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Every Segal algebra satisfies Ditkin's condition

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Abstract. The purpose of this note is to prove the assertion stated in the title. As an easy corollary we obtain the Shilov–Wiener Tauberian theorem for *all* Segal algebras. Warner [5] and the author [6] have obtained special cases of these theorems.

Let G be a locally compact Abelian group with dual group $\check{G}$. Following Reiter ([3], p. 126), a subalgebra $S(G)$ of $L^1(G)$ is called a *Segal algebra* if:

(i) $S(G)$ is dense in $L^1(G)$ in the L^1-norm topology and if $f \in S(G)$ then $f_a \in S(G)$, where $f_a(x) = f(a^{-1}x)$;

(ii) $S(G)$ is a Banach algebra under some norm $\|\cdot\|_S$ which also satisfies $\|f\|_S = \|f_a\|_S$ for all $f \in S(G)$, $a \in G$ (multiplication in $S(G)$ is the usual convolution);

(iii) if $f \in S(G)$, then for any $\varepsilon > 0$ there exists a neighborhood U of the identity element of G such that $\|f_y - f\|_S < \varepsilon$ for all $y \in U$.

Throughout the rest of this note, $S(G)$ will denote an arbitrary Segal algebra. The following facts, which are needed in the sequel, can be found in Reiter ([3], p. 128):

(I) There exists a constant C such that $\|f\|_1 \leqslant C\|f\|_S$ for all $f \in S(G)$.

(II) $S(G)$ is an ideal in $L^1(G)$ and $\|h * f\|_S \leqslant \|h\|_1 \|f\|_S$ for all $f \in S(G)$, $h \in L^1(G)$.

(III) If $f \in L^1(G)$ and the Fourier transform $\hat{f}$ has compact support, then $f \in S(G)$.

(IV) Given any $f \in S(G)$ and $\varepsilon > 0$, there is a $v \in S(G)$ such that $\hat{v}$ has compact support and $\|v * f - f\|_S < \varepsilon$.

LEMMA. *The maximal ideal space Δ of $S(G)$ can be identified with the dual group $\hat{G}$.*

Proof. (II) implies that $\lim_{n\to\infty} \|f^n\|_S^{1/n} \leqslant \|f\|_1$ for each $f \in S(G)$. Now if $\gamma \in \Delta$, then we have $|\gamma(f)|^n = |\gamma(f^n)| \leqslant \|f^n\|_S$ and hence γ is L^1-norm bounded. Thus γ can be extended in a unique fashion to a multiplicative linear functional γ_1 on $L^1(G)$, and, conversely, every multiplicative linear functional γ_1 on $L^1(G)$ determines a $\gamma \in \Delta$. Now recall that the maximal ideal space of $L^1(G)$ is $\hat{G}$ and finally observe that the Gelfand topology on Δ agrees with the usual topology on $\hat{G}$ by (III) above, Rudin ([4], 2.6.8), and Loomis ([1],5G).

COROLLARY. *$S(G)$ is a regular, semi-simple, commutative Banach algebra.*

Proof. Regularity follows from the Lemma, Rudin ([3], 2.6.2) and (III).

Now recall that $S(G)$ satisfies *Ditkin's condition* (or *condition* D) if: For $f \in S(G)$ and $\gamma \in \hat{G}$ such that $\hat{f}(\gamma) = 0$, there is a sequence $\{f_n\}$ in $S(G)$ such that $\hat{f}_n = 0$ on some neighborhood V_n of γ and $\|f * f_n - f\|_S \to 0$; if $\hat{G}$ is non-compact, the condition must also be satisfied at the point at infinity, i. e., for each $f \in S(G)$, there is a sequence $\{f_n\}$ in $S(G)$ such that $\hat{f}_n$ has compact support and $\|f * f_n - f\|_S \to 0$. It is well-known that $L^1(G)$ satisfies condition D (see for example Loomis [1]) and the usual proof of this fact depends on the existence of a bounded approximate unit. It is quite easy to show that $S(G)$ has *no* bounded approximate unit unless $S(G) = L^1(G)$, (see the proof of (3) in [2]), and this makes the next theorem more interesting. (However, the absence of a bounded approximate unit in certain Segal algebras has been the cause of interesting difference between the ideal structure of these algebras and that of $L^1(G)$, see [2], [6].)

THEOREM. *Every Segal algebra $S(G)$ satisfies condition* D.

Proof. Let $\gamma \in \hat{G}$ and $f \in S(G)$ such that $\hat{f}(\gamma) = 0$. Since $L^1(G)$ satisfies condition D, there exist a sequence $\{f_n\}$ in $L^1(G)$ and a sequence $\{V_n\}$ of open neighborhoods of γ such that $\hat{f}_n = 0$ on V_n and $\|f_n * f - f\|_1 \to 0$. By (IV), there exists $v_k \in S(G)$ such that $\|f * v_k - f\|_S < 1/k$ for $k = 1, 2, 3, \ldots$ By (III), we have $\|f_n * f * v_k - f * v_k\|_S \leqslant \|f_n * f - f\|_1 \|v_k\|_S \to 0$ as $n \to \infty$. Thus there is a subsequence $\{n_k\}$ of $1, 2, 3, \ldots$ such that $\|f_{n_k} * f * v_k - f * v_k\|_S < 1/k$ for $k = 1, 2, 3, \ldots$ Putting $g_k = f_{n_k} * v_k$, we have $g_k \in S(G)$, $\hat{g}_k = 0$ on V_{n_k} and $\|g_k * f - f\|_S \leqslant 2/k \to 0$ as $k \to \infty$. Thus $S(G)$ satisfies condition D at every point $\gamma \in \hat{G}$. Finally, by (IV), $S(G)$ also satisfies condition D at the point at infinity.

By virtue of the general Tauberian theorem in Loomis ([1], p. 86), the following result is now obvious.

COROLLARY. *Let I be a closed ideal in $S(G)$. Then I contains every element f in kernel (hull (I)) such that the intersection of the boundary of hull (f) with hull(I) contains no non-void perfect set.*

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Added in the proof. The reader may be interested in a related paper by R. Larsen, *A theorem concerning Ditkin's condition*, Port. Math. (to appear). A generalization of our result has been announced by J. T. Burnham in Notices Amer. Math. Soc. 17 (1970), p. 815.

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