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Distinguished subsets and summability invariants

by

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Abstract. Several subsets of the convergence domain of a matrix have been studied by various authors. The purpose of this paper is to continue this study and to answer some problems raised by Wilansky in [9] and Chang in [3].

1. Notation and preliminary ideas. We shall be concerned with matrix transformations $y = Ax$, where $x = \{x_j\}_{j=1}^{\infty}$ and $y = \{y_j\}_{j=1}^{\infty}$ are complex-valued sequences, $A = \{a_{ij}\}_{i,j=1}^{\infty}$ is an infinite matrix with complex coefficients and

$$y_i = \sum_{j=1}^{\infty} a_{ij} x_j \quad (i = 1, 2, \ldots).$$

ω denotes the space of all complex-valued sequences and any vector subspace of ω is called a *sequence space*. A sequence space E with a locally convex topology τ is a *K-space* if the inclusion map: $(E, \tau) \to \omega$ is continuous when ω is endowed with the topology of coordinate-wise convergence. If, in addition, τ is complete and metrizable, (E, τ) is an *FK-space*. The basic properties of *FK-spaces* may be found, for example, in [10] and [14]; in particular, we recall that a sequence space can have at most one FK-topology. The following FK-spaces will be important in the sequel:

m, the space of all bounded sequences;

c, the space of all convergent sequences;

c_0, the space of all null sequences;

l, the space of all absolutely summable sequences.

$c_A = \{x \in \omega\colon Ax \text{ exists and } Ax \in c\}$ is called the *convergence domain of* A. If $x \in c_A$ we write

$$\text{`}\lim_A x\text{'} \quad \text{in place of} \quad \text{`}\lim_{i\to\infty} \sum_{j=1}^{\infty} a_{ij} x_j.\text{'}$$

By Theorem 4.10 of [12], c_A can be topologized so that it becomes an FK-space and we assume throughout that c_A carries this topology. c_A' denotes the (continuous) dual of c_A. A is called *conservative* if $c \subseteq c_A$ and we deal exclusively with this type of matrix, though certain generalizations are possible. In particular, this means that

$$\lim_{i\to\infty} a_{ij} = a_j \quad \text{exists} \quad (j = 1, 2, \ldots).$$

Associated with A are the following '*distinguished*' *sets*, which play an important role in the theory of summability. (See [9]).

$$B = \Big\{x \in c_A\colon \sup_{i,n} \Big|\sum_{j=1}^{n} a_{ij}x_j\Big| < +\infty\Big\}.$$

$$F = \Big\{x \in c_A\colon \sum_{j=1}^{\infty} x_j f(e^j) \text{ is convergent, for each } f \in c_A'\Big\},$$

where e^j denotes the sequence $(0, \ldots, 0, 1, 0, \ldots)$ with the 'one' in the jth position.

$$W = \{x \in c_A\colon f(x) = \sum_{j=1}^{\infty} x_j f(e^j), \text{ for each } f \in c_A'\}.$$

$$S = \{x \in c_A\colon x = \sum_{j=1}^{\infty} x_j e^j\}.$$

$$L = \{x \in c_A\colon (tA)x = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} t_i a_{ij} x_j \text{ exists, for each } t \in l\}.$$

$P = \{x \in c_A\colon (tA)x = t(Ax)$, for each $t \in l$ which is such that $(tA)y$ exists for each $y \in c_A\}$.

$$I = \{x \in c_A\colon \sum_{j=1}^{\infty} a_j x_j \text{ is convergent}\}.$$

On I we define the linear functional Λ, given by

$$\Lambda(x) = \lim_A x - \sum_{j=1}^{\infty} a_j x_j \quad (x \in I).$$

$$\Lambda^{\perp} = \{x \in c_A\colon \Lambda(x) = 0\}.$$

Letting e denote the sequence $(1, 1, \ldots)$, A is called *conull* if $e \in \Lambda^{\perp}$ or *coregular* if $e \notin \Lambda^{\perp}$. If $I = c_A$, then A is said to have *maximal inset*.

