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Range of operators and regularity of solutions

by

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Abstract. In the paper there are given necessary and sufficient conditions for a linear operator P to have its adjoint surjective. In Theorem 7.2 the conditions are expressed by use of the notion of good location of the image. In the background lies the notion of strongly good location which is pretty complicated. However, Theorem 6.1 states that under certain natural assumptions which are almost always met in practice, the notion of strongly good location is equivalent to that of good location which is much simpler and very intuitive. In Theorem 8.1 there is discussed a certain basic phenomenon accompanying the surjectivity of the adjoint. This phenomenon is explained in Theorems 9.1 and 9.2 by way of the global behaviour of the functions of order of distribution which are defined on the set of all compacts of the underlying space.

0. Introduction. It was the results of [2] which opened a wide field for more of less general investigations of the range of operators in spaces of distributions and other spaces of a similar structure, e.g. [15].

Need for some general approach seems to be quite obvious and still only partly fulfilled (Cf. [14]). In [9] a first attempt has been made to describe the most important case concerning the range of operators of restriction of functionals to fixed subspaces (Cf. also [6]). Subsequently preliminary investigations of the general case appeared in [10] and most crucial points were outlined in [11] and [12].

This paper is in a sense a continuation of [9] aiming to simplify, systematize and generalize some results of [9] explaining in the two last sections interesting situations which arised in consequence of attacking the main problem.

This paper is not a continuation of [9] in the sense that it does not require the knowledge of [9] to understand its content. Also the notation in this paper has been considerably changed.

The present publication exposes only part of the results contained in [10]. The rest of it, sketched also in [12], shall be published separately [13].

1. Preliminary definitions. Given a linear space W and a convex subset B of W, B shall be called a *ball* if it is absolutely convex and

$$B = \{x \in L_B\colon \|x\|_B \leqslant 1\},$$

where $L_B \subset W$ denotes the linear space spanned by B and $\|\cdot\|_B$ denotes Minkowski's functional induced by B. Hence $(L_B, \|\cdot\|_B)$ will denote the seminormed space with B as the unit ball. A ball B is said to be *complete* if the space $(L_B, \|\cdot\|_B)$ is complete. Given a linear subset $L \subset W$ we say that a ball B is *adequate for* L if $L \cap L_B$ is dense in $(L_B, \|\cdot\|_B)$. If W is provided with a locally convex topology ν, we shall consider polars

$$(L \cap B)^\circ = \{w' \in W' \colon |w'x| \leqslant 1 \quad \text{for} \quad x \in L \cap B\},$$

where W' denotes the dual of (W, ν).

Given linear spaces W_i, $i = 1, 2$, we say that P is a *linear mapping* from W_1 to W_2 if in W_1 there is fixed a linear subset D_P called the domain of P such that P is a linear W_2-valued mapping defined on D_P. If W_1 and W_2 are provided with topologies ν_1 and ν_2 respectively, we call P *densely defined* if D_P is dense in (W_1, ν_1) In the case of a densely defined mapping P we define the adjoint P' from W_2' to W_1' setting for $x \in D_P$ and $x' \in W_2'$

$$(\bar{P}'x')x \overset{\text{df}}{=} x'(Px)$$

and

$$D_{P'} = \{x' \in W_2' \colon \bar{P}'x' \text{ is continuous in } (D_{P'}, \nu_1)\}$$

and then putting for $x' \in D_{P'}$

$$P'x' = \text{the continuous extension of } \bar{P}'x'$$

over the whole (W_1, ν_1).

Given a locally convex space (W, ν) with the adjoint W', we write for the closure of zero element

$$0(W, \nu) = \{x \in W \colon w'x = 0 \quad \text{for all } w' \in W'\}.$$

Given topological spaces (W_i, ν_i), $i = 1, 2$, we shall write

$$(W_1, \nu_1) \leqslant (W_2, \nu_2)$$

if W_2 is contained in W_1 and the identical injection of W_2 into W_1 is continuous.

If both (W_i, ν_i) are locally convex and W_1, W_2 are contained in the same linear space we define the operation "*roof*" setting

$$(W_1, \nu_1) \wedge (W_2, \nu_2) = (W_1 + W_2, \nu_1 \wedge \nu_2),$$

where $\nu_1 \wedge \nu_2$ is the finest topology of $W_1 + W_2$ such that $(W_i, \nu_1 \wedge \nu_2) \leqslant (W_i, \nu_i)$ for $i = 1, 2$.

The operation "roof" is also called the operation of inductive limit.

