{\wedge} T(U_i)$ over it taking values in the set of simple m-vectors generating at each point the basic tangent element. The function

$$F := \langle S \circ \pi_1 | \tilde{\theta}\rangle - L \circ S \circ \pi_1$$

is obviously a submersion and $\mathscr{E}' \cap \pi_1^{-1}(U_i) = F^{-1}(\{0\})$. $F^{-1}(\{0\})$ is of course non-void (by the finite-dimensional variant of the Hahn–Banach theorem). Hence $\mathscr{E}' \cap \pi_1^{-1}(U_i)$ is a submanifold of $\pi_1^{-1}(U_i)$ of codimension 1.

LEMMA 10. *Conditions I and II of Corollary 1 are equivalent to the following one:*

III. *ϑ is a section of $\mathscr{E}$, where $\mathscr{E} := \{\omega_X \in \overset{m}{\wedge} T^*_-(J_m(\mathscr{X}))\colon$ a) $\omega_X \in \mathscr{E}'$, b) $\langle \bar{X} | \bar{Y}_{\omega_X} \lrcorner\, d\tilde{\theta}(\omega_X)\rangle = 0$ for every π_2-vertical vector $\bar{Y}_{\omega_X}$ tangent to $\mathscr{E}'$ at ω_X and $\bar{X}$ — a simple m-vector generating $X\}$.*

Proof. First we shall prove the following fact:

(α) $\langle \bar{X} | \bar{Y}_{\omega_X} \lrcorner\, d\tilde{\theta}(\omega_X)\rangle = 0$ if $\bar{Y}_{\omega_X}$ is a π_1-vertical vector tangent to $\mathscr{E}'$ at ω_X and $\bar{X}$ generates X.

(α) follows immediately from the invariant formula for $d\theta$ (θ is π_2-horizontal):

$$\langle \bar{X} | \bar{Y}_{\omega_X} \lrcorner\, d\tilde{\theta}(\omega_X)\rangle = (-1)^{m+1} \bar{Y}_{\omega_X}(\langle \bar{X} | \tilde{\theta}\rangle).$$

But $\langle \bar{X} | \tilde{\theta}\rangle = L(\bar{X}) =$ const. along the π_1-fibre of $\mathscr{E}'$. Thus, for a section ϑ satisfying I and II we have

$$\langle \bar{X} | \bar{Y}_{\vartheta(X)} \lrcorner\, d\tilde{\theta} \circ \vartheta(X)\rangle = \langle \bar{X} | T\pi_1(\bar{Y}_{\vartheta(X)}) \lrcorner\, d\vartheta(X)\rangle = 0, \tag{4}$$

and ϑ is a section of $\mathscr{E}$.

Conversely, if ϑ is a section of $\mathscr{E}$, then it is also a section of $\mathscr{E}'$ and satisfies I. Now (4) shows that II is also satisfied (we put $\bar{Y}_{\vartheta(X)}$ to be an arbitrary lifting of the given π-vertical vector $T\pi_1(\bar{Y}_{\vartheta(X)})$ tangent at X to $J_m(\mathscr{X})$).

THEOREM 3. *$\mathscr{E}$ is a submanifold of $\overset{m}{\wedge} T^*_-(J_m(\mathscr{X}))$, $\pi_1(\mathscr{E}) = J_m(\mathscr{X})$ and $\pi_1|\mathscr{E}$ is a submersion.* $\dim \mathscr{E} = n + \binom{n}{m} - 1$.

Proof. See an elegant proof in [5].

We define θ to be the restriction of $\tilde{\theta}$ to $\mathscr{E}$.

Being often considered is an open subset $\mathcal{O}$ of $\mathscr{E}$ of ordinary points, i. e., of points at which the canonical mapping λ of $\mathscr{E}$ into $\overset{m}{\wedge} T^*(\mathscr{X})$ is an immersion.

LEMMA 11. *$x \in \mathcal{O}$ if and only if the mapping $\overline{Y}_x \to \overline{Y}_x \lrcorner d\theta$ ($\overline{Y}_x$ being a π_2-vertical vector tangent to $\mathscr{E}$ at x) is an isomorphism onto the subspace of $\overset{m}{\wedge} T_{\pi_2(x)}(\mathscr{X})$ containing m-covectors which give zero when acting on simple m-vectors generating $\pi_1(x)$.*

Proof. By the definition of $\mathscr{E}$, $\overline{Y}_x \lrcorner d\theta$ lies in the subspace just described. Since the diagram

$$\begin{array}{ccc} \mathscr{E} & \longrightarrow & \overset{m}{\wedge} T^*(\mathscr{X}) \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \tilde{\omega}} \\ \mathscr{X} & \xrightarrow{\ \mathrm{Id}\ } & \mathscr{X} \end{array}$$

is comutative, it suffices to show that our condition is equivalent to the following one:

(β) λ acts injectively on the π_2-vertical subspace of $T_x\mathscr{E}$.

