STUDIA MATHEMATICA T. XL. (1971)

Linear operators and operational calculus, Part I

by

GREGERS KRABBE (Lafayette, Ind.)

Abstract. To any open interval $\tilde{\omega}$ (containing the origin) there corresponds a linear injection of the space $L^{\text{loc}}(\tilde{\omega})$ (of all the functions which are locally integrable on the interval $\tilde{\omega}$) into a commutative algebra of linear operators; this injection is a useful substitute for the Fourier transformation.

The present paper deals with a commutative algebra of generalized functions on a sub-interval $(\tilde{\omega}_-, \tilde{\omega}_+)$ of the real line (we suppose that $-\infty \leqslant \tilde{\omega}_- < 0 < \tilde{\omega}_+ \leqslant \infty$). This algebra contains all functions which are locally integrable on the open interval $(\tilde{\omega}_-, \tilde{\omega}_+)$; in consequence, equations such as

$$y(t) + \int_t^0 e^{t-u} y(u)\,du = \frac{e^{2t}}{(\lambda - t)^3}$$

and

$$y''(t) + 9\ y(t) = \sec\frac{\pi t}{2\lambda} \qquad (\text{for } -\lambda < t < \lambda)$$

can be solved by calculations entirely similar to the ones that would arise if the Carson–Laplace transformation could be applied to these equations (see 2.29–2.30, 2.40, and 4.9); in some cases, the calculations are shorter than the ones that would arise if Mikusiński's calculus (or the Laplace transformation) were applicable: see 2.41. The unique theorem in this paper depends neither on Titchmarsh's theorem nor on Lerch's theorem.

In case $(\tilde{\omega}_-, \tilde{\omega}_+)$ is the whole real line $(-\infty, \infty)$, our algebra yields an operational calculus which is a definite improvement compared to the one described in [5]; the present operational calculus is a useful substitute for the two-sided Laplace transformation (or the Fourier transformation) of generalized functions: no growth conditions are needed.

Organization of the paper. The only theorem is stated in § 1; some of its consequences are sketched in §§ 2–4. In fact, § 2 deals with algebraic consequences; § 3 concerns limits, derivatives, and the unit impulse; § 4 is devoted to partial differential equations, and § 5 contains the proof of the theorem.

§ 1. The theorem. Throughout, we suppose $-\infty \leqslant \tilde{\omega}_- < 0 < \tilde{\omega}_+ \leqslant \infty$ and denote by $\tilde{\omega}$ the open interval $(\tilde{\omega}_-, \tilde{\omega}_+)$. Let $L^{\mathrm{loc}}(\tilde{\omega})$ be the family of all the complex-valued functions which are Lebesgue integrable on each interval (a, b) with $\tilde{\omega}_- < a < 0 < b < \tilde{\omega}_+$. If $f(\ )$ and $g(\ )$ belong to $L^{\mathrm{loc}}(\tilde{\omega})$, we denote by $f \wedge g(\ )$ the function defined by the equations

$$(1.1) \qquad -\int_t^0 f(t-u)g(u)\,du = f \wedge g(t) = \int_0^t f(t-u)g(u)\,du$$

for almost-all values of t in $\tilde{\omega}$: the first equation is used in case $t < 0$. It can be proved that

$$(1.2) \qquad f \wedge g(\ ) \in L^{\mathrm{loc}}(\tilde{\omega}).$$

1.3. Definition. *Let $W_{\tilde{\omega}}$ be the linear space of all the complex-valued functions $w(\ )$ which are infinitely differentiable on the open interval $\tilde{\omega}$ and are such that $w(0) = 0 = w^{(k)}(0)$ for every integer $k \geqslant 1$. As usual, $w^{(k)}(\ )$ denotes the k-th derivative of the function $w(\ )$.*

1.4. Equivalence of functions. If $f_1(\ )$ and $f_2(\ )$ are functions, the equation $f_1(\ ) = f_2(\ )$ will mean that these functions are equal almost-everywhere on the open interval $\tilde{\omega}$.

1.5. Remark. Suppose that $f(\ )$ and $g(\ )$ belong to $L^{\mathrm{loc}}(\tilde{\omega})$; it is easily verified that

$$(1.6) \qquad f \wedge g(\ ) = g \wedge f(\ ).$$

1.7. Notation and terminology. Henceforth, the word "*operator*" will indicate a linear mapping of $W_{\tilde{\omega}}$ into $W_{\tilde{\omega}}$. If A is an operator and if $w(\ ) \in W_{\tilde{\omega}}$, *we shall denote by $A \cdot w(\ )$ the function that the operator A assigns to $w(\ )$.*

As usual, the operator-product A_1A_2 (of two operators A_1 and A_2) is defined by

$$(1.8) \qquad A_1A_2 \cdot w(\ ) = A_1 \cdot (A_2 \cdot w)(\ ) \quad (\text{for } w(\ ) \text{ in } W_{\tilde{\omega}}),$$

and $A_1 = A_2$ means that $A_1 \cdot w(\ ) = A_2 \cdot w(\ )$ for every $w(\ )$ in $W_{\tilde{\omega}}$. The identity-operator I is defined by

$$(1.9) \qquad I \cdot w(\ ) = w(\ ) \quad (\text{for } w(\ ) \text{ in } W_{\tilde{\omega}}).$$

The space of generalized functions. *Let $\mathscr{A}_{\tilde{\omega}}$ be the family of all the operators A such that the equation*

$$(1.10) \qquad A \cdot (w_1 \wedge w_2)(\ ) = (A \cdot w_1) \wedge w_2(\ )$$

holds whenever both $w_1(\ )$ and $w_2(\ )$ belong to $W_{\tilde{\omega}}$.

It is easily verified that $\mathscr{A}_{\tilde{\omega}}$ is an algebra whose unit-element is the identity-operator I. The algebra $\mathscr{A}_{\tilde{\omega}}$ will be topologized in § 3. For further comments on $\mathscr{A}_{\tilde{\omega}}$, see 1.26.

1.11. Definition. *If $f(\ ) \in L^{\mathrm{loc}}(\tilde{\omega})$, we denote by f the mapping which assigns to each $w(\ )$ in $W_{\tilde{\omega}}$ the function $f \wedge w'(\ )$:*

$$(1.12) \qquad f \cdot w(\ ) = f \wedge w'(\ ) \quad (\text{for } w(\ ) \text{ in } W_{\tilde{\omega}}).$$

We call f the ***operator of the function*** *$f(\ )$; we shall often write*

$$(1.13) \qquad \{f(t)\} \quad \textit{instead of} \quad f.$$

1.14. Remark. The unit constant function $\mathbf{1}(\ )$ is defined by $\mathbf{1}(x) = 1$ for $-\infty < x < \infty$. Let us prove that

$$(1.15) \qquad \mathbf{1} = I.$$

To that effect, it suffices to note that (1.1) gives

$$(1.16) \qquad \mathbf{1} \wedge w'(t) = \int_0^t \mathbf{1}(t-u)w'(u)\,du = \int_0^t w';$$

consequently, $\mathbf{1} \wedge w'(t) = w(t) - w(0) = w(t)$ for $t \in \tilde{\omega}$, whence $\mathbf{1} \wedge w'(\ ) = w(\ )$. Conclusion: (1.15) is now immediate from (1.12) and (1.9).

1.17. Proposition. *If $f_1(\ )$ and $f_2(\ )$ belong to $L^{\mathrm{loc}}(\tilde{\omega})$, then*

$$(1.18) \qquad f_1(\ ) = f_2(\ ) \quad \textit{implies} \quad f_1 = f_2,$$

and the equation

$$(1.19) \qquad \{c_1f_1(t) + c_2f_2(t)\} = c_1f_1 + c_2f_2$$

holds for any two complex numbers c_1 and c_2.

Proof: immediate from (1.12) and (1.1).

1.20. The operator D. We denote by D the restriction to $W_{\tilde{\omega}}$ of the differentiation operator:

$$(1.21) \qquad D \cdot w(\ ) = w'(\ ) \quad (\text{for } w(\ ) \text{ in } W_{\tilde{\omega}}).$$

1.22. Theorem. *The algebra $\mathscr{A}_{\tilde{\omega}}$ is commutative, and $D \in \mathscr{A}_{\tilde{\omega}}$; moreover,*

$$(1.23) \qquad f(\ ) \in L^{\mathrm{loc}}(\tilde{\omega}) \quad \textit{implies} \quad f \in \mathscr{A}_{\tilde{\omega}},$$

and the two properties

$$(1.24) \qquad D\{f_1 \wedge f_2(t)\} = f_1f_2,$$

$$(1.25) \qquad f_1 = f_2 \quad \textit{implies} \quad f_1(\ ) = f_2(\ )$$

hold whenever $f_k(\ ) \in L^{\mathrm{loc}}(\tilde{\omega})$ for $k = 1,2$.