2. $(\mathcal{DF})$-classes. A set of balls ξ contained in the same linear space L is said to be a *semi pre-$(\mathcal{DF})$-class if*

(i) There exists a sequence $\{B_n\}$, $B_n \subset B_{n+1}$ for $n = 1, 2, \ldots$, which is cofinal with ξ with respect to the upward directed inclusion.

(ii) Given $B \in \xi$ and $\{x_n\} \subset L$ with $\{x_n - x_m\} \subset L_B$, $\lim \|x_n - x_m\|_B = 0$. If for some $C \in \xi$, $\{x_n\} \subset L_C$ and $\lim \|x_n\|_C = 0$, then all x_n are in L_B and $\lim \|x_n\|_B = 0$.

(iii) Define

$$L_\xi = \bigcup_{B \in \xi} L_B.$$

If a ball B in L_ξ is absorbed by a ball from ξ and B fulfils (ii), then $B \in \xi$ (i.e. ξ is saturated).

Given a semi pre-$(\mathcal{DF})$-class ξ, any sequence $\{B_n\} \subset \xi$ such that every ball from ξ is absorbed by some B_n, is called a generating sequence of ξ; ξ is said to be complete if all balls from ξ are complete and then ξ is called a *semi $(\mathcal{DF})$-class*.

The following is an immediate consequence of (ii).

(ii') If for $x \in L_B$, $B \in \xi$, there is $\|x\|_B = 0$, then for every $C \in \xi$ it must be $x \in L_C$ and $\|x\|_C = 0$.

It is clear that in the case of semi $(\mathcal{DF})$-classes, (ii) follows from (ii'). We write

$$0(\xi) = \{x \in L_\xi \colon \|x\|_C = 0 \quad \text{for some } C \in \xi\},$$

and we call ξ a *pre-$(\mathcal{DF})$-class* iff $0(\xi)$ consists only of zero. Complete pre-$(\mathcal{DF})$-classes are called *$(\mathcal{DF})$-classes*. Given a semi pre-$(\mathcal{DF})$-class and a linear subset $L \subset L_\xi$, we define new pre-$(\mathcal{DF})$-classes

$$L \cap \xi = \{L \cap B \colon B \in \xi\}, \qquad L \bar{\cap} \xi = \{L \bar{\cap} B \colon B \in \xi\},$$

where $L \bar{\cap} B$ denotes the closure of $L \cap B$ in $(L_B, \|\cdot\|_B)$; L is said to be *closed* in ξ if $L \cap \xi = L \bar{\cap} \xi$. Given $L \subset L_\xi$ closed in ξ, we define

$$\xi / L = \{B/L \colon B \in \xi\},$$

where B/L denotes the unit ball in $(L_B, \|\cdot\|_B)/L$. It is clear that ξ/L is again a semi pre-$(\mathcal{DF})$-class and $L_\xi / L = L_{\xi/L}$. Hence, in particular, $\xi / 0(\xi)$ is always a pre-$(\mathcal{DF})$-class. Given a semi pre-$(\mathcal{DF})$-class ξ and a subspace L of L_ξ we call L *dense* in ξ if there exists in ξ a cofinal sequence of balls each of them adequate for L (cf. [4]).

Given a semi pre-$(\mathcal{DF})$-class ξ, we denote by ι_ξ the inductive topology induced on L_ξ by ξ. Even in the case when ξ is a $(\mathcal{DF})$-class this topology need not be Hausdorff.

Given semi pre-$(\mathcal{DF})$-classes ξ_1 and ξ_2, we write $\xi_1 \leqslant \xi_2$ if $L_{\xi_1} \supset L_{\xi_2}$ and if to every $B_2 \in \xi_2$ there corresponds $B_1 \in \xi_1$ such that $B_2 \subset B_1$.

Consider a locally convex (not necessarily Hausdorff) space (X, τ). A semi pre-$(\mathcal{DF})$-class is said to be a *p-component* of (X, τ) if X is dense in ξ and $(L_\xi, \iota_\xi) \leqslant (X, \tau)$. In particular, a semi pre-$(\mathcal{DF})$-class ξ constitutes a p-component of the space (L_ξ, ι_ξ).

Given semi pre-$(\mathcal{DF})$-classes ξ_i, $i = 1, 2$, and a linear mapping P of a subspace $D_P \subset L_{\xi_1}$ into L_{ξ_2}, P is said to be *continuous* from ξ_1 to ξ_2 if to every $B_1 \epsilon \xi_1$ there corresponds $B_2 \epsilon \xi_2$ such that $P(D_P \cap B_1) \subset B_2$.