When the dependence on A is in doubt we shall write B_A, F_A, etc. Given any distinguished subset X of c_A, we define the *internal* and *external partners* of X respectively as follows:

$$X_{\text{int}} = \bigcap \{X_D\colon c_D = c_A\};$$

$$X_{\text{ext}} = \bigcup \{X_D\colon c_D = c_A\}.$$

If $X_{\text{int}} = X_{\text{ext}}$, then X is said to be an *invariant subset* of c_A; clearly, X_{int} and X_{ext} provide examples of such sets. If X is an invariant subset of c_A we try to find an invariant expression for it, that is, one in terms of c_A rather than of A. It may then be possible to associate the property 'ϵX' with an arbitrary FK-space. As Wilansky [9] remarks, F, W and S are all defined in terms of the topology of c_A and so are invariant subsets — their associated properties being FAK, SAK and AK respectively [13].

From Theorems 5.1, 5.2 and 5.4 of [9], we have the following inclusion relationships:

$$(*) \qquad \left.\begin{array}{c} c_0 \subseteq S \subseteq W = B \cap \Lambda^{\perp} \\ c \subseteq c_A \cap m \end{array}\right\} \subseteq F = I_{\text{int}} = L \cap I \subseteq B = L \subseteq P.$$

The matrix A is said to be *multiplicative* $(-\alpha)$ if, for some scalar α,

$$\lim_A x = \alpha \lim_{j\to\infty} x_j \quad (x \in c).$$

It turns out ([9], p. 329) that A is multiplicative if and only if $a_j = 0$, $j = 1, 2, \ldots$, and that the only possible value for α is $\Lambda(e)$. If there exists a multiplicative matrix D with $c_D = c_A$, then A is said to be *replaceable*. The existence of non-replaceable matrices is not at all obvious. Zeller gave the first example, which appears in [8]; others can be found in [4], [6]. For replaceable matrices A, it is clear that $I_{\text{ext}} = c_A$ and we shall use this fact to derive a new source of non-replaceable matrices.

2. The problems and their solutions. In connection with the foregoing ideas Wilansky [9] poses several problems, listed here for convenience.

(I) Must $L = F$, if A is coregular?

(II) If F is closed, must L be?

(III) Is L invariant?

(IV) Must there exist a matrix D with $c_D = c_A$ and $I_D = F_A$?

(V) If A has maximal inset, must A be replaceable?

(VI) Is $\Lambda^{\perp}$ invariant?

(VII) Can $\sum_{j=1}^{\infty} a_j x_j$ be bounded and divergent if A is coregular?

(VIII) Is P invariant?

(IX) Must $I_{\text{ext}} = c_A$?

(X) It is conjectured that if I is invariant for a certain matrix A, then $I_{\text{int}} = c_A$.

(XI) Must I be closed if Λ is continuous?

(XII) Must a conull matrix be replaceable?

The purpose of this section is to answer all twelve of these problems, save for (VIII), where only partial solutions are given. (XII) has already been solved by Chang, MacPhail, Snyder and Wilansky in [4] — though their solution leads to further problems, raised by Chang in [3]. In order to answer these additional questions, we present a different solution to (XII).

Our first result settles (II) affirmatively.

PROPOSITION 1. *If either F or L is closed, then $F = L$ and both are closed.*

Proof. That $L = F$ whenever L is closed is just Lemma 5.6 of [9]. To prove the other part suppose, if possible, that F is closed and $F \neq L$. Then, by (*), $\bar{c}_0$ (the closure of c_0 in c_A) $\subseteq F \nsubseteq L$. Thus c_0 is not dense in L and Lemma 5.5 of [9] is contradicted.

The next result shows that B is invariant, thus providing an affirmative solution to (III) by (*). We note here that J. J. Sember has discovered this result independently ([7], p. 442).