1.26. Comments. This is the only theorem in this paper; its proof is given in § 5. Equation (1.24) states that the operator-product of D and $\{f_1 \wedge f_2(t)\}$ (that is, the product of the operators D and $f_1 \wedge f_2$: see (1.13)) equals the product of the operators f_1 and f_2. From (1.23), (1.25), and 1.17 it follows that the mapping $f(\ ) \mapsto f$ *is a linear injection* of $L^{\mathrm{loc}}(\tilde{\omega})$ into $\mathscr{A}_{\tilde{\omega}}$; this is analogus to the linear injection of $L^{\mathrm{loc}}(\tilde{\omega})$ into the space $\mathscr{D}'(\tilde{\omega})$ of Schwartz distributions — *the injection of* $L^{\mathrm{loc}}(\tilde{\omega})$ *into* $\mathscr{A}_{\tilde{\omega}}$ *justifies our description of* $\mathscr{A}_{\tilde{\omega}}$ *as a space of generalized functions.*

§ 2. Elementary applications. Recall that $\mathbf{1}(\ )$ is the unit constant ($1 = \mathbf{1}(t)$ for $-\infty < t < \infty$). When c is a complex number, the equation $\{c\mathbf{1}(\mathrm{t})\} = cI$ comes from (1.19) and (1.13) and implies that $cI \in \mathscr{A}_{\tilde{\omega}}$ (in view of (1.23)). Since the correspondence $c \to cI$ is an algebraic isomorphism of the complex field $\boldsymbol{C}$ into $\mathscr{A}_{\tilde{\omega}}$, we shall not distinguish between a complex number c and the operator cI:

(2.1) $$c = cI = \{c\mathbf{1}(\mathrm{t})\} = c\mathbf{1} \quad (\text{for } c \in \boldsymbol{C});$$

in particular,

(2.2) $$1 = I = \{\mathbf{1}(\mathrm{t})\} = \mathbf{1}.$$

Suppose that $c_k \in \boldsymbol{C}$ for $k = 1,2$ and $f_1(\ ) \in L^{\mathrm{loc}}(\tilde{\omega})$; setting $f_2(\ ) = \mathbf{1}(\ )$ in (1.19), the equation

(2.3) $$\{c_1 f_1(\mathrm{t}) + c_2\} = c_1 f_1 + c_2$$

follows directly from (2.1). In view of (2.1) and the commutativity of the algebra $\mathscr{A}_{\tilde{\omega}}$, we see that

$$Ac = cA \quad (\text{for } A \in \mathscr{A}_{\tilde{\omega}} \text{ and every } c \text{ in } \boldsymbol{C}).$$

Substituting $f_1(\ ) = \mathbf{1}(\ )$ in (1.24), we obtain

(2.4) $$D\{\mathbf{1} \wedge f_2(\mathrm{t})\} = f_2 \quad (\text{for } f_2(\ ) \in L^{\mathrm{loc}}(\tilde{\omega})).$$

2.5. Notation. If $f(\ ) \in L^{\mathrm{loc}}(\tilde{\omega})$ is a function such that $|f^{(k)}(0-)| < \infty$ for $0 \leqslant k < m$, we set

(2.6) $$\partial^m f \stackrel{\mathrm{def}}{=} D^m f - \sum_{k=0}^{m-1} f^{(k)}(0-)D^{m-k}.$$

In particular, if $|f(0-)| < \infty$ then

(2.7) $$\partial f \stackrel{\mathrm{def}}{=} Df - f(0-)D.$$

One last definition:

(2.8) $$|t_+^0| = \begin{cases} 0 & \text{for } t \leqslant 0, \\ 1 & \text{for } t > 0; \end{cases}$$

the function $t \mapsto |t_+^0|$ is the Heaviside unit jump function. Note that Definition (2.7) gives

(2.9) $$\partial\{|\mathrm{t}_+^0|\} = D\{|\mathrm{t}_+^0|\};$$

we shall see in § 3 that $D\{|\mathrm{t}_+^0|\}$ corresponds to the unit impulse applied at the origin.

2.10. Proposition. *Suppose that* $f(\ )$ *is a function which is continuous on the open interval* $\tilde{\omega}$. *If* $f'(\ )$ *has at most countably-many discontinuities in each compact sub-interval of the open interval* $\tilde{\omega}$, *then*

(2.11) $$f'. = \partial f = Df - f(0-)D \quad (\text{if } f'(\ ) \in L^{\mathrm{loc}}(\tilde{\omega})).$$

Proof. From [4, p. 143] it follows that the equations

$$f(t) - f(0) = \int_0^t f' = \mathbf{1} \wedge f'(t)$$

hold for both $0 < t < \tilde{\omega}_+$ and $\tilde{\boldsymbol{\omega}}_- < t < 0$: the second equation is from (1.16); consequently,

(1) $$f(t) - f(0-) = \mathbf{1} \wedge f'(t) \quad (\text{for } t \in \tilde{\omega}).$$

From (1), (1.18), and (2.3) it follows that

(2) $$\{f(\mathrm{t})\} - f(0-) = \{\mathbf{1} \wedge f'(\mathrm{t})\};$$

multiplying by D both sides of (2), we obtain

(3) $$D\{f(\mathrm{t})\} - f(0-)D = D\{\mathbf{1} \wedge f'(\mathrm{t})\} = f':$$

the last equation is from (2.4). Conclusion (2.11) is immediate from (3), (1.13), and (2.7).

2.12. Definition. *For* $m \geqslant 1$ *let* $\mathscr{K}_m(\tilde{\omega})$ *be the family of all the functions* $y(\ )$ *such that* $y^{(m-1)}(\ )$ *is continuous on the open interval* $\tilde{\omega}$, *such that* $y^{(m)}(\ ) \in L^{\mathrm{loc}}(\tilde{\omega})$, *and such that* $y^{(m)}(\cdot)$ *has at most countably-many discontinuities in each compact sub-interval of the open interval* $\tilde{\omega}$.

2.13. Proposition. *If* $y(\ ) \in \mathscr{K}_m(\tilde{\omega})$ *then* $\partial^m y = y^{(m)}$.

Proof: by induction (based on 2.10).

2.14 Remarks. If $f(\ ) \in \mathscr{K}_1(\tilde{\omega})$, it follows immediately from 2.10 and (1.13) that

(2.15) $$\partial f = f' = \left\{\frac{\mathrm{d}}{\mathrm{dt}} f(\mathrm{t})\right\};$$

it is necessary that $f(\ )$ be continuous, since $\partial\{|\mathrm{t}_+^0|\} \neq \{\mathrm{d}|\mathrm{t}_+^0|/\mathrm{dt}\} = 0$. It will be shown in another paper that the operation $f(\ ) \mapsto \partial f$ corresponds to the distributional derivative (see 3.10).

2.16. Invertibility. An operator A is called *invertible* if $A \in \mathscr{A}_{\tilde{\omega}}$ and if there exists an operator X such that $AX = 1$. Suppose that A is invertible; since $\mathscr{A}_{\tilde{\omega}}$ is a commutative algebra, there exists exactly one operator A^{-1} such that $A^{-1} \in \mathscr{A}_{\tilde{\omega}}$ and $AA^{-1} = 1$.

Setting $f(t) = t$ in (2.11), we obtain $1 = Df$ (since $f'(\ ) = \mathbf{1}(\ )$). Consequently, D is invertible, and $D^{-1} = \{f(t)\}$; since $f(t) = t$ we can write

$$D^{-1} = \{t\} = \frac{1}{D}. \tag{2.17}$$

Substituting $y(t) = t^n/n!$ into 2.13, we can use (2.6) to obtain $D^n\{t^n/n!\} = \partial^n y = 1$, so that

$$D^{-n} = \left\{\frac{t^n}{n!}\right\} = \frac{1}{D^n}. \tag{2.18}$$

We may now multiply by D^{-1} both sides of (1.24) and use the commutativity of the algebra $\mathscr{A}_{\tilde{\omega}}$ to obtain

$$\left\{\int_0^t f_1(t-u) f_2(u)\,du\right\} = f_1 D^{-1} f_2. \tag{2.19}$$

Substituting $f_1(\ ) = \mathbf{1}(\ )$ into (2.19), we can use (2.2) to obtain

$$\left\{\int_0^t f_2(u)\,du\right\} = D^{-1} f_2. \tag{2.20}$$

2.21. Proposition. *Suppose that $Y \in \mathscr{A}_{\tilde{\omega}}$ and $V \in \mathscr{A}_{\tilde{\omega}}$. If the equation $VY = R$ holds for some invertible R, then V is invertible and $Y = R/V$, where R/V denotes RV^{-1}.*

Proof: easy; see 1.76 in [5].

2.22. Remarks. Let a be a complex number. The equations

$$a\{e^{at}\} = \left\{\frac{d}{dt} e^{at}\right\} = \partial\{e^{at}\} = D\{e^{at}\} - D$$

are from (2.15) and (2.7): consequently, $(D-a)\{e^{at}\} = D$; we can use 2.21 (with $R = D$) to solve this equation for $\{e^{at}\}$:

$$\{e^{at}\} = \frac{D}{D-a}. \tag{2.23}$$

Formulas (2.18) and (2.23) can be compared with the Laplace-transform formulas

$$\mathfrak{L}\left\{|t_+^0| \frac{t^n}{n!}\right\} = \frac{1}{s^{n+1}} \quad\text{and}\quad \mathfrak{L}\{|t_+^0| e^{at}\} = \frac{1}{s-a};$$

recall that the function $t \mapsto |t_+^0|$ is the Heaviside unit jump function. If $F(\ ) \in L^{\rm loc}(\tilde{\omega})$ the equations

$$\frac{1}{D-a} F = \frac{D}{D-a} D^{-1} F = \left\{\int_0^t e^{a(t-u)} F(u)\,du\right\} \tag{2.24}$$

are from (2.19) and (2.23). Setting $a = c$ (a complex number) and $F(t) = c$ in (2.24):

$$\frac{c}{D-c} = \{e^{ct} - 1\}. \tag{2.25}$$

Let us derive another formula: from (2.11) we see that

$$D^2\{\sin at\} = D\partial\{\sin at\} = aD\{\cos at\},$$

and another application of (2.11) now gives

$$D^2\{\sin at\} = a(-a\{\sin at\} + D):$$

solving this equation for $\{\sin at\}$:

$$\left\{\frac{\sin at}{a}\right\} = \frac{D}{D^2 + a^2}. \tag{2.26}$$

On the other hand, the equations

$$\{\cos at\} = \partial\left\{\frac{\sin at}{a}\right\} = D\left\{\frac{\sin at}{a}\right\} = \frac{D^2}{D^2 + a^2} \tag{2.27}$$

are from (2.15), (2.7), and (2.26). The equations

$$\frac{1}{D^2+a^2} f_2 = \frac{D}{D^2+a^2} D^{-1} f_2 = \left\{\int_0^t \frac{\sin a(t-u)}{a} f_2(u)\,du\right\} \tag{2.28}$$

are from (2.26) and (2.19).