But the subspace tangent to the fibre of $\overset{m}{\wedge} T^*(\mathscr{X})$ can be canonically identified with the fibre. Let $t \to x(t)$ be a curve in $\mathscr{E}$ and $x(0) = x$, $\pi_2(x(t)) \equiv \pi_2(x)$. We have

$$\left\langle \overline{X}_{\pi_2(x)} \middle| T_x\lambda\left(\frac{dx(t)}{dt}\bigg|_{t=0}\right)\right\rangle = \left\langle \overline{X}_{\pi_2(x)} \middle| \frac{d\lambda(x(t))}{dt}\bigg|_{t=0}\right\rangle =$$

$$= \frac{d}{dt}\left(\langle \overline{X}_{\pi_2(x)} | \lambda(x(t))\rangle\right)\big|_{t=0} = \frac{d}{dt}\left(\langle \overline{X}_{\pi_2(x)} | \theta(x(t))\rangle\right)\big|_{t=0}$$

$$= (-1)^{p+1}\left\langle \overline{X}_{\pi_2(x)} \middle| \frac{dx(t)}{dt}\bigg|_{t=0} \lrcorner\, d\theta(x)\right\rangle$$

for an arbitrary m-vector $\overline{X}_{\pi_2(x)} \in \overset{m}{\wedge} T_{\pi_2(x)}(\mathscr{X})$. Thus, as the dimensions agree, we have proved (β) and the condition of the lemma to be equivalent.

As the section s_c of Π_c, we take a mapping s_σ which assigns to every submanifold ($\in X_c$) its image under a section σ of $J_m(\mathscr{X})$ over $\mathscr{X}$.

LEMMA 12. *s_σ is continuous.*

LEMMA 13. *s_σ is a C^∞-morphism.*

The proofs of these two lemmas proceed on a full analogy to those of Lemmas 3 and 4.

LEMMA 14. *$T(g_\vartheta \circ s_\sigma)|\Delta_c = 0$ if and only if $d(\sigma^*\vartheta) = 0$.*

Proof. By 3 we have

$$T_\Omega(g_\vartheta \circ s_\sigma)([u]) = \int_\Omega u \lrcorner d(\sigma^*\vartheta) - \int_{\partial\Omega} u \lrcorner \sigma^*\vartheta.$$

Since if $u|\partial\Omega = 0$ then $[u] \in \Delta_c$, we have $d(\sigma^*\vartheta) = 0$. Conversely, if $d(\sigma^*\vartheta) = 0$ then $T_\Omega(g_\vartheta \circ s_\sigma)([u]) = 0$ for u such that $u|\partial\Omega = 0$.

From Corollary 1 and Lemma 14 we get

PROPOSITION 1. *(ϑ, σ) generates a g. f. $(\Pi_c, D_c, g_\vartheta, s_\sigma)$ of the problem with common boundary if and only if ϑ is a section of $\mathscr{E}$ and $d(\sigma^*\vartheta) = 0$.*

LEMMA 15. *For every $\Omega \in X_c$ there exists a neigbourhood U of Ω in X_c such that if $\Omega' \in U$, $\partial\Omega' = \partial\Omega$ then Ω' is connectable with Ω by means of an integral C^∞-curve of Δ_c.*

Proof. We take at Ω a chart from the atlas defined in [2] determined by a six $(N, \Phi, H, \Psi, C, \xi)$. Then Ω' with $\partial\Omega' = \partial\Omega$ is represented by a section u over Ω of a vector bundle N having an extension of Ω as the base. The superposition of the curve $t \to tu$ with the inverse chart satisfies the requirements of Lemma 15.