PROPOSITION 2. *B is invariant.*

Proof. It is easy to check that $x \in B$ if and only if x has *bounded sections* in c_A; that is, the set $\{\sum_{j=1}^{n} x_j e^j \colon n = 1, 2, \ldots\}$ is bounded in c_A. Thus B, being defined in terms of the (unique) FK-topology on c_A, must be invariant.

COROLLARY 1. $B \cap I_{\mathrm{ext}} = F$.

Proof.
$$\begin{aligned} B \cap I_{\mathrm{ext}} &= B_A \cap \bigcup \{I_D \colon c_D = c_A\} \\ &= \bigcup \{B_D \cap I_D \colon c_D = c_A\}, \quad \text{by Proposition 2} \\ &= F, \quad \text{by } (*). \end{aligned}$$

COROLLARY 2. *If $I_{\mathrm{ext}} = c_A$, then $B = F$. In particular, if A is replaceable, then $B = F$.*

Proof. Direct application of Corollary 1 and the remarks made earlier on replaceable matrices.

The next result gives an affirmative solution to (V).

PROPOSITION 3. *If A has maximal inset, then A is replaceable.*

Proof. Consider the matrix D derived from A as follows:

$$D = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdot \\ a_{11}-a_1 & a_{12}-a_2 & a_{13}-a_3 & \cdot \\ a_{21}-a_1 & a_{22}-a_2 & a_{23}-a_3 & \cdot \\ a_{31}-a_1 & a_{32}-a_2 & a_{33}-a_3 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

Since A has maximal inset, it is clear that $c_D = c_A$ and

$$\lim_D x = \lim_A x - \sum_{j=1}^{\infty} a_j x_j \qquad (x \in c_A).$$

If $x \in c$, it follows from [9], p. 329 that

$$\lim_A x = \Lambda_A(e) \lim_{j\to\infty} x_j + \sum_{j=1}^{\infty} a_j x_j \qquad (x \in c),$$

and so

$$\lim_D x = \Lambda_A(e) \lim_{j\to\infty} x_j \qquad (x \in c).$$

Thus D is multiplicative and A replaceable.

We now give a matrix which provides solutions to (I), (VII) and (IX).

EXAMPLE 1.

$$\text{Let } A = \begin{bmatrix} 1 & 0 & \cdot \\ 1 & \frac14 & 1 & 0 & \cdot \\ 1 & \frac14 & 0 & \frac19 & \frac12 & \frac12 & 0 & \cdot \\ 1 & \frac14 & 0 & \frac19 & 0 & 0 & \frac1{16} & 1 & 0 & \cdot \\ 1 & \frac14 & 0 & \frac19 & 0 & 0 & \frac1{16} & 0 & \frac1{25} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix};$$

then A is coregular. The sequence

$$x = (1, -4, 1, 9, 1, -1, -16, 1, 25, 1, -1, \ldots)$$

is a member of c_A, yet

$$\sum_{j=1}^{\infty} a_j x_j$$

is bounded and divergent, so that (VII) is answered affirmatively. It is easy to check that x has bounded sections so that $x \in B \setminus I$. Thus, by (*) and Corollary 2 to Proposition 2, $L \setminus F$ is non-empty and $I_{\mathrm{ext}} \neq c_A$, so that (I) and (IX) are also answered negatively. It is interesting to note that A is not replaceable.

Next, we give a counter-example to (IV).

EXAMPLE 2.

$$\text{Let } A = \begin{bmatrix} \frac12 & \frac12 & 0 & \cdot \\ 0 & 0 & \frac12 & \frac12 & 0 & \cdot \\ 0 & 0 & 0 & 0 & \frac12 & \frac12 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix};$$

that is,

$$a_{ij} = \begin{cases} \frac{1}{2}, & \text{if } j = 2i-1 \text{ or } j = 2i, \\ 0, & \text{otherwise} \end{cases} \qquad (i, j = 1, 2, \ldots).$$