2.29. An integral equation. Take $\lambda > 0$ and let $G(\ )$ be a function in $L^{\rm loc}(-\lambda, \lambda)$; for example, $G(t) = e^{2t}/(\lambda - t)^3$. Let us find a function $y(\ ) \in L^{\rm loc}(-\lambda, \lambda)$ such that

$$y(t) + \int_t^0 e^{t-u} y(u)\,du = G(t) \qquad (\text{whenever } |t| < \lambda);$$

if $y(\ )$ is such a function, it follows from the first equation in (1.1), from (1.18), and from (2.24) that

$$y - \frac{1}{D-1} y = G,$$

whence

$$y = \frac{D-1}{D-2}G = \left(1+\frac{1}{D-2}\right)G = G+\frac{1}{D-2}G,$$

and another application of (2.24) now gives

$$y(t) = G(t)+\int_0^t e^{2(t-u)}G(u)\,du \qquad (\text{for } -\lambda<t<\lambda):$$

this comes from (1.25). Throughout this problem and the next, we use $\tilde{\omega} = (-\lambda, \lambda)$.

2.30. An initial-value problem. Given two complex numbers c_0 and c_1, let us find a function $y(\ )$ in $L^{\mathrm{loc}}(-\lambda, \lambda)$ such that $y(0-) = c_0$, $y'(0-) = c_1$, and

$$\partial^2 y+9y = \left\{\sec\frac{\pi t}{2\lambda}\right\}.$$

We are again dealing the case $\tilde{\omega} = (-\lambda, \lambda)$. If $y(\ )$ is such a function, it follows from (2.6) that

$$(D^2+9)y = c_0D^2+c_1D+\left\{\sec\frac{\pi t}{2\lambda}\right\}.$$

Solving for y:

$$y = c_0\frac{D^2}{D^2+9}+c_1\frac{D}{D^2+9}+\frac{1}{D^2+9}\left\{\sec\frac{\pi t}{2\lambda}\right\};$$

from (2.27), (2.26), and (2.28) we see that

$$y(t) = c_0\cos 3t+c_1\frac{\sin 3t}{3}+\int_0^t\frac{\sin 3(t-u)}{3}\left(\sec\frac{\pi u}{2\lambda}\right)du$$

(for $-\lambda<t<\lambda$).

2.31. Translates. Henceforth, suppose that $0\leqslant a\leqslant\infty$ and $G(\ )\in L^{\mathrm{loc}}(\tilde{\omega})$. We set

(2.32) $$G^a(t) = \begin{cases} G(t-a) & \text{for } t>a, \\ 0 & \text{for } t\leqslant a. \end{cases}$$

In particular, $\mathbf{1}^a(\ )$ is the characteristic function of the open interval (a, ∞). In case $a\geqslant\tilde{\omega}_+$ we have $G^a(\ ) = 0 = \mathbf{1}^a(\ )$ (see 1.4), so that

(2.33) $$G^a = \mathbf{1}^a = \mathbf{1}^\infty = 0 \qquad \text{whenever} \qquad a\geqslant\tilde{\omega}_+.$$

2.34. Proposition. $\mathbf{1}^aG = G^a$.

Proof. In view of (2.33), $\mathbf{1}^aG = 0 = G^a$ in case $a\geqslant\tilde{\omega}_+$. It only remains to consider the case $a<\tilde{\omega}_+$. To begin with, observe that

(1) $$\mathbf{1}^a\wedge G(t) = 0 = G^a\wedge\mathbf{1}(t) \qquad (\text{for } \tilde{\omega}_-<t<a):$$

this is easily verified. Next, suppose that $a<t<\tilde{\omega}_+$. From (1.1) and (2.32) it follows that

(2) $$\mathbf{1}\wedge G^a(t) = \int_a^t G(u-a)\,du = \int_0^{t-a}G(x)\,dx.$$

On the other hand,

(3) $$\mathbf{1}^a\wedge G(t) = \left(\int_0^{t-a}+\int_{t-a}^t\right)\mathbf{1}^a(t-u)G(u)\,du,$$

and, since

(4) $$\mathbf{1}^a(t-u)G(u) = \begin{cases} G(u) & \text{if } t-u>a, \\ 0 & \text{if } t-u\leqslant a, \end{cases}$$

we see that $\mathbf{1}^a(t-u)G(u) = 0$ if $u\geqslant t-a$; Equations (3)–(4) now imply

(5) $$\mathbf{1}^a\wedge G(t) = \int_0^{t-a}G(u)\,du \qquad (\text{for } a<t<\tilde{\omega}_+).$$

We can now combine (5), (2), and (1) to obtain

(6) $$\mathbf{1}^a\wedge G(t) = G^a\wedge\mathbf{1}(t) = \mathbf{1}\wedge G^a(t):$$

the last equation is from (1.6). The equations

$$\mathbf{1}^aG = D\{\mathbf{1}^a\wedge G(\mathrm{t})\} = D\{\mathbf{1}\wedge G^a(\mathrm{t})\} = \mathbf{1}G^a = G^a$$

are from (1.24), (6), (1.24), and (2.2): this concludes the proof.

2.35. Remarks. Setting $G(\ ) = \mathbf{1}^x(\ )$ in 2.34, we obtain

(2.36) $$\mathbf{1}^a\mathbf{1}^x = \mathbf{1}^{a+x} \qquad (\text{for } 0\leqslant x\leqslant\infty).$$

Setting $a = 0$ in (2.32), we see that $G^0(\ )$ is the function which vanishes on the half-open interval $(-\infty, 0]$ and such that $G^0(t) = G(t)$ for $t>0$. Note that

(2.37) $$G^a(t) = G^0(t-a) \qquad (\text{for } t\in\tilde{\omega}).$$

From (2.8) it follows that

$$\mathbf{1}^0(t) = |t_+^0| = \begin{cases} 1 & \text{for } t>0, \\ 0 & \text{for } t\leqslant 0; \end{cases}$$

consequently,

(2.38) $$G^a(t) = |(t-a)_+^0|G(t-a) \qquad (\text{for } t\in\tilde{\omega}).$$

In view of our notation (1.13), we can use (2.38) to write 2.34 in the form

(2.39) $$\mathbf{1}^a\{G(\mathrm{t})\} = \{|(\mathrm{t}-a)_+^0|G(\mathrm{t}-a)\} \qquad (\text{in case } 0\leqslant a\leqslant\infty).$$

Formula (2.39) corresponds to the Laplace-transform identity

$$e^{-as}\mathscr{L}\{|t_+^0|G(t)\} = \mathscr{L}\{|(t-a)_+^0|G(t-a)\}$$

— and can be applied to the same type of problems.

2.40. For example, take $\tilde{\omega} = (-\infty, \lambda)$ and consider the equation

$$(7) \qquad \partial y - cy = \left\{\begin{matrix} c & \text{for } 0 < t < a \\ 0 & \text{when } -\infty < t < 0 \text{ or when } a < t < \lambda \end{matrix}\right\}:$$

here c is a given complex number. In view of (2.7), Equation (7) means that

$$(8) \qquad Dy - y(0-)D - cy = c(\mathbf{1}^0 - \mathbf{1}^a).$$

Let us solve (8) subject to the condition $y(0-) = 0$:

$$(9) \qquad y = (\mathbf{1}^0 - \mathbf{1}^a)\frac{c}{D-c} = (\mathbf{1}^0 - \mathbf{1}^a)\{e^{ct} - 1\}:$$

the last equation is from (2.25). From (9) we can use (2.39) and (1.25) to infer that

$$y(t) = |t_+^0|(e^{ct} - 1) - |(t-a)_+^0|(e^{ct}e^{-ca} - 1) \qquad (\text{for } -\infty < t < \lambda).$$

2.41. Concluding remarks. In Mikusiński's calculus [7–8], the equation (9) above would involve the ratio $c/s(s-c)$, which would have to be decomposed into partial fractions. Formulas such as

$$(2.42) \qquad \frac{\mathbf{1}^\lambda G}{(1 - c\mathbf{1}^a)^{m+1}} = \left\{\sum_{k=0}^{\infty} \frac{(k+m)!}{k!\,m!} c^k G^0(t - \lambda - ka)\right\}$$

(proved in [5, 11.58.1]) are useful to solve more complicated problems.

2.43. Proposition. *If $A \in \mathscr{A}_{\tilde{\omega}}$ and $v(\ ) \in W_{\tilde{\omega}}$, then $A\cdot v = Av$.*

Proof. Take any $w(\ )$ in $W_{\tilde{\omega}}$; the equations

$$(1) \qquad (A\cdot v)\cdot w(\ ) = (A\cdot v) \wedge w'(\ ) = A\cdot(v \wedge w')(\ ) = A\cdot(v\cdot w)(\ )$$

are from (1.12), (1.10), and (1.12). In view of (1.8), Equation (1) gives:

$$(A\cdot v)\cdot w(\ ) = Av\cdot w(\ ) \qquad (\text{for } w(\ ) \in W_{\tilde{\omega}});$$

the conclusion $A\cdot v = Av$ is now at hand.