A semi $(\mathcal{DF})$-class ξ is said to be *reflexive* iff there exists a cofinal $\{B_n\} \subset \xi$ consisting of adequate balls such that the spaces $(L_{B_n}, \|\cdot\|_{B_n})$ factorized by $0(L_{B_n}, \|\cdot\|_{B_n})$ are reflexive Banach spaces.

3. $(\mathcal{F})$-classes. According to the definition given in [7] a sequence $\{(V_n, \|\cdot\|_n)\}$ of seminormed spaces is called a *pre-$(\mathcal{F})$-sequence* if $(V_n, \|\cdot\|_n) \leqslant (V_{n+1}, \|\cdot\|_{n+1})$ and $\sup_n \|z\|_n = 0$ for $z \epsilon \bigcap_{n=1}^{\infty} V_n$ implies $z = 0$. The functions

$$\varrho_n(x) = \begin{cases} \|x\|_n/(1+\|x\|_n) & \text{for } x \epsilon V_n, \\ 1 & \text{otherwise,} \end{cases}$$

provide a translation invariant metric function $\varrho(x-y)$

$$\varrho(x) = \sum_{n=1}^{\infty} 2^{-n} \varrho_n(x),$$

making out of V an additive topological group. If this group is complete, we call $\{V_n, \|\cdot\|_n\}$ an *$(\mathcal{F})$-sequence*.

We define a pre-$(\mathcal{F})$-class (ϱ) as a pair consisting of a translation invariant metric function ϱ defined over a linear space $[\varrho]$ such that there exists a countable set of balls [1] in $[\varrho]$ providing a basis of neighbourhoods of zero in $([\varrho], \varrho)$, and the family $\boldsymbol{L}_\varrho$ of all clopen [2] linear subspaces of $([\varrho], \varrho)$. It is easy to see that for every ball $B \subset [\varrho]$ which constitutes a neighbourhood of zero in $([\varrho], \varrho)$, L_B belongs to $\boldsymbol{L}_\varrho$. Conversely, every $L \epsilon \boldsymbol{L}_\varrho$ is itself an absorbing ball B which is a neighbourhood of zero in $([\varrho], \varrho)$.

Given a basis of neighbourhoods of zero in $([\varrho], \varrho)$ consisting of a sequence of balls $\{B_n\}$ such that $B_n \supset B_{n+1}$ for $n = 1, 2, \ldots$, $\{(L_{B_n}, \|\cdot\|_{B_n})\}$ constitutes a pre-$(\mathcal{F})$-sequence which provides the metric equivalent to ϱ. Such pre-$(\mathcal{F})$-sequences shall be called generating sequences for (ϱ).

A pre-$(\mathcal{F})$-class shall be called an *$(\mathcal{F})$-class* iff the corresponding topological group is complete.

We shall consider pre-$(\mathcal{F})$-classes (ϱ_1) and (ϱ_2) equivalent if there exists an $L \epsilon \boldsymbol{L}_{\varrho_1} \cap \boldsymbol{L}_{\varrho_2}$ such that (L, ϱ_1) and (L, ϱ_2) are the same topological groups. Clearly for equivalent (ϱ_1) and (ϱ_2) the spaces $[\varrho_1]$ and $[\varrho_2]$ are generally different.

(1) Balls are here meant as defined in Section 0, not as balls with respect to ϱ.

(2) Notice that open linear subspaces are automatically closed.

Given pre-$(\mathcal{F})$-classes (ϱ_i), $i = 1, 2$, we say that T is a linear mapping from (ϱ_1) to (ϱ_2) if there is fixed a domain D_T which is a subspace of $[\varrho_1]$ and T is a linear transformation of D_T into $[\varrho_2]$; T is said to be continuous (respectively, open, nearly open, closed) if T is continuous (respectively open, nearly open, closed) as mapping from $([\varrho_1], \varrho_1)$ to $([\varrho_2], \varrho_2)$; T is said to be surjective if to every $L_1 \epsilon \boldsymbol{L}_{\varrho_1}$ there corresponds $L_2 \epsilon \boldsymbol{L}_{\varrho_2}$ such that $T(L_1 \cap D_T) \supset L_2$; T is said to be nearly surjective if to every $L_1 \epsilon \boldsymbol{L}_{\varrho_1}$ there corresponds $L_2 \epsilon \boldsymbol{L}_{\varrho_2}$ such that $T(L_1 \cap D_T)$ is dense in (L_2, ϱ_2).