Let $\mathscr{E}_c$ be a positively homogeneous function on the manifold of simple non-zero m-vectors tangent to $\mathscr{X}$ defined as follows:

$$\mathscr{E}_c(\overline{X}) := L(\overline{X}) - \langle \overline{X} | \sigma^*\vartheta\rangle.$$

Thus

$E_c(\Omega) = \int_\Omega \mathscr{E}_c$ — where the integral is understood as the integral of the corresponding tensor density generated by $\mathscr{E}_c$ on Ω. Let V be a neighbourhood of $D(\Omega_0)$ in $J_m(\mathscr{X})$. If $\mathscr{E}_c > 0$ on $\eta^{-1}(V \setminus D(\Omega_0))$ (where η is the canonical mapping from the manifold of simple non-zero m-vectors onto $J_m(\mathscr{X})$), then $E_c(\Omega) > 0$ for $\Omega \neq \Omega_0$ such that $D(\Omega) \subset V$. If $\mathscr{E}_c \geq 0$ on $\eta^{-1}(V)$, then $E_c(\Omega) \geq 0$ for Ω such that $D(\Omega) \subset V$.

Let us summarize the results concerning classical variational problems with common boundary:

THEOREM 4. *Let Ω_0 be embedded in a Lepage g. f. $(\Pi_c, D_c, g_\vartheta, s_\sigma)$ of (f_c, Δ_c). Let $\mathscr{E}_c \geq 0$ ($\mathscr{E}_c > 0$) on $\eta^{-1}(V)$ $(\eta^{-1}[V \setminus \sigma(\Omega_0)])$, V being a neighbourhood of $\sigma(\Omega_0)$ in $J_m(\mathscr{X})$. Then f_c has at Ω_0 a relative (essential) minimum in the class of submanifolds with common boundary.*

REMARK 3. A. Liesen gave in [5] a local construction of a broad class of Lagrange geodesic fields in which a given critical submanifold was embedded. The variational problems considered there were assumed to lead to non-void ordinary submanifolds of $\mathscr{E}$.

4. Classical integral variational problems with movable boundary. Assume we are given an integral functional on the set of compact oriented submanifolds with boundary. The search for its extremal points, while the class of submanifolds boundaries of which lie on a given surface is being compared, is the aim of the theory of classical variational problems with movable boundary. In the present section we shall formulate those problems in the language of Section 2 and give the construction of some geodesic fields which lead (through Theorem 2) to reasonable necessary conditions for a minimum (maximum).

Let $X_m := X_c$, $f_m := f_c$. Let $\mathscr{B}$ be a closed submanifold of $\mathscr{X}$, which is to contain the boundaries of submanifolds which are being compared. We set $\Delta_m(\Omega)$ to be the set of vectors tangent to C^1-curves in $\mathscr{P}_m(\mathscr{X})$, the boundaries of whose points lie in $\mathscr{B}$ if $\partial\Omega \subset \mathscr{B}$ and $\{0\}$ for other Ω. Let $\Omega_0 \in X_m$ be a critical point of the problem $(f_m, \Delta_m$ with $\partial\Omega_0 \subset \mathscr{B})$. We set:

$Y_m := Y_c \times X'$, where X' is the open submanifold of $\mathscr{P}_{m-1}(\mathscr{B})$ consisting of all submanifolds Σ for which $\partial\Sigma = \emptyset$,

$D_m := (D_c, \partial\Omega_0)$ — being constant in the second factor,

$\Pi_m := \Pi_c \circ \mathrm{pr}_1$ where pr_1 is the projection of $Y_c \times X'$ onto the first factor.

Let Ω_0 be embedded in a g. f. (Π_c, D_c, g_c, s_c) of (f_m, Δ_c) — the problem with common boundary.

LEMMA 16. *$T_{\Omega_0}(g_c \circ s_c)(e) = 0$ if $T_{\Omega_0} f_m(e) = 0$.*

Proof.

$$0 = T_{\Omega_0} f_m(e) = T_{\Omega_0}(g_c \circ D_c)(e) = T_{D_c(\Omega_0)} g_c\big(T_{\Omega_0} D_c(e)\big)$$
$$= T_{s_c(\Omega_0)} g_c\big(T_{\Omega_0} s_c(e)\big) = T_{\Omega_0}(g_c \circ s_c)(e)$$

as $D_c(\Omega_0) = s_c(\Omega_0)$ and $\big(T_{\Omega_0} s_c(e) - T_{\Omega_0} D_c(e)\big)$ is Π_c-vertical.

