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# Any separable Banach space with the bounded approximation property is a complemented subspace of a Banach space with a basis

by

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**Abstract.** The result announced in the title is proved.

Recall that a separable Banach space $X$ has the bounded approximation property (=Banach approximation property in the terminology of [4] and [8]) if the identity operator on $X$ is a pointwise limit of a sequence of finite-dimensional operators, equivalently if there exists a sequence $(A_n)$ of finite-dimensional operators such that $A_n \neq 0$ for $n = 1, 2, \ldots$ and

$$\lim_n \Big\| x - \sum_{p=1}^{n} A_p(x) \Big\| = 0 \quad \text{for } x \in X. \tag{1}$$

Clearly (1) and the Banach–Steinhaus Principle imply

$$\sup_n \Big\| \sum_{p=1}^{n} A_p \Big\| = K < +\infty. \tag{2}$$

The purpose of the present note is to prove the following:

THEOREM 1. *A separable Banach space $X$ has the bounded approximation property iff $X$ is a complemented subspace of a Banach space with a Schauder basis.*

J. Lindenstrauss ([5], Corollary 4) proved recently the same fact under the additional assumption that $X$ is isomorphic to a separable conjugate space. In this case, by a result of Grothendieck (cf. [1], Chap. I, § 5, No 2), the condition "$X$ has the bounded approximation property" is equivalent to the (formally weaker) condition "$X$ has the approximation property". Theorem 1 improves also Theorem 1.1 of [8] which states that a separable Banach space has the bounded approximation property iff it is a complemented subspace of a Banach space with a Schauder decomposition into finite-dimensional subspaces. The last paragraph of the proof of Theorem 1 is in fact the same argument as is used in the proof of Theorem 1.1 of [8].

Proof of Theorem 1. The part "if" is trivial. To prove the "only if" part, let us put $E_p = A_p(X)$; $m_0 = 0$; $m_p = \dim E_p$ for $p = 1, 2, \ldots$ It follows from the Auerbach Lemma (cf. Taylor [9]) that there exist one-dimensional operators $B_j^{(p)}\colon E_p \to E_p$ with $\|B_j^{(p)}\| = 1$ $(j = 1, 2, \ldots, m_p)$ such that $\sum_{j=1}^{m_p} B_j^{(p)}(e) = e$ for $e \in E_p$. Let us set

$$C_i^{(p)} = m_p^{-1} B_j^{(p)} \quad \text{for } i = rm_p + j$$
$$(r = 0, 1, \ldots, m_p - 1;\ j = 1, 2, \ldots, m_p).$$

Clearly we have

$$(3) \qquad \sum_{i=1}^{m_p^2} C_i^{(p)}(e) = e \quad \text{for } e \in E_p \quad \text{and} \quad \max_{1 \leqslant q \leqslant m_p^2} \Big\| \sum_{i=1}^{q} C_i^{(p)} \Big\| \leqslant 2.$$

Let us set

$$\tilde{A}_s = C_i^{(p)} A_p \quad \text{for } s = m_0^2 + m_1^2 + \ldots + m_{p-1}^2 + i$$
$$(i = 1, 2, \ldots, m_p^2;\ p = 1, 2, \ldots).$$

By (2) and (3), we get

$$(4) \qquad \sup_n \Big\| \sum_{s=1}^{n} \tilde{A}_s \Big\| \leqslant 4K.$$

Thus, by (1), (3) and (4), we get

$$\lim_n \Big\| \sum_{s=1}^{n} \tilde{A}_s(x) - x \Big\| = \lim_n \Big\| \sum_{p=1}^{n} \sum_{i=1}^{m_p^2} C_i^{(p)} A_p(x) - x \Big\| = 0 \quad \text{for } x \in X.$$

Now let us consider the space $Y$ of all sequences $(y(s))_{s=1}^{\infty}$ such that $y(s) \in \tilde{A}_s(X)$ for $s = 1, 2, \ldots$ and the series $\sum_{s=1}^{\infty} y(s)$ converges. The operations of addition and multiplication by scalars in $Y$ are defined coordinatewise and the norm is defined by

$$|||(y(s))||| = \sup_n \Big\| \sum_{s=1}^{n} y(s) \Big\| \quad \text{for } (y(s)) \in Y.$$

Clearly $Y$ is a Banach space. Since $\dim \tilde{A}_s(X) = 1$, one can pick a $y_s \in \tilde{A}_s(X)$ so that $\|y_s\| = 1$ and any $y \in \tilde{A}_s(X)$ is of the form $y = cy_s$ for some scalar $c$. Define $\tilde{y}_s \in Y$ by $\tilde{y}_s(t) = 0$ for $t \neq s$ and $\tilde{y}_s(s) = y_s$ $(s = 1, 2, \ldots)$. The sequence $(\tilde{y}_s)$ forms a monotone basis for $Y$ because linear combinations of the $\tilde{y}_s$'s are dense in $Y$ and for any sequence of scalars $(c_s)_{s=1}^{\infty}$ we have

$$\Big|\Big|\Big| \sum_{s=1}^{n} c_s \tilde{y}_s \Big|\Big|\Big| \leqslant \Big|\Big|\Big| \sum_{s=1}^{n+1} c_s \tilde{y}_s \Big|\Big|\Big| \quad \text{for } n = 1, 2, \ldots$$



Furthermore the map $T\colon X \to Y$ defined by

$$T(x) = \big(\tilde{A}_s(x)\big)_{s=1}^{\infty} \quad \text{for } x \in X$$

is an isomorphic embedding with $\|T\|\,\|T^{-1}\| \leqslant 4K$. Finally the map $P\colon Y \to T(X)$ defined by

$$P\big(y(s)\big) = \Big(\tilde{A}_s\Big(\sum_{t=1}^{\infty} y(t)\Big)\Big) \quad \text{for } (y(s)) \in Y$$

is a projection from $Y$ onto $T(X)$ with $\|P\| \leqslant 4K$. Thus $X$ is isomorphic to a complemented subspace of the space $Y$ with a basis. By a suitable renorming of $Y$ (cf. [6], Proposition 1), we infer that $X$ is isometric to a complemented subspace of a Banach space with a Schauder basis. This completes the proof.