2.44. Corollary. *If $0 \leqslant a \leqslant \infty$ then $\mathbf{1}^a$ is the translation operator:*

$$(2.45) \qquad \mathbf{1}^a\cdot v(\ ) = v^a(\ ) \qquad (\text{for each } v(\ ) \text{ in } W_{\tilde{\omega}}).$$

Proof. The equations $\mathbf{1}^a\cdot v = \mathbf{1}^a v = v^a$ are from 2.43 and 2.34; consequently, $\mathbf{1}^a\cdot v = v^a$, so that (1.25) implies our conclusion (2.45).

2.46. Remark. Suppose $0 \leqslant x \leqslant \infty$ and $w(\ ) \in W_{\tilde{\omega}}$. Combining (2.45) with (2.32), we obtain

$$(2.47) \qquad \mathbf{1}^x\cdot w(t) = \begin{cases} w(t-x) & \text{for } t > x, \\ 0 & \text{for } t \leqslant x. \end{cases}$$

§ 3. The topological space $\mathscr{A}_{\tilde{\omega}}$. Let us associate with the linear space $W_{\tilde{\omega}}$ the topology of pointwise convergence on the open interval $\tilde{\omega}$. Henceforth, the algebra $\mathscr{A}_{\tilde{\omega}}$ will be equipped with the topology of pointwise convergence on $W_{\tilde{\omega}}$ (this makes sense, since $\mathscr{A}_{\tilde{\omega}}$ is a space of mappings of the topological space $W_{\tilde{\omega}}$ into itself). Suppose that $B \in \mathscr{A}_{\tilde{\omega}}$ and let $(A_h)_{h\in J}$ be a *family of elements of* $\mathscr{A}_{\tilde{\omega}}$ (that is, a function of J into $\mathscr{A}_{\tilde{\omega}}$): the relation

$$(1) \qquad B = \lim_{h\to\lambda} A_h$$

means that

$$(2) \qquad B\cdot w(\ ) = \lim_{h\to\lambda} A_h\cdot w(\ ) \qquad (\text{for each } w(\ ) \text{ in } W_{\tilde{\omega}});$$

to simplify matters, we suppose that J is a subset of $(-\infty, \infty)$ having λ as an adherent point. The equivalence (1) $\Leftrightarrow$ (2) is an immediate consequence of the topology that $\mathscr{A}_{\tilde{\omega}}$ has been equipped with. We denote by

$$\lim_{h\to\lambda} A_h$$

the mapping that assigns to each $w(\ )$ in $W_{\tilde{\omega}}$ the function $B\cdot w(\ )$ defined by (2); consequently,

$$(3.1) \qquad (\lim_{h\to\lambda} A_h)\cdot w(\ ) = \lim_{h\to\lambda} A_h\cdot w(\ ) \qquad (\text{for } w(\ ) \in W_{\tilde{\omega}}).$$

It is easily seen that the topological space $\mathscr{A}_{\tilde{\omega}}$ is locally convex and Hausdorff; it can be proved [1] that multiplication is sequentially continuous and that $\mathscr{A}_{\tilde{\omega}}$ is topologically complete in the following sense: if $\lim A_h\cdot w(t)$ exists (as $h \to \lambda$) for every $t \in \tilde{\omega}$ and for every $w(\ ) \in W_{\tilde{\omega}}$, then the equation (2) defines an element B of $\mathscr{A}_{\tilde{\omega}}$.

3.2. Derivatives. Let J be an open sub-interval of $(-\infty, \infty)$. If $(F(x))_{x\in J}$ is a family of elements of $\mathscr{A}_{\tilde{\omega}}$, we set

$$(3.3) \qquad \frac{d}{dx}F(x) \overset{\text{def}}{=} \lim_{h\to 0}\frac{1}{h}\big(F(x+h) - F(x)\big);$$

in view of (3.1), this means that $dF(x)/dx$ is the operator defined by

$$(3.4) \qquad \left(\frac{d}{dx}F(x)\right)\cdot w(\ ) = \frac{\partial}{\partial x}\big(F(x)\cdot w(\ )\big) \qquad (\text{for } w(\ ) \in W_{\tilde{\omega}}).$$

[1] This has been done by Harris Shultz.

3.5. Proposition. *If* $0 \leqslant x < \infty$ *then*

$$\frac{d}{dx}\mathbf{1}^x = -\mathbf{1}^x D. \tag{3.6}$$

Proof. Take any $w(\ )$ in $W_{\tilde{\omega}}$. Setting $F(x) = \mathbf{1}^x$ in (3.4), we obtain

$$\left(\frac{d}{dx}\mathbf{1}^x\right)\cdot w(\ ) = \frac{\partial}{\partial x}\left(\mathbf{1}^x \cdot w(\ )\right). \tag{3}$$

Let us verify the equation

$$\frac{\partial}{\partial x}\left(\mathbf{1}^x \cdot w(t)\right) = -\mathbf{1}^x \cdot w'(t) \quad (\text{for } t \neq x). \tag{4}$$

Indeed, if $t < x$, then both sides of (4) equal zero (by (2.47)); if $t > x$ then

$$\frac{\partial}{\partial x}\left(\mathbf{1}^x \cdot w(t)\right) = \frac{\partial}{\partial x} w(t-x) = -w'(t-x) = -\mathbf{1}^x \cdot w'(t):$$

the first and last equations are both immediate from (2.47). Combining (3) and (4), we can use (1.21) and (1.8) to obtain

$$\left(\frac{d}{dx}\mathbf{1}^x\right)\cdot w(\ ) = -\mathbf{1}^x \cdot D \cdot w(\ ) = -\mathbf{1}^x D \cdot w(\ );$$

since $w(\ )$ is an arbitrary element of $W_{\tilde{\omega}}$. Conclusion (3.6) is at hand.

3.7. The unit impulse. In view of (3.3), Equation (3.6) can be written

$$\lim_{h\to 0} \frac{1}{h}\left(\mathbf{1}^{x+h} - \mathbf{1}^x\right) = -\mathbf{1}^x D,$$

which implies that

$$\lim_{\substack{h\to 0\\ h>0}} \frac{1}{h}\left(\mathbf{1}^x - \mathbf{1}^{x+h}\right) = D\mathbf{1}^x. \tag{5}$$

Since

$$\frac{1}{h}\left(\mathbf{1}^x - \mathbf{1}^{x+h}\right)(t) = \begin{cases} h^{-1} & \text{for } x < t < x+h, \\ 0 & \text{otherwise,} \end{cases}$$

we can re-write (5) as follows:

$$D\mathbf{1}^x = \lim_{\substack{h\to 0\\ h>0}} \left\{\begin{matrix} h^{-1} & \text{for } x < \mathrm{t} < x+h \\ 0 & \text{all other } \mathrm{t} \in \tilde{\omega} \end{matrix}\right\} = \partial \mathbf{1}^x: \tag{3.8}$$

the second equation is from (2.11). In consequence of (3.8), $D\mathbf{1}^x$ *represents the unit impulse* applied at the time $t = x$. Recall that $0 \leqslant x < \infty$ and

$$\mathbf{1}^x = \{|(\mathrm{t}-x)_+^0|\} \quad (\text{by } (2.38)). \tag{3.9}$$

3.10 The Dirac delta. The equation $\partial y = D\mathbf{1}^0$ governs the velocity y of a particle of unit mass subjected to a unit impulse applied at the time $t = 0$; if the initial velocity $y(0-) = -1$, then Definition (2.7) gives $Dy + D = D\mathbf{1}^0$, so that $y = -\mathbf{1} + \mathbf{1}^0$, which implies that $y(t) = 0$ for $t > 0$. Although this example is extremely simple, it illustrates the fact that the answer is given directly (without the need for another look at the problem): in this way it contrasts with the calculus described in [5] (the answer in [5, 2.50] is not obtained as automatically: it requires an additional step).

In case $\tilde{\omega}$ is the whole real line $(-\infty, \infty)$, it can be proved that the correspondence $f(\ ) \mapsto f$ (of $L^{\mathrm{loc}}(\tilde{\omega})$ into $\mathscr{A}_{\tilde{\omega}}$) can be extended to the space of all the distributions which are regular on the negative axis; under this extended correspondence, the Dirac distribution $\delta_x(\ )$ (concentrated at the point x) corresponds to $D\mathbf{1}^x$; it might be added that the distributional derivative corresponds to the operation $f(\ ) \mapsto \partial f$ defined in (2.7).

3.11 Application. When $c = 0$ the equation

$$\partial^4 y = m(\mathbf{1}^0 - \mathbf{1}^3) + cD\mathbf{1}^8 \tag{6}$$

governs the upwards deflection of a beam subjected to a uniform load of density m applied to the interval $(0, 3)$; when $c = 6$ the beam is also subjected to a load of magnitude 6 concentrated at the point $t = 8$ (compare with [7, p. 128] and [5, 6.68–6.86]). In case $m = 0$ and

$$y(0-) = y'(0-) = y^{(2)}(0-) = y^{(3)}(0-) = 0,$$

we see from Definition (2.6) that $\partial^4 y = D^4 y$, so that (6) gives

$$y = \frac{6}{D^3}\mathbf{1}^8 = \mathbf{1}^8\left\{\frac{6\mathrm{t}^3}{3!}\right\} = \{|(\mathrm{t}-8)_+^0|(\mathrm{t}-8)^3\}: \tag{7}$$

the last two equations are from (2.18) and (2.39). From (7) and (1.25) it follows that $y(t) = (t-8)^3$ when $8 < t < \tilde{\omega}_+$: observe that $y = 0$ when $\tilde{\omega}_+ \leqslant 8$.