Given a pre-$(\mathcal{F})$-class (ϱ) and a linear subspace W of some of $L \epsilon \boldsymbol{L}_\varrho$ we define a new pre-$(\mathcal{F})$-class $(W \cap \varrho)$ which we obtain by restricting the metric function ϱ to W and then getting $\boldsymbol{L}_{W \cap \varrho} \overset{\mathrm{df}}{=} \{V \cap W\colon V \epsilon \boldsymbol{L}_\varrho\}$; the subspace W is said to be closed in (ϱ) iff all $W \cap V$, $V \epsilon \boldsymbol{L}_\varrho$, are closed. It is easy to see that in the case when (ϱ) is an (F)-class and W is closed in (ϱ), $(W \cap \varrho)$ is an $(\mathcal{F})$-class as well.

If (V, ϱ) is a locally convex metric space, then we can look upon it as on a pre-$(\mathcal{F})$-class (ϱ), where $\boldsymbol{L}_\varrho$ consists of only one element V.

4. Polar $(\mathcal{F})$-classes. Given a semi pre-$(\mathcal{DF})$-class ξ and a linear subspace $X \epsilon L_\xi$ dense in ξ, we define the dual $(\mathcal{F})$-class (ϱ_ξ^*) of ξ with respect to X, taking the algebraic dual X^* of X and providing it with a translation invariant metrization ϱ_ϱ^* of the following convergence.

> $\{x_n'\}$ tends to zero if for every adequate ball $B \epsilon \xi$ almost all x_n' belong to the polar of $X \cap B$ in X^*.

Then we assign to $\boldsymbol{L}_{\varrho_\xi^*}$ all clopen linear subspaces of (X^*, ϱ_ξ^*).

Suppose in the sequel that X is provided with a locally convex topology τ and that ξ is a p-component of (X, τ). Then we define the polar pre-$(\mathcal{F})$-class (ϱ_ξ°) of ξ by providing the dual X' of (X, τ) with a translation invariant metrization ϱ_ξ° of the following convergence:

> $\{x_n'\}$ tends to zero if for every adequate ball B almost all x_n' belong to $(X \cap B)^\circ$.

Then we assign to $\boldsymbol{L}_{\varrho_\xi^\circ}$ all clopen linear subspaces of (X', ϱ_ξ°). It is clear that (ϱ_ξ°) is generated by any pre-$(\mathcal{F})$-sequence $\{(L_{(X \cap B_n)^\circ}, \|\cdot\|_{(X \cap B_n)^\circ})\}$, where $\{B_n\}$, $B_n \subset B_{n+1}$ forms a generating sequence of ξ. We have

$$(\varrho_\xi^\circ) = (X' \cap \varrho_\xi^*)$$

and if (X, τ) is barreled, all the sequences of functionals from X', which are pointwise convergent, have their limits again in X' which makes (ϱ_ξ°) complete.

Looking for ξ as for a p-component of (L_ξ, ι_ξ) we can see that its polar pre-$(\mathscr{F})$-class identifies with the $(\mathscr{F})$-space (L'_ξ, ϱ'_ξ), where L'_ξ denotes the adjoint to (L_ξ, ι_ξ) and ϱ'_ξ denotes the topology of uniform convergence on balls from ξ.

5. Strongly good location and openness. Given a locally convex (X, τ) and its p-components $\xi_0 \leqslant \xi_1 \leqslant \xi_2$, a linear subset U of X is said to be strongly well located in X with respect to $\xi_0 \leqslant \xi_1 \leqslant \xi_2$ if the following condition holds (cf. [9], p. 213, [10], p. 5.7 and [11], p. 110).

(A_0) To every adequate ball $B_2 \epsilon \xi_2$ there corresponds an adequate ball $B_1 \epsilon \xi_1$, $B_1 \supset B_2$, such that to every $\varepsilon > 0$ every $B_0 \epsilon \xi_0$ and every $z' \epsilon L'_{B_1}$ vanishing on $U \cap B_1$ there corresponds $x' \epsilon X'$ bounded on $X \cap B_2$ and vanishing on $U \cap B_0$ such that $\|x'_{|2} - z'_{|2}\|'_{B_2} < \varepsilon$, where $x'_{|2} \epsilon L'_{B_2}$ denotes the extension of the restriction of x' to $X \cap L_{B_2}$ and $z'_{|2}$ denotes the restriction of z' to L_{B_2}.

Consider a barreled space (Y, σ) with a p-component ζ and a barreled space (X, τ) with p-components $\xi_1 \leqslant \xi_2$. Let in the sequel P be a continuous linear mapping with arguments in Y and values in X.