Let (Π_c, D_c, g_c, s_c) be of the Lepage type, i. e., let $g_c = g_\vartheta$ and let $s_c = s_\sigma$. We have $d(\sigma^* \vartheta) = 0$. Suppose that there exists an $(m-1)$-form Ψ on $\mathscr{X}$ for which $d\Psi = \sigma^* \vartheta$ (e. g. let $\mathscr{X}$ be contractable). We set

$$g_m\big((\Omega, \Sigma)\big) := g_c(\Omega) - \int_\Sigma \Psi + \int_{\partial\Omega_0} \Psi,$$

$$s_m(\Omega) := \begin{cases} \big(s_c(\Omega), \partial\Omega\big) & \text{if } \partial\Omega \subset B, \\ \big(s_c(\Omega), \partial\Omega_0\big) & \text{otherwise}. \end{cases}$$

LEMMA 17. *(Π_m, D_m, g_m, s_m) is a w. g. f. of the problem (f_m, Δ_m).*

Proof. That $T_{D_m(\Omega)} g_m(e) = 0$ for a Π_m-vertical vector e follows directly from an analogous property of D_c and g_c, and the following consideration. $T_{\Omega_0} f_m | \Delta_m = 0$ as Ω_0 is a critical point of (f_m, Δ_m). Thus $T_{\Omega_0}(g_\vartheta \circ s_\sigma) | \Delta_m = 0$. But $g_\vartheta \circ s_\sigma(\Omega) = \int_\Omega \sigma^* \vartheta = \int_{\partial\Omega} \Psi$. Let us take at Ω_0 a chart $\varkappa$ from the atlas defined in [2] determined by a six $(N, \Phi, H, \Psi, C, \xi)$, H being trivial. Let $t \to \Sigma_t$ be a C^∞-curve in X' with $\Sigma_0 = \partial\Omega_0$. Utilising the connection C, we can treat $\pi_N^{-1}(H) =: G$ as a vector bundle isomorphic to $\pi_N^{-1}(\partial\Omega_0) \oplus H$ ($\partial\Omega_0$ being its zero section). Thus we have a mapping of a neighbourhood

of $\partial\Omega_0$ in X' into $\Gamma(\partial\Omega_0, G)$ induced by β — the canonical injection of $\mathscr{B}$ into $\mathscr{X}$. This mapping is of class C^∞ (the proof of the fact proceeds on a full analogy to that of class of Π_c in Section 3). Thus Σ_t generates C^∞-curves θ_t and β_t in $\Gamma(\partial\Omega_0, N)$ and $\Gamma(\partial\Omega_0, H) = C^\infty(\partial\Omega_0)$ respectively. Let $\zeta \in C_0^\infty(R^1)$ with $\zeta(0) = 1$ and $\zeta'(0) = 1$. Then the mapping Z from $\Gamma(\partial\Omega_0, N)$ to $\Gamma(\Omega_0, N)$

$$\big(Z(\theta)\big)(x) := \begin{cases} \theta\big(\pi_H(x)\zeta(t_x)\big) & \text{for } x \in H, \\ 0 & \text{for } x \notin H, \end{cases}$$

is linear and continuous (t_x is the fibre coordinate in H of x). Thus $t \to \big(\beta_t, Z(\theta_t)\big)$ is an integral C^∞-curve of Δ_m.

We have shown that every C^∞-curve in X' passing through $\partial\Omega_0$ can be obtained locally at $\partial\Omega_0$ by taking boundaries of points of an integral C^∞-curve of Δ_m in X_m. Hence the derivative of the C^1-function $X' \ni \Sigma \to \int_\Sigma \Psi$ is equal to 0 at $\partial\Omega_0$.

REMARK 4. Let us take Ω' lying in the domain of the chart $\varkappa$ generated by $(N, \Phi, H, \Psi, C, \xi)$ and such that $\partial\Omega'$ $(\subset \mathscr{B})$ is connectable with $\partial\Omega_0$ by means of a curve which can be obtained by taking boundaries of an integral C^∞-curve of Δ_m $t \to \Omega_t$ running in the domain of $\varkappa$. Of course the end-point of Ω_t (i. e., the point Ω_{t_e} for which $\partial\Omega_{t_e} = \partial\Omega'$) can be connected with Ω' by a C^∞-curve of submanifolds with common boundary. Thus Ω' can be connected with Ω_0 by a continuous curve being of class C^∞ and integral of Δ_m everywhere beyond one point. Moreover, left- and right-hand limits of first derivatives at that point exist for both pieces of the curve which were obtained by restrictions of longer curves. Thus, changing suitably the parametrisation, we can obtain an integral C^1-curve of Δ_m connecting Ω' and Ω_0.

Before we can show the differentiability of s_m in the weak sense, a classification of integral curves of Δ_m must be established.