Remark. 1. Theorem 1 can be easily generalized to the case of separable Fréchet spaces (= locally convex complete metric linear spaces). The proof presented here requires only a few minor changes.

Remark 2. In the particular case of finite dimensional spaces Theorem 1 admits the following improvement:

For any $n$-dimensional Banach space $E$ there exist a Banach space $F$ with $\dim F = n^2 = N$ and a constant $C$ with $1 \leqslant C \leqslant 1 + n^{-1/2}$ such that

(i) $F \supset E$,

(ii) there is a projection $P\colon F \xrightarrow{\text{onto}} E$ with $\|P\| \leqslant 1$,

(iii) $F$ has a basis, say $(f_i)_{i=1}^{N}$, with the norm $\leqslant C$, i. e.

$$\Big\| \sum_{i=1}^{n} c_i f_i \Big\| \leqslant C \Big\| \sum_{i=1}^{N} c_i f_i \Big\| \quad \text{for } n = 1, 2, \ldots, N.$$

Sketch of the proof. By a result of F. John [3], there exists a linear operator $T\colon E \to l_2^n$ such that $\|T^{-1}\| = 1$ and $\|T\| \leqslant n^{1/2}$. Let $(z_j)_{j=1}^{n}$ be an orthonormal basis in $l_2^n$. Let us put $B_j(e) = \big(T(e), z_j\big) T^{-1}(z_j)$ for $e \in E$ and for $j = 1, 2, \ldots, n$. Then $\|\sum_{j=1}^{k} B_j\| \leqslant n^{1/2}$ for $k = 1, 2, \ldots, n$. Next define $C_i\colon E \to E$ by

$$C_i = n^{-1} B_j \quad \text{for} \quad i = rn + j \quad (r = 0, 1, \ldots, n-1;\ j = 1, 2, \ldots, n).$$

Then $\|\sum_{i=1}^{p} C_i\| \leqslant 1 + n^{-1/2}$ for $p = 1, 2, \ldots, N$. We define $F$ to be the space of all sequences $(y_i)_{i=1}^{N}$ such that $y_i \in C_i(E)$ for $i = 1, 2, \ldots, N$ with the norm defined as the gauge of the convex body

$$\operatorname{conv}\Big(\Big\{(y_i) \in F\colon \sup_{1 \leqslant k \leqslant N} \Big\| \sum_{i=1}^{k} y_i \Big\| \leqslant 1\Big\} \cup \big\{(y_i) \in F\colon y_i = C_i(e);$$
$$i = 1, 2, \ldots, N;\ \|e\| \leqslant 1\big\}\Big).$$

We identify $E$ with the subspace of $F$ consisting of the sequences $(C_i(e))_{i=1}^N$ for $e \in E$.

Remark 3. It follows from 2 that for a given finite-dimensional $E$ and given $\varepsilon > 0$ there exists a finite-dimensional $F$ satisfying (i)–(iii) with $C < 1+\varepsilon$.

Remark 4. For any finite-dimensional $E$ there exists a Banach space $F$ (in general infinite dimensional) satisfying (i)-(iii) with $C = 1$.

Indeed, this is equivalent to the existence on $E$ of a sequence of one-dimensional operators, say $(A_s)_{s=1}^{\infty}$, such that $\sum_{s=1}^{\infty} A_s(e) = e$ for $e \in E$ and $\sup_k \|\sum_{s=1}^{k} A_s\| \leqslant 1$. We put

$$A_s = (n2^{k+1})^{-1} B_j$$

for $s = kn^2+rn+j$ $(k = 0, 1, \ldots; r = 0, 1, \ldots, n-1;\ j = 1, 2, \ldots, n)$ where $B_j$ are defined as in 2.

We do not know whether one can construct for a given finite-dimensional $E$ a finite-dimensional $F$ satisfying (i)–(iii) with $C = 1$.

Remark 5. We do not know whether the unconditional analogue of Theorem 1 is true even for finite-dimensional spaces (cf. [8], Theorem 1.1, the "unconditional part").

Remark 6. Combining Theorem 1 with Corollary 1 of [7], and Theorem 3.3 and Remark 4.1 of [8] we get

COROLLARY. *The* (*separable*) *Banach space $B$ with a complementably universal basis constructed in* [7] *has the following property: any separable Banach space with the bounded approximation property is isomorphic to a complemented subspace of $B$. Hence $B$ is isomorphic to the complementably universal space constructed by Kadec in* [4].

Added in proof. Essentially the same result as our Theorem 1 and the Corollary has been independently discovered by W. B. Johnson, H. P. Rosenthal and M. Zippin [3]. Their proof is entirely different than ours; it generalizes Lindenstrauss' method for conjugate spaces. An interesting application of this approach is a result of W. Johnson [10] that every reflexive Banach space with the bounded approximation property is isomorphic to a complemented subspace of a reflexive space with a basis.

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