THEOREM 5.1. (Cf. [9], Prop. 5.4 and [10], Prop. 5.3.) *If for a p-component $\xi_0 \leqslant \xi_1$ the image PY is strongly well located in X with respect to $\xi_0 \leqslant \xi_1 \leqslant \xi_2$, P' is nearly surjective from $(\varrho^{\circ}_{\xi_1})$ to $(\varrho^{\circ}_{\zeta})$, P^{-1} exists and is continuous from ξ_1 to ζ and P is continuous from ζ to ξ_0, then P' is open from $(\varrho^{\circ}_{\xi_2})$ to $(\varrho^{\circ}_{\zeta})$.*

Proof. From Proposition 12 of [7] it follows that it is sufficient to prove that P' is nearly open from $(\varrho^{\circ}_{\xi_2})$ to $(\varrho^{\circ}_{\zeta})$, i.e. that

(NO) To every adequate ball $B_2 \epsilon \xi_2$ there corresponds an adequate ball $C_1 \epsilon \zeta$ such that for every $y' \epsilon (Y \cap C_1)^{\circ}$ every adequate ball $C \epsilon \zeta$ and every $\varepsilon > 0$ there corresponds $x' \epsilon D_{P'} \cap (X \cap B_2)^{\circ}$ such that $\|y' - P'x'\|_{(Y \cap C)^{\circ}} < \varepsilon$.

Furthermore, P' is nearly surjective from $(\varrho^{\circ}_{\xi_1})$ to $(\varrho^{\circ}_{\zeta})$ iff

(NS) To every adequate ball $B_1 \epsilon \xi_1$ there corresponds an adequate ball $C_1 \epsilon \zeta$ such that for every $y' \epsilon L_{(Y \cap C_1)^{\circ}}$, every adequate ball $C \epsilon \zeta$ and every $\varepsilon > 0$ there corresponds $x' \epsilon D_{P'} \cap L_{(X \cap B_1)^{\circ}}$ such that $\|y' - P'x'\|_{(Y \cap C)^{\circ}} < \varepsilon$.

Take an adequate ball $B_2 \epsilon \xi_2$ and adjust an adequate ball $B_1 \epsilon \xi_1$, $B_1 \supset B_2$, to fulfil the requirements of (A_0). Subsequently, adjust $C_1 \epsilon \zeta$ to B_1 such that (NS) holds. Due to the continuity of P^{-1} we can choose C_1 in such a way that

$$(*) \qquad Px \epsilon B_1 \quad \text{implies} \quad x \epsilon \tfrac{1}{4} C_1$$

for all $Px \epsilon L_{B_1}$.

Fix $\frac{1}{2} > \varepsilon > 0$ and an adequate ball $C \epsilon \zeta$. Given $y' \epsilon (Y \cap C_1)^{\circ}$ we apply (NS) and find $u' \epsilon D_{P'} \cap L_{(X \cap B_1)^{\circ}}$ such that $\|y' - P'u'\|_{(Y \cap C)^{\circ}} < \varepsilon$. Denoting by $\|u'\|$ the sup norm of u' in $(PY \cap L_{B_1}, \|\cdot\|_{B_1})$, we have from (*)

$$\|u'\| \leqslant \tfrac{1}{4} \|P'u'\|_{(Y \cap C_1)^{\circ}}$$

and denoting by $z' \epsilon L'_{B_1}$ the norm preserving extension of the restriction of u' to $(PY) \cap L_{B_1}$, we obtain

$$\|z'\|'_{B_1} \leqslant \tfrac{1}{4} \|P'u'\|_{(Y \cap C_1)^{\circ}}$$

so that denoting by $z'_{|2}$ the restriction of z' to L_{B_2} we have

$$\|z'_{|2}\|'_{B_2} \leqslant \|z'\|'_{B_1} \leqslant \tfrac{1}{4} \|P'u'\|_{(Y \cap C_1)^{\circ}} \leqslant \tfrac{1}{4} (\|y' - P'u'\|_{(Y \cap C_1)^{\circ}} + \|y'\|_{(Y \cap C_1)^{\circ}}) \leqslant \tfrac{1}{2}.$$