PROPOSITION 2. *Either integral C^1-curves of Δ_m are constant or the boundaries of their points are contained in $\mathscr{B}$.*

Proof of Proposition 2. Let $I \ni t \to \Omega_t$ be an integral C^1-curve of Δ_m, I being an open interval. Let I_1 consists of points $t \in I$ for which $\partial\Omega_t \notin \mathscr{B}$ and $I_2 := I \setminus I_1$. As $\mathscr{B}$ is closed in $\mathscr{X}$, I_1 is open ($t \to \Omega_t$ is of course continuous). I_1 is convex as well. Indeed, let $t_i \in I_1$, $i = 1, 2$, $t_1 < t_2$, and let $t' := \inf\{t: t_1 < t < t_2,\ t \in I_2\}$; if t' exists then $t_1 < t' < t_2$ and $t' \in I_2$. But $\partial\Omega_t/\partial t = 0$ for $t_1 \leqslant t < t'$ whence $\Omega_t = \text{const.}$, contrary to the continuity of $t \to \Omega_t$; thus t' does not exist and $t \in I_1$ for $t_1 \leqslant t < t_2$. Now let $t_n \xrightarrow[n\to\infty]{} \tau \in I$, $t_n \in I_1$. Then $\Omega_{t_n} = \Omega_{t_1}$, $\Omega_\tau = \lim \Omega_{t_n} = \Omega_{t_1}$, whence $\partial\Omega_\tau \notin \mathscr{B}$ and $\tau \in I_1$ and I_2 is open. Thus either I_1 or I_2 is void, which completes the proof of Proposition 2.

PROPOSITION 3. *$\partial: \mathscr{P}_m(\mathscr{X}) \to \mathscr{P}_{m-1}(\mathscr{X})$ is of class C^∞.*

Proof of Proposition 3. $\mathscr{P}_m(\mathscr{X})$ is equipped with the finest topology in which the canonical injections I_Ω of all $\mathscr{E}_\Omega$ are continuous. But $\partial \circ I_\Omega$ is the superposition of the continuous mapping $\mathscr{E}_\Omega \ni \varkappa \to \varkappa|\partial\Omega$ with $I_{\partial\Omega}$ and thus is continuous. Let us take at $\Omega_0 \in \mathscr{P}_m(\mathscr{X})$ a chart $\varkappa$ determined by a six $(N, \Phi, H, \Psi, C, \xi)$. $G := \pi_N^{-1}(H)$ can be treated as a vector bundle isomorphic to $\pi_N^{-1}(\partial\Omega_0) \oplus H$ over $\partial\Omega_0$, $\pi_H \circ \pi_N$ being its projection (the linear structure in fibres being defined by means of parallel displacement in N). Thus the couple $(G, \Phi|G)$ defines at $\partial\Omega_0$ a chart ν for $\mathscr{P}_{m-1}(\mathscr{X})$ ($\partial\Omega_0$ is a submanifold without boundary).

$$\Gamma(\Omega, N) \times \Gamma(\partial\Omega, H) \xrightarrow{\nu\circ\partial\circ\varkappa^{-1}} \Gamma(\partial\Omega, G) \cong \Gamma(\partial\Omega, N) \times \Gamma(\partial\Omega, H).$$

$(\nu \circ \partial \circ \varkappa^{-1})\big((\theta, \varphi)\big) = (\theta|\partial\Omega, \varphi)$. Thus $\nu \circ \partial \circ \varkappa^{-1}$ is of class C^∞ as a linear continuous mapping (provided all spaces are equipped with the topology of uniform convergence of all derivatives).

PROPOSITION 4. *The set X' of all $\Sigma \in \mathscr{P}_{m-1}(\mathscr{X})$, $\Sigma \subset \mathscr{B}$, for which $\partial\Sigma = \emptyset$ is a C^∞-submanifold of $\mathscr{P}_{m-1}(\mathscr{X})$ whose differentiable structure coincides with that induced from $\mathscr{P}_{m-1}(\mathscr{B})$.*