Letting D be any matrix with $c_D = c_A$, we show that $I_D \neq F$. Since D is conservative, $d \in l$ where $d = \{d_j\}_{j=1}^{\infty}$ denotes the sequence of column limits of D. Thus we can choose a strictly increasing sequence, $\{j_n\}_{n=1}^{\infty}$, of *odd* positive integers so that

$$|d_j| \leqslant 2^{-j} \quad \text{whenever} \quad j \geqslant j_n.$$

Define the sequence x by

$$x_j = \begin{cases} n, & \text{if } j = j_n \text{ for some positive integer } n, \\ -n, & \text{if } j = j_n + 1 \text{ for some positive integer } n, \\ 0, & \text{otherwise.} \end{cases} \qquad (j = 1, 2, \ldots).$$

Then clearly $x \in c_A \setminus B_A = c_D \setminus B_D$, by Proposition 2. Furthermore, it is easy to check that $x \in I_D$ so that $x \in I_D \setminus B_D$, which, together with (*), gives $I_D \neq F$.

For $a \in \omega$, we define a^β, a^γ as follows:

$$a^\beta = \Big\{x \in \omega: \quad \sum_{j=1}^{\infty} a_j x_j \text{ converges}\Big\};$$

$$a^\gamma = \Big\{x \in \omega: \sup_n \Big|\sum_{j=1}^{n} a_j x_j\Big| < +\infty\Big\}.$$

The next result is just Theorem 9, (i) (a) → (ii) (a), of [5].

LEMMA. *For* $a, b \in \omega$, $a^\gamma \subseteq b^\beta$ *whenever* $a^\beta \nsubseteq b^\beta$.

Our next matrix provides counter-examples to (IX), (X), (XI) and (XII).

EXAMPLE 3.

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 0 & . & & & & & \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & . & & & \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & . & \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & . \\ . & . & . & . & . & . & . & . & . \end{bmatrix};$$

then A is conull. Let $x = (1, -1, 2, -2, 4, -4, \ldots)$; then $x \in B \setminus I$ and, as in Example 1, it follows that $I_{\text{ext}} \neq c_A$ and that A is not replaceable. Thus (IX) and (XII) are answered.

Clearly,

$$I = \{x \in \omega: x \in (1, 1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, \ldots)^\beta\}$$

and, if

$$p(x) = \sup_i \Big|\sum_{j=1}^{\infty} a_{ij} x_j\Big|,$$

p is a continuous seminorm on c_A ([12], Theorem 4.10). Since

$$\Big|\sum_{j=1}^{\infty} a_j x_j\Big| = \Big|\sum_{j=1}^{\infty} \frac{x_{2j-1} + x_{2j}}{2^{j-1}}\Big| \leqslant \sup_i \Big|\sum_{j=1}^{i} \frac{x_{2j-1} + x_{2j}}{2^{j-1}}\Big| = p(x),$$

it follows that the mapping:

$$x \to \sum_{j=1}^{\infty} a_j x_j \qquad (x \in I)$$

is continuous on I. By Theorem 4.4 (c) of [12], the mapping:

$$x \to \Lambda(x) = \lim_A x - \sum_{j=1}^{\infty} a_j x_j \qquad (x \in I)$$

is also continuous on I. Now

$$B = \{x \in c_A: x \in (1, 1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, \ldots)^\gamma\}$$

and so, by (*), $F = I$. It follows from Proposition 1, (*), and the fact that $B \neq F$ that I is not closed in c_A. Thus (XI) is also answered.

Finally, we show that I is invariant. This, together with the fact that $B \neq F$, contradicts the conjecture (X). To do this, let D be any matrix with $c_D = c_A$ and suppose, if possible, that $I_D \neq I_A$. Then, since $I_A = F$, we must have $I_D \nsupseteq I_A$ by (*). This gives

$$d^\beta \supsetneqq c_D \cap d^\beta = I_D \supsetneqq I_A = a^\beta$$

and so, using the lemma,

$$d^\beta \supseteq a^\gamma.$$

Thus $I_D = c_D \cap d^\beta \supseteq c_A \cap a^\gamma = B$, which contradicts Corollary 1 to Proposition 2 since $B \neq F$. It follows that $I_D = I_A$ so that I is invariant.