Writing $u'_{|1}$ for the restriction of u' to $X \cap L_{B_1}$, we notice that $z' - u'_{|1} \epsilon L'_{B_1}$ vanishes on $PY \cap L_{B_1}$. By virtue of the continuity of P we can find $B_0 \epsilon \xi_0$ such that $PC \subset B_0$, and then applying (A_0) we find $v' \epsilon X'$ bounded on $X \cap B_2$ and vanishing on $PY \cap B_0$ such that

$$\|(z'_{|2} - u'_{|2}) - v'_{|2}\|'_{B_2} < \varepsilon,$$

where $u'_{|2}, v'_{|2} \epsilon L'_{B_2}$ denote the extensions of the restrictions to $X \cap L_{B_2}$ of u' and v' respectively. Since $PC \subset B_0$, we have $\|P'v'\|_{(Y \cap C)^{\circ}} = 0$ and setting

$$x' = u' + v',$$

we obtain

$$\|y' - P'x'\|_{(Y \cap C)^{\circ}} \leqslant \|y' - P'u'\|_{(Y \cap C)^{\circ}} + \|P'v'\|_{(Y \cap C)^{\circ}} < \varepsilon$$

and

$$\|x'\|_{(X \cap B_2)^{\circ}} \leqslant \|v'_{|2} - (z'_{|2} - u'_{|2})\|'_{B_2} + \|z'_{|2}\|_{B_2} \leqslant \varepsilon + \|z'_{|2}\|'_{B_2} \leqslant 1$$

and thus (NO) holds and the Proposition follows.

Consider a locally convex space (X, τ) and a p-component ξ of (X, τ). We say that ξ admits (D) if the following condition holds (cf. [9], p. 216).

(D) Given $B \epsilon \xi$, there exists an adequate $C \epsilon \xi$, $C \supset B$, such that the set of extensions to functionals from L'_C of the restrictions to $X \cap L_C$ of functionals from X' which are bounded on $X \cap C$ is dense in $(L'_C, \|\cdot\|_C)$.

If (X, τ) is an $(\mathscr{LF})$-space in the sense of [1], then every p-component of (X, τ) admits (D).

Take a densly defined P from (Y, σ) to (Y, τ).

THEOREM 5.2. (Cf. [9], Prop. 5.5.) *If ξ_1 admits (D), P is continuous from ζ to ξ_1 and P' is surjective from $(\varrho^{\circ}_{\xi_2})$ to $(\varrho^{\circ}_{\zeta})$, then the image PD_P is strongly well located in X with respect to $\xi_0 \leqslant \xi_1 \leqslant \xi_2$ for every p-component $\xi_0 \leqslant \xi_1$.*

Proof. Notice at first that by virtue of Theorem 3 of [7], P' being surjective must be open, i.e.

(O) To every adequate ball $B_2 \in \xi_2$ there corresponds an adequate ball $C_1 \in \zeta$ such that for every $y' \in (Y \cap C_1)^\circ$ there corresponds $x' \in D_{P'} \cap (X \cap B_2)^\circ$ such that $y' = P'x'$.

Fix $B_2\in\xi_2$ and assign C_1 to B_2 according to (O). Since D_P is dense in ζ, C_1 can be chosen adequate for D_P. By continuity of P we can find $B_1\in\xi_1$, $B_1\supset B_2$, such that $PC_1 \subset B_1$. Take an arbitrary $\varepsilon>0$ and $z'\in L'_{B_1}$ vanishing on $(PD_P)\cap L_{B_1}$. By virtue of (D) we can find $u'\in X'$ such that $\|u'_{|1} - z'\|'_{B_1} < \frac12\varepsilon$, where $u'_{|1}\in L'_{B_1}$ denotes the extension of the restriction of u' to $X\cap L_{B_1}$. We have for $x\in C_1\cap D_P$ $|u'Px| = |u'Px - z'Px| \leqslant \|u'_{|1} - z'\|'_{B_1} < \frac12\varepsilon$ so that $\|P'u'\|_{(Y\cap C_1)^\circ} < \frac12\varepsilon$. Applying (O) we can find $v'\in(\frac12\varepsilon)(D_{P'}\cap(X\cap B_2)^\circ)$ such that $P'u' = P'v'$.

Setting $x' = u' - v'$, we have

$$\|z'_{|2} - x'_{|2}\|'_{B_2} \leqslant \|z'_{|2}-u'_{|2}\|'_{B_2} + \|v'_{|2}\|'_{B_2} \leqslant \|z' - u'_{|1}\|'_{B_1} + \tfrac12\varepsilon < \varepsilon$$

and since x' vanishes on PD_P, it also vanishes on $L_{B_0}\cap(PD_P)$ for every p-component ξ_0 and every $B_0\in\xi_0$. The Proposition is proved.