Proof of Proposition 4. It is obvious that the topologies of X' induced from $\mathscr{P}_{m-1}(\mathscr{X})$ and $\mathscr{P}_{m-1}(\mathscr{B})$ coincide. Now let us take a tubular neighbourhood of $\mathscr{B}$ in $\mathscr{X}$ i. e., a vector boundle $\bar{M}$ over $\mathscr{B}$ and a diffeomorphism of a neighbourhood of its zero-section onto an open neighbourhood of $\mathscr{B}$ in $\mathscr{X}$ coinciding on the zero-section with $\pi_{\bar{M}}$. Let C be a linear connection in $\bar{M}$, $\Sigma_0 \in X'$, and let ν be a chart for $\mathscr{P}_{m-1}(\mathscr{B})$ at Σ_0 determined by a couple (N, Φ). $\pi_{\bar{M}}^{-1}(N)$ can be treated as a vector bundle over Σ_0 isomorphic to $\pi_{\bar{M}}^{-1}(\Sigma_0) \oplus N$, $\pi_N \circ \pi_{\bar{M}}$ being its projection (the linear structure in fibres being defined by means of parallel displacement in $\bar{M}$ along radii in corresponding fibres of N). Thus there exists a neighbourhood $\mathcal{O}$ of Σ_0 in $\mathscr{X}$ and its diffeomorphism Ψ^{-1} onto an open neighbourhood of the zero-section of a bundle $M \oplus N$ coinciding with the zero-section of $M \oplus N$ on Σ_0 and such that $x \in \mathscr{B} \cap \mathcal{O}$ if and only if $\Psi^{-1}(x) \in N$ (N is a subbundle of $M \oplus N$). As Σ_0 is compact, we may assume that Ψ^{-1} maps onto $M \oplus N$. The couple $(M \oplus N, \Psi)$ determines a chart $\varkappa$ for $\mathscr{P}_{m-1}(\mathscr{X})$ at Σ_0. Locally, in the chart $\varkappa$, $\mathscr{P}_{m-1}(\mathscr{X}) = \Gamma(\Sigma_0, M) \times \Gamma(\Sigma_0, N)$ and $\Sigma \in X'$ if and only if $\Sigma \in \{0\} \times \Gamma(\Sigma_0, N)$ with $\varkappa|X' = \nu$ which completes the proof.

Now the fact that $t \to s_m(\Omega_t) \in Y_c \times X'$ is of class C^1 for an integral C^1-curve $t \to \Omega_t$ of Δ_m follows immediately from Propositions 2, 3, 4 and the definition of s_m.

Moreover, $g_m\big(s_m(\Omega_t)\big) = \text{const}$ if $\partial\Omega_t \not\subset \mathscr{B}$ for every t, or $g_m\big(s_m(\Omega_t)\big) = g_\vartheta \circ s_\sigma(\Omega_t) - \int_{\partial\Omega_t} \Psi + \int_{\partial\Omega_0} \Psi = \int_{\Omega_t} \sigma^*\vartheta - \int_{\partial\Omega_t} \Psi + \int_{\partial\Omega_0} \Psi = \int_{\Omega_t} d\Psi - \int_{\partial\Omega_t} \Psi + \int_{\partial\Omega_0} \Psi = \text{const.}$ if $\partial\Omega_t \subset \mathscr{B}$ for every t. The other assertions of Lemma 17 can be verified immediately.

LEMMA 18. *There exists an open neighbourhood U of $\partial\Omega_0$ in $\mathscr{B}$ and a closed $(m-1)$-form ψ on it for which $z^*\beta^*\Psi = z^*\psi$ on U (where z, β are the canonical injections of $\partial\Omega_0$ into $\mathscr{B}$ and of $\mathscr{B}$ into $\mathscr{X}$ respectively).*

Proof. We take a neighbourhood U of $\partial\Omega_0$ in $\mathscr{B}$ which is isomorphic to a vector bundle N, $\partial\Omega_0$ corresponding to the zero-section of N (the so called tubular neighbourhood). We put $\psi := \pi_N^* z^* \beta^* \Psi$, π_N being the projection in N.

Let $\mathscr{E}_b$ be a function on the manifold of simple non-zero $(m-1)$-vectors tangent to U defined as follows:

$$\mathscr{E}_b(\bar{X}) := -\langle \bar{X} | \beta^*\Psi - \psi \rangle.$$

We have for Ω sufficiently close to Ω_0, $\partial\Omega \subset \mathscr{B}$:

$$E_m(\Omega) = \int_\Omega \mathscr{E}_c + \int_{\partial\Omega} \mathscr{E}_b, \tag{5}$$

where $E_m = f_m - g_m \circ s_m$ is the Weierstrass function for (Π_m, D_m, g_m, s_m) (the last component of (5) has been obtained by virtue of the Stokes theorem).

Finally we give

LEMMA 19. *There exists a neighbourhood U' of Ω_0 in X_m such that every $\Omega' \in U'$, $\partial\Omega' \subset \mathscr{B}$, is connectable with Ω_0 by means of integral C^1-curves of Δ_m.*

Proof. See Remark 4.