§ 4. Partial differential equations. As before, J is an open sub-interval of $(-\infty, \infty)$; again as before, $0 < \tilde{\omega}_+ \leqslant \infty$, but from now on $\tilde{\omega}$ is the open interval $(0, \tilde{\omega}_+)$. Consider a complex-valued function $(x, t) \mapsto F(x, t)$ on the open rectangle $J \times \tilde{\omega}$: we shall denote by $F(x)(\ )$ the function defined on the open interval $\tilde{\omega}$ by

$$F(x)(t) = F(x, t) \quad (\text{for } x \in J \text{ and } 0 < t < \tilde{\omega}_+). \tag{4.1}$$

If $F(x)(\ )\in L^{\mathrm{loc}}(\tilde{\omega})$ for all $x\in J$, we set

(4.2) $$\{F(x, \mathrm{t})\} \overset{\mathrm{def}}{=} F(x),$$

where $F(x) = \{F(x)(\mathrm{t})\}$ is the operator of the function $F(x)(\ )$ (recall the definitions (1.12)–(1.13) with $f = F(x)$). From (4.2) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}\{F(x, \mathrm{t})\} = \frac{\mathrm{d}}{\mathrm{d}x}F(x):$$

the right-hand side is defined in (3.3). If the function $(x, t)\mapsto \partial F(x, t)/\partial x$ is continuous on the open rectangle $J\times\tilde{\omega}$, then

(1) $$\frac{\mathrm{d}}{\mathrm{d}x}\{F(x, \mathrm{t})\} = \left\{\frac{\partial}{\partial x}F(x, \mathrm{t})\right\} \quad (\text{for } x\in J):$$

this can be proved as in [5] (see 9.15.1 in [5]). Note that $\partial|(t-x)_+^0|/\partial x$ has no meaning when $x = t$, but

(2) $$\frac{\mathrm{d}}{\mathrm{d}x}\{|(\mathrm{t}-x)_+^0|\} = \frac{\mathrm{d}}{\mathrm{d}x}\{\mathbf{1}^x(\mathrm{t})\} = \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{1}^x = -D\mathbf{1}^x \quad (\text{for } x\geqslant 0):$$

see (3.9) and (3.6).

4.3. The time derivative. As before, we consider a complex-valued function $(x, t)\mapsto F(x, t)$ defined on the open rectangle $J\times\tilde{\omega}$, this function being such that $F(x)(\ )\in L^{\mathrm{loc}}(\tilde{\omega})$. For $x\in J$ we set

(4.4) $$\frac{\partial}{\partial \mathrm{t}}\{F(x, \mathrm{t})\} \overset{\mathrm{def}}{=} D\{F(x, \mathrm{t})\} - F(x, 0+)D,$$

and

(4.5) $$\left(\frac{\partial}{\partial \mathrm{t}}\right)^2\{F(x, \mathrm{t})\} \overset{\mathrm{def}}{=} D^2\{F(x, \mathrm{t})\} - F(x, 0+)D^2 - F'_{\mathrm{t}}(x, 0+)D,$$

where

(4.6) $$F'_{\mathrm{t}}(x, 0+) \overset{\mathrm{def}}{=} \lim_{\substack{t\to 0\\ t>0}} \frac{\partial}{\partial t}F(x, t).$$

If $x > 0$ and $F(x, t) = |(t-x)_+^0|$ then $F(x, 0+) = 0 = F'_{\mathrm{t}}(x, 0+)$, so that

(3) $$\frac{\partial}{\partial \mathrm{t}}\{|(\mathrm{t}-x)_+^0|\} = D\{|(\mathrm{t}-x)_+^0|\} = D\mathbf{1}^x:$$

the last equation is from (3.9). Again we remark that the ordinary derivative $\partial|(t-x)_+^0|/\partial t$ has no meaning when $t = x$. From (2) and (4.5) it follows that

(4.7) $$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2\mathbf{1}^x = \left(\frac{\partial}{\partial \mathrm{t}}\right)^2\mathbf{1}^x = D^2\mathbf{1}^x \quad (\text{for } 0 < x < \infty).$$

It is readily proved that the equation

(5) $$\frac{\partial}{\partial \mathrm{t}}\{F(x, \mathrm{t})\} = \left\{\frac{\partial}{\partial \mathrm{t}}F(x, \mathrm{t})\right\} \quad (\text{for } x\in J)$$

holds when $F(x)(\ )$ belongs to the family $\mathscr{K}_1(\tilde{\omega})$ (that was defined in 2.12); in particular, it holds when the function $(x, t)\mapsto \partial F(x, t)/\partial t$ is continuous on the open interval $\tilde{\omega}$.

4.8. Motivation. The rôle of the equations (1), (2), (3), and (5) is to justify utilizing the operations $\mathrm{d}/\mathrm{d}x$ and $\partial/\partial\mathrm{t}$ (defined for families of operators) rather than their classical counterparts.

4.9. Vibrating string. The equation

$$\left(\frac{\partial}{\partial \mathrm{t}}\right)^2\{U(x, \mathrm{t})\} = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2\{U(x, \mathrm{t})\} \quad (\text{for } 0 < x < l)$$

governs the vertical displacement $U(x, t)$ of a point with coordinates x and $U(x, t)$ at a time $t > 0$; the point lies on a string with end-points at $x = 0$ and at $x = l\leqslant\infty$. Using the notations defined in (4.1)–(4.2), the equation can be written

(1) $$\left(\left(\frac{\partial}{\partial \mathrm{t}}\right)^2 - \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2\right)U(x) = 0 \quad (\text{for } 0 < x < l):$$

recall that $U(x)$ is the operator of the function $U(x)(\ )$ defined by $U(x)(t) = U(x, t)$.

Let us solve the equation (1) subject to the initial conditions

(2) $$U(x, 0+) = 0 = U'_{\mathrm{t}}(x, 0+) \quad (0 < x < l)$$

and subject to the boundary conditions

(3) $$0 = U(0) \quad\text{and}\quad U(l) = G,$$

where G is the operator of a given function $G(\ )\in L^{\mathrm{loc}}(\tilde{\omega})$. From (4.5) we see that the equations (1)–(2) imply

(4) $$\left(D^2 - \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2\right)U(x) = 0 \quad (0 < x < l).$$

If X and Y belong to $\mathscr{A}_{\tilde{\omega}}$, it follows easily from (4.7) that the equation

(5) $$U(x) = X\mathbf{1}^x + Y\mathbf{1}^{l-x} \quad (0\leqslant x\leqslant l)$$

defines a solution of (4); if the initial conditions (2) are satisfied, then (5) implies (1). Let us determine the parameters X and Y to satisfy the boundary conditions: setting $x = 0$ and $x = l$ in (5), we obtain

(6) $$U(0) = X\,\mathbf{1}^0 + Y\mathbf{1}^l \quad\text{and}\quad U(l) = X\mathbf{1}^l + Y\mathbf{1}^0;$$

next, we substitute the boundary conditions (3) into (6) and use (2.36) to solve for X and Y:

$$X = \frac{-\mathbf{1}^{l}G}{\mathbf{1}^{0}-\mathbf{1}^{2l}} \quad \text{and} \quad Y = \frac{G}{\mathbf{1}^{0}-\mathbf{1}^{2l}};$$

substituting into (5):

$$U(x) = \frac{\mathbf{1}^{l-x}}{\mathbf{1}^{0}-\mathbf{1}^{2l}}G - \frac{\mathbf{1}^{l+x}}{\mathbf{1}^{0}-\mathbf{1}^{2l}}G,$$

from which (2.32) readily gives the answer

$$(7) \qquad U(x) = \Big\{\sum_{k=0}^{\infty}\big(G^{0}(\mathrm{t}-2kl-l+x)-G^{0}(\mathrm{t}-2kl-l-x)\big)\Big\}.$$

Equation (7) verifies the initial conditions (2); since it also satisfies (4), Conclusion (1) is an immediate consequence of our definition (4.5):

$$(4.10) \qquad \Big(\frac{\partial}{\partial \mathrm{t}}\Big)^{2} U(x) = D^{2}U(x) - U(x, 0+)D^{2} - U_{\mathrm{t}}'(x, 0+)D.$$

If the function $G(\ )$ is not continuous, then the solution $(x, t) \mapsto U(x, t) = U(x)(t)$ (defined by (7)) is not differentiable: the classical equation

$$\frac{\partial^{2}}{\partial x^{2}}U(x, t) = \frac{\partial^{2}}{\partial t^{2}}U(x, t)$$

has no meaning in this case.