Given an $(\mathcal{LF})$-space (X,τ) (cf. [1] for the definition), a p-component ξ of (X,τ) is said to be not overrunning (X,τ) if to every $B\in\xi$ there corresponds $C\in\xi$, $C\supset B$, and a linear subspace Z of X which is a Fréchet space under the topology induced by τ such that $X\cap L_C\subset Z$ and that $\|\cdot\|_C$ is continuous in $(X\cap L_C,\tau)$ (cf. [10], p. N3 and [9], p. 108).

We shall write briefly "$(Z,\varrho)\in(\mathcal{F})$" instead of "$(Z,\varrho)$ is a Fréchet space".

Consider a pair (Y,σ), (X,τ) of $(\mathcal{LF})$-spaces and a linear continuous mapping P from (Y,σ) to (X,τ). Let ζ and ξ_1 be p-components of (Y,σ) and (X,τ), respectively. We shall say that P^{-1} is $\wedge$-continuous from ξ_1 to ζ if P is one-to-one and if the following condition holds.

(M) To every $B\in\xi_1$ there corresponds a $C\in\zeta$ such that to every $(U,\vartheta)\in(\mathcal F)$, $(U,\vartheta)\geqslant(X,\tau)$ there can be assigned $(V,\nu)\in(\mathcal F)$, $(V,\nu)\geqslant(Y,\sigma)$, in such a way that P^{-1} maps $PY\cap(L_B+U)$ into L_C+V and is continuous from $(L_B,\|\,\|_B)\wedge(U,\vartheta)$ to $(L_C,\|\,\|_C)\wedge(V,\nu)$.

PROPOSITION 5.1[3] *If P^{-1} is $\wedge$-continuous from ξ_1 to ζ and ζ does not overrun (Y,σ), then P' is nearly surjective from $(\varrho^\circ_{\xi_1})$ to (ϱ°_ζ).*

Proof. Take $B\in\xi_1$ and adjust $C_1\in\zeta$ according to (M). Fix any $C\in\zeta$, $C\supset C_1$ and adjust $(Z,\varrho)\in(\mathcal F)$, $(Z,\varrho)\geqslant(Y,\sigma)$ such that $L_C\cap Y\subset Z$.

(3) E. Simonsen pointed out a mistake in the proof of Theorem 5.1 of [9]. The present Proposition substitutes the wrong Theorem.

Subsequently, take $(U,\vartheta)\in(\mathcal F)$, $(U,\vartheta)\geqslant(X,\tau)$ with $PZ\subset U$ and assign $(V,\nu)\in(\mathcal F)$, $(Z,\varrho)\geqslant(V,\nu)\geqslant(Y,\sigma)$ according to (M). For $y'\in L_{(Y\cap C_1)^\circ}$, $y'P^{-1}$ is continuous in $(L_B,\|\,\|_B)\wedge(U,\vartheta)$ and admits continuous extension $\bar x'$ over L_B+U. Take the extension $x'\in X'$ of $\bar x'_{|U}$. Since $y\in L_C\cap Y$ implies $x = Py\in U$, we have $y'y = y'P^{-1}x = \bar x'x = x'Py$ and thus $\|y' - P'x'\|_{(Y\cap C)^\circ} = 0$, and the Proposition follows.

THEOREM 5.3. *Fix additional p-components ξ_0 and ξ_2 of (X,τ). If PY is strongly well located in X with respect to $\xi_0\leqslant\xi_1\leqslant\xi_2$, P^{-1} is $\wedge$-continuous from ξ_1 to ζ and P is continuous from ξ_0 to ζ, then P' is open from $(\varrho^\circ_{\xi_2})$ to (ϱ°_ζ).*

Proof. It follows directly from Theorem 5.1 and Proposition 5.1.

Remark. The openness of P' from $(\varrho^\circ_{\xi_2})$ to (ϱ°_ζ) implies $\wedge$-continuity of P from ξ_2 to ζ. The proof of this is postponed till the next paper.

6. Good location. Consider a linear locally convex space (X,τ) and its p-components $\xi_0\leqslant\xi_1\leqslant\xi_2$.

A linear subset $U\subset X$ is said to be well located in X with respect to $\xi_0\leqslant\xi_1\leqslant\xi_2$ if the following condition holds. (Cf. [9], p. 207, [10], p. 5.11 and [11], p. 110).

(ACC) To every adequate ball $B_2\in\xi_2$ there corresponds an adequate ball $B_1\in\xi_1$ such that for every adequate ball $B_0\in\xi_0$ we have

(*) $$L_{B_2}\cap(U\cap L_{B_0})^- \subset (U\cap L_{B_1})^-,$$

where the closures $^-$ are taken subsequently in $(L_{B_2},\|\cdot\|_{B_2})\wedge(X,\tau)$ and $(L_{B_1},\|\cdot\|_{B_1})$.