Summarizing the results concerning classical variational problems with movable boundary, we get

THEOREM 5. *Let Ω_0 be a critical point of (f_m, Δ_m) embedded in a Lepage geodesic field $(\Pi_c, D_c, g_\vartheta, s_\sigma)$. Let $\mathscr{E}_c \geqslant 0$ ($\mathscr{E}_c > 0$) on $\eta^{-1}(V)$ $\big(\eta^{-1}[V \setminus D_c(\Omega_0)]\big)$ and $\mathscr{E}_b \geqslant 0$ on $\eta^{-1}(V')$, V being a neighbourhood of $D_c(\Omega_0)$ in $J_m(\mathscr{X})$ and V' — of $D_c(\partial\Omega_0)$ in $J_{m-1}(\mathscr{B})$. Then f_m has at Ω_0 a relative (essential) minimum in the class of submanifolds with a movable (longswise $\mathscr{B}$) boundary.*

Suppose that our g. f. $(\Pi_c, D_c, g_\vartheta, s_\sigma)$ is of the Carathéodory type, i. e., that rang $\sigma^*\vartheta = \text{const.} = n-m$. Then a somewhat more geometric condition for an extremum of f_m can be given. Let

$$\text{Centr}_x \sigma^*\vartheta := \{\bar{X}_x \in T_x(\mathscr{X}) : \bar{X}_x \lrcorner \sigma^*\vartheta(x) = 0\}.$$

Let $\text{Centr}\,\sigma^*\vartheta := \bigcup_{x\in\mathscr{X}} \text{Centr}_x \sigma^*\vartheta$. $\text{Centr}\,\sigma^*\vartheta$ is a distribution (of subspaces of tangent to $\mathscr{X}$ spaces).

PROPOSITION 5. *$\text{Centr}\,\sigma^*\vartheta$ is a distribution of class C^∞ (i. e., a subbundle of $T(\mathscr{X})$) and is totally integrable.*

For proof of Proposition 5 see for example [6].

At points $x \in \Omega_0$ $\text{Centr}_x \sigma^*\vartheta$ is transversal to $T_x\Omega_0$ as $\langle \bar{X}_x | \sigma^*\vartheta \rangle = L(\bar{X}_x) > 0$, $\bar{X}_x$ being a p-vector tangent at x to Ω_0 with $\eta(\bar{X}_x) = \sigma(x)$.

Thus for every $x \in \tilde{\Omega}_0$ ($\tilde{\Omega}_0$ being an extension of Ω_0) there exist a neighbourhood U_x of x in $\mathscr{X}$ and a diffeomorphism χ_x: $U'_x \times K(0, R_x) \to U_x$ for which $(U_x \cap \tilde{\Omega}_0) = \chi_x(U'_x \times \{0\}) = U'_x$ and $\chi_x(\{y\} \times K(0, R_x))$ is an integral submanifold of Centr$\sigma^*\vartheta$ (U'_x is open in $\tilde{\Omega}_0$ and $K(0, R_x)$ is a ball in R^{n-m}). Let W be a neighbourhood of Ω_0. By R_W we shall denote a relation $\subset W \times \tilde{\Omega}_0$ defined as follows:

$$(x \overset{R_W}{\sim} x') \Leftrightarrow \begin{pmatrix}\text{there exists a connected integral submanifold} \subset W \text{ of} \\ \text{Centr}\,\sigma^*\vartheta \text{ of dimension } n-m, \text{ containing } x \text{ and } x'\end{pmatrix}$$

LEMMA 20. *There exists a neighbourhood W of Ω_0 such that R_W defines a function P: $W \to \tilde{\Omega}_0$, $P(W) \supset \Omega_0$. P is a submersion.*

Proof. Let (x_i) be a finite set of points $\in \Omega_0$ such that $\bigcup_i U'_{x_i} \supset \Omega_0$. Let (U''_i) be a family of sets open in $\tilde{\Omega}_0$, relatively compact and covering Ω_0, and let $\overline{U''_i} \subset U'_{x_i}$ hold. Then there exists a sequence (r_i) such that

$$\big[\bigcup_i \chi_{x_i}(U''_{x_i} \times K(0, r_i))\big] \cap \big[\bigcup_i \chi_{x_i}(\overline{U''_i} \times [\overline{K(0, \tfrac{3}{4}R_{x_i})} \setminus K(0, \tfrac{1}{2}R_{x_i})])\big] = \emptyset.$$

It can easily be verified that $W := \bigcup_i \chi_{x_i}(U''_i \times K(0, r_i))$ possesses the required properties.

LEMMA 21. $P^*\sigma^*\vartheta = \sigma^*\vartheta$.