4.11. A fundamental solution. Let p be a fixed complex number; if $0 \leqslant x \leqslant \infty$ we set $p_{1}^{x}(t) = [\exp(-pt)]\operatorname{cerf}(x/2\sqrt{t})$, where cerf denotes the complementary error function, and

$$(4.12) \qquad p_{2}^{x}(t) = \mathbf{1}^{x}(t) = \mathbf{1}^{x}(t)e^{-px} \qquad (\text{for } t \text{ in } \tilde{\omega}).$$

As usual, p_{m}^{x} denotes the operator of the function $t \mapsto p_{m}^{x}(t)$ defined on $\tilde{\omega}$. For $m = 1, 2$ it is not hard to verify that

$$(4.13) \qquad \Big((p+D)^{m} - \Big(\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big)^{2}\Big)p_{m}^{\lambda} = 0 \qquad (\text{for } 0 < \lambda < \infty).$$

4.14 A more general problem. Given a family $\big(h(x)(\ )\big)_{x\in J}$ of elements of $L^{\mathrm{loc}}(\tilde{\omega})$, and two families $\big(g(x)\big)_{x\in J}$ and $\big(G(x)\big)_{x\in J}$ of elements of $\mathscr{A}_{\tilde{\omega}}$; the index-set J is an open interval $J = (0, l)$ with $l \leqslant \infty$. Given $1 \leqslant m \leqslant 2$ and $a > 0$, consider the initial-value problem

$$(4.15) \qquad \Big(\Big(p+\frac{\partial}{\partial \mathrm{t}}\Big)^{m} - a^{-2}\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^{2}\Big)U(x) = h(x),$$

$$(4.16) \qquad U(x, 0+) = g(x),$$

$$(4.17) \qquad U_{\mathrm{t}}'(x, 0+) = G(x) \qquad (\text{in case } m = 2 \text{ only}).$$

In view of (4.4) and (4.10), this initial-value problem implies that

$$(4.18) \qquad \Big((p+D)^{m} - a^{-2}\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^{2}\Big)U(x) = R(x)D \qquad (\text{for } x\in J),$$

where

$$(4.19) \qquad R(x) = \frac{h(x)}{D} + (m-1)\big(g(x)D + G(x) + 2pg(x)\big) + (2-m)g(x).$$

Suppose that there exists a number c such that

$$(4.20) \qquad \Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^{2}R(x) = cR(x) \qquad (\text{for all } x \text{ in } J);$$

if $X(\ )$ and $Y(\ )$ are any two elements of $L^{\mathrm{loc}}(\tilde{\omega})$, it can be shown that the equation

$$(4.21) \qquad U(x) = \frac{R(x)D}{(p+D)^{m} - a^{-2}c} + Xp_{m}^{ax} + Yp_{m}^{al-ax} \qquad (\text{for } x\in J)$$

defines a solution of the initial-value problem (4.15)-(4.17): see 4.25. The parameters X and Y can be adjusted to make (4.21) satisfy the usual boundary conditions. Throughout, J denotes an open interval $(0, l)$ with $l \leqslant \infty$.

4.22. Case $l = \infty$. If $x\in J$, then $0 < x < l$ (since $J = (0, l)$); if $l = \infty$ then $al - ax = \infty$ in (4.21), so that

$$(1) \qquad Yp_{m}^{al-ax} = Yp_{m}^{\infty} = 0 \qquad (\text{for } x\in J);$$

the last equation is obtained by verifying that $p_{m}^{\infty} = 0$ for both $m = 1$ and $m = 2$ (in case $m = 2$ this is immediate from (2.33) and (4.12)).

From (1) and (4.21) it follows that, for any $X(\ )$ in $L^{\mathrm{loc}}(\tilde{\omega})$, the equation

$$(4.23) \qquad U(x) = \frac{R(x)D}{(p+D)^{m} - a^{-2}c} + Xp_{m}^{ax} \qquad (\text{with } 0 < x < \infty)$$

defines a solution of the initial-value problem (4.15)–(4.17) in case J is the open interval $(0, \infty)$. *Recall that the number c is determined by* (4.20); we can take $c = 0$ when there exist two operators A and B such that $R(x) = Ax + B$ for all x in J.

4.24. Application. Let us apply the above procedure to the initial-value problem

$$(2) \qquad \Big(\frac{\partial}{\partial \mathrm{t}}\Big)^{2}U(x) = \Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^{2}U(x) \qquad (0 < x < \infty)$$

with

$$(3) \qquad U(x, 0+) = \mathrm{e}^{-x} \quad \text{and} \quad U_{\mathrm{t}}'(x, 0+) = 0 \qquad (\text{for } 0 < x < \infty),$$

and subject to the boundary condition

(4) $$0 = \lim_{x\to 0} \frac{d}{dx} U(x).$$

From (1) and (4.15) we see that this is the case $p = 0$, $a = 1$, $m = 2$, and $h = 0$ of (4.15); since $l = \infty$ we conclude from (4.23) that the equation

(5) $$U(x) = \frac{R(x)D}{D^2-c} + X\mathbf{1}^x \quad (\text{with } 0 < x < \infty)$$

defines a one-parameter solution of the initial-value problem (2)–(3): the last term on the right-hand side was obtained by substituting $m = 2$ and $p = 0$ into (4.12). From (3) and (4.16)–(4.17) we see that $g(x) = \exp(-x)$ and $G(x) = 0$; substituting into (4.19) gives

$$R(x) = g(x)D = e^{-x}D;$$

consequently, (4.20) implies $c = 1$: Equation (5) becomes

(6) $$U(x) = \frac{e^{-x}D^2}{D^2-1} + X\mathbf{1}^x \quad (\text{with } 0 < x < \infty).$$

Let us determine the parameter X to satisfy the boundary condition (4); in view of (3.6), Equation (6) gives

$$\frac{d}{dx} U(x) = \frac{-e^{-x}D^2}{D^2-1} - D\mathbf{1}^x X;$$

combining with (4), we obtain

(7) $$\mathbf{1}^0 X = \frac{-D}{D^2-1} = -\frac{1}{2}\left(\frac{D}{D-1} - \frac{D}{D+1}\right) = \{-\sinh t\}:$$

the second equation is from (2.23). Since $\tilde{\omega} = (0, \tilde{\omega}_+)$, we have $\mathbf{1}^0 = \mathbf{1}$ and $\mathbf{1}^0 X = X$; substituting (7) into (6), we can use (2.39) to obtain the conclusion

$$U(x) = \{e^{-x}\cosh t - |(t-x)_+^0| \sinh (t-x)\}.$$

More precisely, given any $\tilde{\omega}_+ \leqslant \infty$, the equation

$$U(x, t) = e^{-x}\cosh t - |(t-x)_+^0| \sinh (t-x)$$

(with $0 < t < \tilde{\omega}_+$) clearly satisfies both the initial conditions (3); in consequence,

$$\left(\frac{\partial}{\partial t}\right)^2 \{U(x, t)\} = \left(\frac{d}{dx}\right)^2 \{U(x, t)\} \quad (\text{for } 0 < x < \infty).$$

4.25 Existence proof. The fact that (4.21) satisfies the initial-value problem (4.15)–(4.17) can be proved exactly as in [5]: here is a sketch of the calculations. It follows from (4.13) that the equation (4.21) implies (4.18). Consequently, if (4.21) also implies the initial conditions (4.16)–(4.17), then it satisfies the initial–value problem: to see that this is so, replace $g(x)$ (respectively, $G(x)$) in (4.19) by $U(x, 0+)$ (respectively, by $U_t'(x, 0+)$), and combine the result with (4.18); the definitions (4.10) and (4.4) now show that (4.15) has been obtained.

In short: the answer (4.21) can be verified by checking that it satisfies the initial conditions.

§ 5. Proof of the theorem. Let $e_t(\ )$ be the function defined by

(5.1) $$e_t(u) = \begin{cases} 1 & \text{for } 0 \leqslant u < t, \\ -1 & \text{for } t < u < 0, \end{cases}$$

and by $e_t(u) = 0$ for all other values of u. It will be convenient to denote by e_t the support of the function $e_t(\ )$; thus, e_t is the interval having end-points 0 and t. Observe that

(5.2) $$f \wedge g(t) = \int_{e_t} f(t-u)e_t(u)g(u)\,du \quad (\text{for } t \in \tilde{\omega}).$$

5.3. Definition. *For any integer $n \geqslant 1$ we denote by $q_n(\ )$ the function defined by $q_n(0) = 0$ and*

(1) $$q_n(t) = \exp\left(\frac{-1}{nt}\right) \quad (\textit{when } t \neq 0).$$

5.4. Proposition. *Suppose that $f(\ ) \in L^{\mathrm{loc}}(\tilde{\omega})$. If*

(2) $$f \wedge q_n(t) = 0 \quad \textit{for} \quad t \in \tilde{\omega} \quad \textit{and every} \quad n \geqslant 1,$$

then $f(\ ) = 0$.

Proof. From (2), (1.6), and (5.2) it follows that

$$0 = \lim_{n\to\infty} q_n \wedge f(t) = \lim_{n\to\infty} \int_{e_t} q_n(t-u)e_t(u)f(u)\,du;$$

since $|q_n(\ )| \leqslant 1$ (by (1)), we may apply the Lebesgue Dominated Convergence Theorem:

$$0 = \int_{e_t} \lim_{n\to\infty}\left(\exp\frac{-1}{n(t-u)}\right)e_t(u)f(u)\,du = \int_{e_t} e_t(u)f(u)\,du;$$

in view of (5.1), this means that

$$0 = \int_0^t f \quad (\text{for } t > 0), \quad \text{and} \quad 0 = -\int_t^0 f \quad (\text{for } t < 0),$$

which implies our conclusion $f(\ ) = 0$.