PROPOSITION 4. *If* I *is invariant, then so is* $\Lambda^\perp$ *and* $\Lambda^\perp = W$.

Proof. If I is invariant, then $I = F$ and the conclusion follows from (*).

For coregular matrices the converse result is true and this leads to a counter-example to (VI).

PROPOSITION 5. *If A is a coregular matrix, then $A^{\perp}$ is invariant if and only if I is invariant.*

Proof. $A^{\perp}$ is a hyperplane in I and so $I = A^{\perp}+\{e\}$ for coregular matrices. Coregularity is an invariant property of c_A ([9], p. 337) and so therefore is the above equation.

It is easy to construct a coregular matrix for which I is not invariant (Example 2) and any such matrix provides a counter-example to (VI). I have been unable to decide whether Proposition 5 holds for all conservative matrices.

To answer (XII), Chang, MacPhail, Snyder and Wilansky [4] consider two classes of matrices introduced by Yurimyae [11]. A is said to belong to the *class* J if c is dense in $m \cap c_A$ in the subspace topology inherited from c_A. A belongs to the *class* 0 if, for every matrix D with $c_D \supseteq m \cap c_A$ and $\lim_D = \lim_A$ on c, we have $\lim_D = \lim_A$ on $m \cap c_A$. It is shown in [4] that $J \setminus 0$ is non-empty and that, if $A \in J\setminus 0$, then A is not replaceable. On the other hand, Chang [3] points out that if $A \notin J$, then A is automatically replaceable by Theorem 9.1 of [9] and (*). Chang goes on to ask what can be said about the replaceability of A when $A \in 0$. We point out that the zero matrix and, by Theorem 3 of [4], the matrix of Example 3 both belong to 0. The former is replaceable, whereas the latter is not.

The only remaining question from Wilansky's list is (VIII), and we close this section by presenting some partial solutions. First we observe, using Theorem 6.3 of [9], that if A is coregular, then P is invariant and has the invariant expression $P = \bar{c}$. Also, by Theorem 9.1 of [9], if A is not replaceable, then P is again invariant and has the invariant expression $P = \bar{c}_0$. Thus, to answer (VIII), we need only consider multiplicative-0 matrices. The next result enables us to restrict this class even further.

PROPOSITION 6. *If $A^{\perp} \nsupseteq B$, then $\bar{B} = P$.*

Proof. Theorem 6.3 of [9] and (*) show that $\bar{B} \subseteq P$ for any matrix. To establish the converse we let $f \in c_A'$ with f vanishing on B. By (4), p. 330 of [9], f has a representation of the form:

$$f(x) = a\lim_A x + t(Ax) - \sum_{j=1}^{\infty}\Big(a\, a_j + \sum_{i=1}^{\infty} t_i a_{ij}\Big) x_j \qquad (x \in c_A),$$

for some scalar a and some $t \in l$. By Lemma 4.1 of [9] and (*),

$$f(x) = a\Lambda(x) \qquad (x \in \dot{B}).$$

Since $A^{\perp} \nsupseteq B$, $a = 0$ and so

$$f(x) = t(Ax) - (tA)x \qquad (x \in c_A).$$

It follows that $f = 0$ on P and then, by the Hahn–Banach theorem, that $\bar{B} = P$.

COROLLARY 1. *If $B \neq W$, then P is invariant.*

Proof. This follows from Propositions 2 and 6 since, by (*), $B = W$ if and only if $B \subseteq A^{\perp}$.

COROLLARY 2. *If c_0 is not dense in B, then P is invariant.*

Proof. This follows from Lemma 5.5 of [9], Corollary 1 and (*).