Proof. Let $\overline{X}_0, \ldots, \overline{X}_m$ be vector fields on W such that $TP(\overline{X}_1), \ldots, TP(\overline{X}_m)$ are vector fields on $\tilde{\Omega}_0$ and $TP(\overline{X}_0) = 0$. Lemma 21 is an immediate consequence of the following formula:

$$0 = \langle \overline{X}_0(x) \wedge \overline{X}_1(x) \wedge \ldots \wedge \overline{X}_m(x) \,|\, d(\sigma^*\vartheta)(x)\rangle = \overline{X}_0(x)(\langle \overline{X}_1 \wedge \ldots \wedge \overline{X}_m \,|\, \sigma^*\vartheta\rangle).$$

LEMMA 22. *If $P(\mathscr{B}) \cap \Omega_0 \subset \partial\Omega_0$, then there exists a neighbourhood $\mathcal{O}$ of $\partial\Omega_0$ in $\mathscr{P}_{m-1}(\mathscr{B})$ such that for $\Sigma \in \mathcal{O}$ $\int_\Sigma \Psi - \int_{\partial\Omega_0} \Psi \geqslant 0$.*

Proof. Using Proposition 4 and Lemmas 2 and 3, we can show that $\mathcal{O}$ may be chosen so that, for $\Sigma \in \mathcal{O}$, $P(\Sigma)$ will be a compact submanifold without boundary which can be represented by a function φ on $\partial\Omega_0$ if we choose a diffeomorphism of a neighbourhood of $\partial\Omega_0$ in $\tilde{\Omega}_0$ onto $]-\varepsilon, \varepsilon[\times \partial\Omega_0$. Let us put

$$h(t, p) := t\varphi(p) \quad \text{for} \quad p \in \partial\Omega_0,\ t \in [0, 1].$$

The diffeomorphism

$$\partial\Omega_0 \ni p \to (t\varphi(p), p) \in]-\varepsilon, \varepsilon[\times \partial\Omega_0$$

defines a curve $t \to \Sigma_t$ of compact submanifolds without boundary of $\tilde{\Omega}_0$ ($\Sigma_0 = \partial\Omega_0$, $\Sigma_1 = \Sigma$).

$$\int_\Sigma \Psi - \int_{\partial\Omega_0} \Psi = \int_0^1 \frac{d}{dt}\Big(\int_{\Sigma_t} \Psi\Big)\, dt = \int_0^1 \Big(\int_{\Sigma_t} \frac{\partial h}{\partial t} \,\lrcorner\, d\Psi\Big)\, dt = \int_0^1 \Big(\int_{\Sigma_t} \frac{\partial h}{\partial t} \,\lrcorner\, \sigma^*\vartheta\Big)\, dt.$$

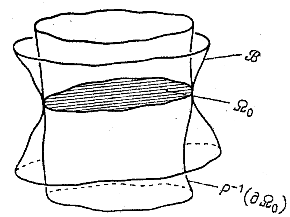
Since Ω_0 is embedded in the field, we have

$$\langle \overline{X}_x | \sigma^*\vartheta\rangle = L(\overline{X}_x) > 0,$$

$\overline{X}_x$ being a p-vector tangent at x to Ω_0 comformable to the orientation. If we choose $\tilde{\Omega}_0$ suitably, this holds for $\tilde{\Omega}_0$ also, and thus $\int_{\Sigma_t} \partial h/\partial t \,\lrcorner\, \sigma^*\vartheta \geqslant 0$, what completes the proof.

Now the condition for a minimum which uses the concept of a Carathéodory g. f. can be put in the following way:

THEOREM 6. *Let Ω_0 be a critical point of (f_m, Δ_m) embedded in a Carathéodory geodesic field $(\Pi_c, D_c, g_\vartheta, s_\sigma)$. Let $\mathscr{E}_c \geqslant 0$ ($\mathscr{E}_c > 0$) on $\eta^{-1}(V)$ $(\eta^{-1}[V \setminus D_c(\Omega_0)])$, V being some neighbourhood of $D_c(\Omega_0)$ in $J_m(\mathscr{X})$. Let W, P be as in Lemma 20, and let $P(\mathscr{B}) \cap \Omega_0 \subset \partial\Omega_0$. Then f_m has at Ω_0 a relative (essential) minimum in the class of submanifolds with movable (longswise $\mathscr{B}$) boundary.*



The condition assumed in Lemma 22 has a geometric interpretation. It means that $\mathscr{B}$ (in a neighbourhood of $\partial\Omega_0$) runs outside the surface $P^{-1}(\partial\Omega_0)$.

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