5.5. LEMMA. *Suppose that the functions* $f(\ )$, $g(\ )$, *and* $h(\ )$ *all belong to* $L^{\mathrm{loc}}(\tilde{\omega})$. *If the function* $|f|\wedge(|g|\wedge|h|)(\ )$ *is continuous on the open interval* $\tilde{\omega}$, *then*

(5.6) $$f\wedge(g\wedge h)(x) = (f\wedge g)\wedge h(x) \quad (\text{for } x\in\tilde{\omega}).$$

Proof. From (5.2) it follows that

(3) $$F\wedge(G\wedge H)(x) = \int_{e_x}\int_{e_t} F(x-t)G(t-u)H(u)\,du\,dt.$$

Since our hypothesis implies $|f|\wedge(|g|\wedge|h|)(x)<\infty$, it follows from (3) that

$$\int_{e_x}\int_{e_t} |f(x-t)g(t-u)h(u)|\,du\,dt<\infty;$$

we may therefore apply Tonelli's Theorem to write

(4) $$f\wedge(g\wedge h)(x) = \int_{e_x}\Big(\int_{x_u} f(x-t)g(t-u)\,dt\Big)h(u)\,du,$$

where x_u is an appropriate interval. Let us prove that

(5) $$f\wedge(g\wedge h)(x) = \int_0^x\Big(\int_u^x f(x-t)g(t-u)\,dt\Big)\,h(u)\,du$$

in case $x<0$ (the case $x>0$ is analogous): the double integral is taken over the triangular region

$$\{(u,t):\ x<t<0 \quad\text{and}\quad t<u<0\};$$

consequently, the range of t (in the integral (4)) is the interval $x_u=[x,u]$; the equation (4) becomes

$$f\wedge(g\wedge h)(x) = \int_x^0\Big(\int_x^u f(x-t)g(t-u)\,dt\Big)\,h(u)\,du,$$

which implies (5). The change of variable $v=t-u$ changes (5) into

$$f\wedge(g\wedge h)(x) = \int_0^x\Big(\int_0^{x-u} f(x-u-v)g(v)\,dv\Big)\,h(u)\,du;$$

consequently, (1.1) gives

$$f\wedge(g\wedge h)(x) = \int_0^x\big(f\wedge g(x-u)\big)h(u)\,du:$$

Conclusion (5.6) is now immediate from (1.1).

5.7. Notation. *Let* $C_0(\tilde{\omega})$ *be the space of all the functions which are continuous on the open interval* $\tilde{\omega}$ *and vanish at the origin.*

5.8. Remarks. *If* $g(\ )\in L^{\mathrm{loc}}(\tilde{\omega})$, *then* $\mathbf{1}\wedge g(\ )\in C_0(\tilde{\omega})$. Indeed, it follows from (1.1) that

(5.9) $$\mathbf{1}\wedge g(t) = \int_0^t \mathbf{1}(t-u)g(u)\,du = \int_0^t g:$$

the conclusion $\mathbf{1}\wedge g(\ )\in C_0(\tilde{\omega})$ is now at hand.

5.10. Remark. *If* $g(\ )$ *is continuous on* $\tilde{\omega}$, *then* $(\mathbf{1}\wedge g)'(\ )=g(\ )$: this is immediate from (5.9).

5.11. LEMMA. *Suppose that* $v(\ )\in C_0(\tilde{\omega})$. *If* $v'(\ )$ *is continuous on* $\tilde{\omega}$, *then* $v(\ )=\mathbf{1}\wedge v'(\ )$.

Proof. Take t in $\tilde{\omega}$. If $t>0$ the equations

$$v(t) = v(t)-v(0) = \int_0^t v' = \mathbf{1}\wedge v(t)$$

are from $v(0)=0$ and (5.9). Same reasoning for $t<0$.

5.12. LEMMA. *If* $G'(\ )\in C_0(\tilde{\omega})$ *and* $f(\ )\in L^{\mathrm{loc}}(\tilde{\omega})$, *then* $G\wedge f(\ )\in C_0(\tilde{\omega})$ *and*

(5.13) $$G\wedge f(\ ) = \mathbf{1}\wedge(G'\wedge f)(\ ).$$

Proof. Clearly, the function $G(\ )$ belongs to $C_0(\tilde{\omega})$; consequently, (5.11) (with $v=G$) gives $G(\ )=\mathbf{1}\wedge G'(\ )$, so that

(1) $$G\wedge f(\ ) = (\mathbf{1}\wedge G')\wedge f(\ ).$$

From (1.2) it follows that $|G'|\wedge|f|(\ )\in L^{\mathrm{loc}}(\tilde{\omega})$: we can therefore use 5.8 (with $g=|G'|\wedge|f|$) to conclude that the function $\mathbf{1}\wedge(|G'|\wedge|f|)(\ )$ is continuous on $\tilde{\omega}$, whence the equation

(2) $$(\mathbf{1}\wedge G')\wedge f(\ ) = \mathbf{1}\wedge(G'\wedge f)(\ )$$

now comes from (5.6). Conclusion (5.13) is immediate from (1)–(2). Set

(3) $$g_1(\ ) \overset{\mathrm{def}}{=} G'\wedge f(\ );$$

from (1.2) we see that $g_1(\ )\in L^{\mathrm{loc}}(\tilde{\omega})$, so that 5.8 gives

(4) $$\mathbf{1}\wedge g_1(\ )\in C_0(\tilde{\omega}).$$

Since we have already proved (5.13), we may combine it with (3) to obtain $G\wedge f(\ )=\mathbf{1}\wedge g_1(\ )$: the conclusion $G\wedge f(\ )\in C_0(\tilde{\omega})$ is now immediate from (4).

5.14. The space of test-functions. From 1.3 it follows that $w(\ )\in W_{\tilde{\omega}}$ if (and only if) $w^{(k)}(\ )\in C_0(\tilde{\omega})$ for every integer $k\geqslant 0$.

5.15. LEMMA. *If* $f(\ )\in L^{\mathrm{loc}}(\tilde{\omega})$ *and* $w_1(\ )\in W_{\tilde{\omega}}$ *then*

(5.16) $$w_1\wedge f(\ )\in C_0(\tilde{\omega}),$$

and

(5.17) $$(w_1 \wedge f)'(\ ) = w_1' \wedge f(\ ).$$

Proof. Since $w_1(\ )\in C_0(\tilde{\omega})$ (by 5.14), we can set $G = w_1$ in 5.12 to obtain (5.16). From (5.13) (with $G = w_1$) we obtain

(4) $$w_1 \wedge f(\ ) = \mathbf{1} \wedge (w_1' \wedge f)(\ ).$$

If

(5) $$g(\ ) \stackrel{\text{def}}{=} w_1' \wedge f(\ ),$$

then (4) gives $w_1 \wedge f(\ ) = \mathbf{1} \wedge g(\ )$, whence

(6) $$(w_1 \wedge f)'(\ ) = (\mathbf{1} \wedge g)'(\ ).$$

Setting $G = w_1'$ in 5.12 we obtain $w_1' \wedge f(\ )\in C_0(\tilde{\omega})$: from (5) we therefore have $g(\ )\in C_0(\tilde{\omega})$; the equation

(7) $$(\mathbf{1} \wedge g)'(\ ) = w_1' \wedge f(\ )$$

is from 5.10 and (5). Conclusion (5.17) is immediate from (6)–(7).

5.18. LEMMA. *If $f(\ )\in L^{\text{loc}}(\tilde{\omega})$ and $w(\ )\in W_{\tilde{\omega}}$, then $f \wedge w(\ )\in W_{\tilde{\omega}}$ and*

(5.19) $$(f \wedge w)'(\ ) = w' \wedge f(\ ) = f \wedge w'(\ ).$$

Proof. If the equation

(8) $$(w \wedge f)^{(k)}(\ ) = w^{(k)} \wedge f(\ )$$

holds for $k = n$, then

$$((w \wedge f)^{(n)})'(\ ) = (w^{(n)} \wedge f)'(\ ) = w^{(n+1)} \wedge f(\ ):$$

the second equation is from (5.17). Thus, the equation (8) holds for $k = n+1$ whenever it holds for $k = n$; since (8) also holds for $k = 0$, we conclude that it holds for any integer $k \geqslant 0$. From (8) and (5.16) (with $w_1 = w^{(k)}$) it follows that $(w \wedge f)^{(k)}(\ )$ belongs to $C_0(\tilde{\omega})$ for any integer $k \geqslant 0$; therefore, $w \wedge f(\ )$ belongs to $W_{\tilde{\omega}}$, and the conclusion $f \wedge w(\ )\in W_{\tilde{\omega}}$ now comes from (1.6). The proof is concluded by noting that (5.19) is a consequence of (5.17) and (1.6).

5.20. First conclusion. $D\in\mathscr{A}_{\tilde{\omega}}$. Indeed, D is clearly an operator, and the equations

$$D\cdot(w_1 \wedge w_2)(\ ) = (w_1 \wedge w_2)'(\ ) = w_1' \wedge w_2(\ ) = (D\cdot w_1) \wedge w_2(\ )$$

come from (1.21), (5.19), and (1.21). The conclusion $D\in\mathscr{A}_{\tilde{\omega}}$ is immediate from (1.8).