As a special case of Corollary 2, we note that, if $A \notin J$, then P is invariant. Though the invariance of P is left in doubt, the set P_{int} is certainly invariant and is closed by Theorem 6.3 of [9]. It would be interesting to have an invariant expression for P_{int}.

3. Some new summability invariants. The following result is established in [1].

THEOREM. *A subset K of c_A is relatively compact if and only if the following conditions hold:*

(i) $\sup_{x \in K} |x_j| < +\infty \qquad (j = 1, 2, \ldots);$

(ii) $\sup_{x\in K} \sup_{n} \Big|\sum_{j=1}^{n} a_{ij}x_j\Big| < +\infty \qquad (i = 1, 2, \ldots);$

(iii) $\lim_{n\to\infty} \sup_{x\in K} \Big|\sum_{j=n}^{\infty} a_{ij}x_j\Big| = 0 \qquad (i = 1, 2, \ldots);$

(iv) $\sup_{x\in K} \sup_{i} \Big|\sum_{j=1}^{\infty} a_{ij}x_j\Big| < +\infty;$

(v) $\lim_{i\to\infty} \sup_{x\in K} \Big|\sum_{j=1}^{\infty} a_{ij}x_j - \lim_A x\Big| = 0.$

It follows that a subset K of c_A with properties (i)–(v) must also have (i)–(v) in terms of any matrix D with $c_D \supseteq c_A$.

In [2] it is shown that

$$\lim_{j\to\infty} \sup_{i} |a_{ij}| = 0$$

is an invariant property of c_A, and this leads us to study the socalled wedge spaces.

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Every Segal algebra satisfies Ditkin's condition

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Abstract. The purpose of this note is to prove the assertion stated in the title. As an easy corollary we obtain the Shilov–Wiener Tauberian theorem for *all* Segal algebras. Warner [5] and the author [6] have obtained special cases of these theorems.

Let G be a locally compact Abelian group with dual group $\check{G}$. Following Reiter ([3], p. 126), a subalgebra $S(G)$ of $L^1(G)$ is called a *Segal algebra* if:

(i) $S(G)$ is dense in $L^1(G)$ in the L^1-norm topology and if $f \in S(G)$ then $f_a \in S(G)$, where $f_a(x) = f(a^{-1}x)$;

(ii) $S(G)$ is a Banach algebra under some norm $\|\cdot\|_S$ which also satisfies $\|f\|_S = \|f_a\|_S$ for all $f \in S(G)$, $a \in G$ (multiplication in $S(G)$ is the usual convolution);

(iii) if $f \in S(G)$, then for any $\varepsilon > 0$ there exists a neighborhood U of the identity element of G such that $\|f_y - f\|_S < \varepsilon$ for all $y \in U$.

Throughout the rest of this note, $S(G)$ will denote an arbitrary Segal algebra. The following facts, which are needed in the sequel, can be found in Reiter ([3], p. 128):

(I) There exists a constant C such that $\|f\|_1 \leqslant C\|f\|_S$ for all $f \in S(G)$.

(II) $S(G)$ is an ideal in $L^1(G)$ and $\|h * f\|_S \leqslant \|h\|_1 \|f\|_S$ for all $f \in S(G)$, $h \in L^1(G)$.

(III) If $f \in L^1(G)$ and the Fourier transform $\hat{f}$ has compact support, then $f \in S(G)$.

(IV) Given any $f \in S(G)$ and $\varepsilon > 0$, there is a $v \in S(G)$ such that $\hat{v}$ has compact support and $\|v * f - f\|_S < \varepsilon$.

LEMMA. *The maximal ideal space Δ of $S(G)$ can be identified with the dual group $\hat{G}$.*

Proof. (II) implies that $\lim\limits_{n\to\infty} \|f^n\|_S^{1/n} \leqslant \|f\|_1$ for each $f \in S(G)$. Now if $\gamma \in \Delta$, then we have $|\gamma(f)|^n = |\gamma(f^n)| \leqslant \|f^n\|_S$ and hence γ is L^1-norm