5.21. DEFINITION. *If $f(\ )\in L^{\text{loc}}(\tilde{\omega})$ we denote by $[f]$ the operator that assigns to each $w(\ )$ in $W_{\tilde{\omega}}$ the function $f \wedge w(\ )$:*

(5.22) $$[f]\cdot w(\ ) = f \wedge w(\ ) \quad (\textit{for } w(\ ) \textit{ in } W_{\tilde{\omega}}).$$

5.23 PROPOSITION. *If $f_1(\ )$ and $f_2(\ )$ belong to $L^{\text{loc}}(\tilde{\omega})$, then*

(5.24) $$[f_1][f_2] = [f_1 \wedge f_2];$$

further, if $w_2(\ )\in W_{\tilde{\omega}}$, then

(5.25) $$f_1 \wedge (f_2 \wedge w_2)(\ ) = (f_1 \wedge f_2) \wedge w_2(\ ).$$

Proof. From 5.18 we see that $|f_2| \wedge |w_2|(\ )\in W_{\tilde{\omega}}$; consequently, we can set $w = |f_2| \wedge |w_2|$ and $f = |f_1|$ in 5.18 to conclude that the function $|f_1| \wedge (|f_2| \wedge |w_2|)(\ )$ belongs to $W_{\tilde{\omega}}$: Conclusion (5.25) therefore follows from (5.6). From (5.25) and Definition (5.22) we see that

(1) $$[f_1]\cdot([f_2]\cdot w_2)(\ ) = [f_1 \wedge f_2]\cdot w_2(\ );$$

since $w_2(\ )$ is an arbitrary element of $W_{\tilde{\omega}}$, Conclusion (5.24) is immediate from (1) and (1.8).

5.26. Remark. *If $f(\ )\in L^{\text{loc}}(\tilde{\omega})$, then $[f]\in\mathscr{A}_{\tilde{\omega}}$.* Indeed, $[f]$ is an operator (by (5.22) and 5.18): it only remains to prove that (1.10) holds when $A = [f]$. Setting $f_1 = f$ and $f_2 = w_1$ in (5.25), we obtain

$$f \wedge (w_1 \wedge w_2)(\ ) = (f \wedge w_1) \wedge w_2(\ );$$

in view of Definition (5.22), this means that

$$[f]\cdot(w_1 \wedge w_2)(\ ) = ([f]\cdot w_1)\cdot w_2(\ ):$$

therefore, (1.10) holds for $A = [f]$.

5.27. Remark. *If $A_k\in\mathscr{A}_{\tilde{\omega}}$ for $k = 1, 2$, then $A_1A_2\in\mathscr{A}_{\tilde{\omega}}$* (this is easily verified).

5.28. LEMMA. *If $f(\ )\in L^{\text{loc}}(\tilde{\omega})$ then $f\in\mathscr{A}_{\tilde{\omega}}$ and*

(5.29) $$f = [f]D.$$

Proof. Equation (5.29) is immediate from the three definitions (1.12), (5.22), and (1.21) (see also Definition (1.8)). In view of 5.27, the conclusion $f\in\mathscr{A}_{\tilde{\omega}}$ comes from (5.29), 5.26, and 5.20.

5.30. LEMMA. *If $f_1(\ )\in L^{\text{loc}}(\tilde{\omega})$ and $f_2(\ )\in L^{\text{loc}}(\tilde{\omega})$, then $f_1 = f_2$ implies $f_1(\ ) = f_2(\ )$.*

Proof. Set $f_0(\ ) = f_1(\ ) - f_2(\ )$; from (1.19) we see that $f_0 = f_1 - f_2 = 0$; consequently, the equation

(1) $$f_0\cdot w(t) = 0 \quad (\text{for every } t\in\tilde{\omega})$$

holds for every $w(\ )$ in $W_{\tilde{\omega}}$: the proof will be completed by showing that $f_0(\ ) = 0$. Take any integer $n \geqslant 1$ and let $q_n(\ )$ be the function defined in 5.3; since $q_n(\ )\in W_{\tilde{\omega}}$ (this is easily verified), it follows by setting $f = \mathbf{1}$ in 5.18 that $\mathbf{1} \wedge q_n(\ )\in W_{\tilde{\omega}}$: we may therefore set $w(\ ) = \mathbf{1} \wedge q_n(\ )$ in (1) to obtain

(2) $$f_0\cdot(\mathbf{1} \wedge q_n)(t) = 0 \quad (\text{for every } t\in\tilde{\omega}).$$

The equations

$$(3)\qquad f_0\cdot(\mathbf{1}\wedge q_n)(\ ) = f_0\wedge(\mathbf{1}\wedge q_n)'(\ ) = f_0\wedge q_n(\ )$$

are from Definition (1.12) and 5.10. Combining (2) and (3), we see that $f_0\wedge q_n(t) = 0$ for $t\in\tilde{\omega}$ and every $n\geqslant 1$: the conclusion $0 = f_0(\ )$ now comes from 5.4. Since $0 = f_0(\cdot) = f_1(\ )-f_2(\ )$, we have proved that $f_1(\ ) = f_2(\ )$.

5.31. LEMMA. *If $B\in\mathscr{A}_{\tilde{\omega}}$ then the equation*

$$(5.32)\qquad B\cdot(p_1\wedge p_2)(\ ) = p_1\wedge(B\cdot p_2)(\ )$$

holds for every $p_1(\ )$ and $p_2(\ )$ in $W_{\tilde{\omega}}$.

Proof. The equations

$$B\cdot(p_1\wedge p_2)(\ ) = B\cdot(p_2\wedge p_1)(\ ) = (B\cdot p_2)\wedge p_1(\ )$$

are from (1.6) and (1.10): Conclusion (5.32) is now immediate from another application of (1.6).

5.33. PROPOSITION. *The algebra $\mathscr{A}_{\tilde{\omega}}$ is commutative.*

Proof. Take A_1 and A_2 in $\mathscr{A}_{\tilde{\omega}}$: it will suffice to demonstrate that $A_1A_2-A_2A_1 = 0$. Let $w_1(\ )$ and $w_2(\ )$ be any two elements of $W_{\tilde{\omega}}$: we begin by observing that

$$(4)\qquad A_1A_2\cdot(w_1\wedge w_2')(\ ) = A_1\cdot((A_2\cdot w_1)\wedge w_2')(\ ) = (A_2\cdot w_1)\wedge(A_1\cdot w_2')(\ ):$$

these equations are from (1.8), (1.10), and (5.32) (with $p_1 = A_2\cdot w_1$). On the other hand, the equations

$$(5)\qquad A_2A_1\cdot(w_1\wedge w_2')(\ ) = A_2\cdot(w_1\wedge(A_1\cdot w_2'))(\ ) = (A_2\cdot w_1)\wedge(A_1\cdot w_2')(\ )$$

are from (1.8), (5.32), and (1.10). We now subtract (5) from (4) to obtain

$$(6)\qquad A\cdot(w_1\wedge w_2')(\ ) = 0,\quad \text{where}\quad A \stackrel{\text{def}}{=} A_1A_2-A_2A_1.$$

From (6) and (1.10) it results that

$$(7)\qquad 0 = (A\cdot w_1)\wedge w_2'(\ ) = (A\cdot w_1)\cdot w_2(\ )\quad (\text{for } w_2(\ )\in W_{\tilde{\omega}}):$$

the last equation is from Definition (1.12). From (7) it follows that $A\cdot w_1 = 0$; setting $f_1(\ ) = A\cdot w_1(\ )$ and $f_2(\ ) = 0$ in 5.30, we obtain $A\cdot w_1(\ ) = 0$ for all $w_1(\ )\in W_{\tilde{\omega}}$: the desired conclusion $A = 0$ is at hand.

5.34. Proof of the theorem. The algebra $\mathscr{A}_{\tilde{\omega}}$ is commutative (by 5.33); $D\in\mathscr{A}_{\tilde{\omega}}$ (by 5.20); Property (1.23) was proved in 5.28, and (1.25) has also been proved (see 5.30). Consequently, it only remains to prove (1.24): to that effect, set $f(\ ) = f_1\wedge f_2(\ )$ and note that (5.29) gives $f = [f_1\wedge f_2]D$; it will therefore suffice to prove that

$$(8)\qquad D([f_1\wedge f_2]D) = f_1f_2.$$

To prove (8), observe that the equations

$$(9)\qquad D[f_1\wedge f_2]D = D[f_1][f_2]D = ([f_1]D)([f_2]D)$$

come from (5.24) and by utilizing both the associativity and the commutativity of the multiplication in $\mathscr{A}_{\tilde{\omega}}$. Conclusion (8) comes from (9) by two more applications of (5.29).

References

[1] L. Berg, *Introduction to the operational calculus*, New York 1967.
[2] H. S. Dunn, *A generalization of the Laplace transform*, Proc Cambridge Phil. Soc., 63, I (1966), pp. 155–160.
[3] A. Erdélyi, *Operational calculus and generalized functions*, New York 1962.
[4] R. L. Jeffery, *The theory of functions of a real variable*, Toronto 1953.
[5] G. Krabbe, *Operational Calculus*, 1970.
[6] J. Mikusiński, *Sur les fondements du caclul operatoire*, Studia Math. 11 (1950) pp. 41–70.
[7] *Operational calculus*, 1959.
[8] — *Le calcul operationnel d'intervalle fini*, Studia Math. 15 (1956), pp. 225–251.
[9] — *L'anneau algébrique et ses applications dans l'analyse fonctionelle*, Annales Univ. Mariae Curie-Skłodowska, II, 1 (1947), pp. 1–48.
[10] G. Krabbe, *An algebra of generalized functions on an open interval; two-sided operational calculus*, Bull. Amer. Math. Soc. 77 (1971), pp. 78–84.
[11] — *Initial-value problems involving generalized functions; two-sided operational calculus*, to appear in the Archiv der Mathematik.
[12] — *An algebra of generalized functions on an open interval; two-sided operational calculus*, to appear in the Pacific J. Math.

Received March 18, 1970 (185)