PROPOSITION 6.1. (Cf. [9], Prop. 5.3, I.) *Given p-components $\xi_0\leqslant\xi_1\leqslant\xi_2$ of (X,τ), every linear subset strongly well located in X with respect to $\xi_0\leqslant\xi_1\leqslant\xi_2$ is well located in X with respect to $\xi_0\leqslant\xi_1\leqslant\xi_2$ as well.*

Proof. Suppose (A_0) is satisfied and (ACC) does not hold. Hence, there exists an adequate $B_2\in\xi_2$ such that (*) does not hold for any adequate $B_1\in\xi_1$. Put $(M_2,\mu_2)\overset{\mathrm{df}}{=}(L_{B_2},\|\cdot\|_{B_2})\wedge(X,\tau)$.

Assign to such a B_2 an adequate $B_1\in\xi_1$ such that all the requirements of (A_0) are fulfilled. Since negation of (*) produces $B_0\in\xi_0$ and $z_0\in L_{B_2}\cap(U\cap L_{B_0})^-$ such that z_0 non $\in(U\cap L_{B_1})^-$, we can produce $z'\in L'_{B_1}$ vanishing on $U\cap L_{B_1}$ such that $z'z_0 = \|z_0\|_{B_2}$. Clearly, $\|z_0\|_{B_2}$ must be positive. Using (A_0), we produce $x'\in X'$ vanishing on $U\cap L_{B_0}$, $\|x'_{|2} - z'_{|2}\|'_2 < \frac12$. Setting for $y = z+x$, $z\in L_{B_2}$, $x\in X$, $y'y \overset{\mathrm{df}}{=} x'_{|2}z + x'x$, we have $y'\in M'_2$, and thus $y'z_0 = 0$ which means that $x'_{|2}z_0 = 0$. But $z'_{|2}z_0 = \|z_0\|_{B_2}$ and $|x'_{|2}z_0 - z'_{|2}z_0| < \frac12\|z_0\|_{B_2}$ which is contradictory.

PROPOSITION 6.2. (Cf. [9], Prop. 5.3, II, and [10], Prop. 5.5.) *Given p-components $\xi_0\leqslant\xi_1\leqslant\xi_2$ of (X,τ). If ξ_2 is reflexive, then every linear* *subset $U\subset X$ well located in X with respect to $\xi_0\leqslant\xi_1\leqslant\xi_2$ is strongly well located with respect to $\xi_0\leqslant\xi_1\leqslant\xi_2$ as well.*

Proof. Assume that the second component is reflexive and that (ACC) holds. Take an adequate ball $B_2\in\xi_2$. Due to the reflexivity of the component we can at once assume that $(L_{B_2},\|\cdot\|_{B_2})/K$, where $K=0(L_{B_2},\|\cdot\|_{B_2})$, is a reflexive space. Assign an adequate ball B_1 to B_2 according to (ACC). Then (A_0) amounts to the statement that given $B_0\in\xi_0$, the subspace

$$V_1=\{x'_{|2}\in L'_{B_2}\colon x'\in L_{(X\cap B_2)^{\bigcirc}},\ x'(U\cap L_{B_0})=\{0\}\},$$

is dense with respect to $\|\cdot\|'_{B_2}$ in the subspace

$$V_2=\{z'_{|2}\in L'_{B_2}\colon z'\in L'_{B_1},\ z'(U\cap L_{B_1})=\{0\}\}.$$

Due to the reflexitivity of $(L_{B_2},\|\cdot\|_{B_2})/K$ it is sufficient to show that V_1 is weak* dense, i.e. that if for $z\in L_{B_2}$ all functionals from V_1 vanish on z, then all functionals from V_2 vanish on z as well. The space V_1 consists of the restriction to L_{B_2} of functionals from

$$V_1^{\sim}=\{x'\in M'_2\colon x'(U\cap L_{B_0})=\{0\}\},$$

where $(M_2,\mu_2)=(L_{B_2},\|\cdot\|_{B_2})\wedge(X,\tau)$. Hence we have to prove that if for $z\in L_{B_2}$ all functionals from $V_1^{\sim}$ vanish on z, then all functionals from V_2 vanish on z as well. The first part means that $z\in(U\cap L_{B_0})^-$ where the closure $^-$ is taken in (M_2,μ_2) and the second part amounts to $z\in(U\cap L_{B_1})^-$ where the closure $^-$ is taken in $(L_{B_1},\|\cdot\|_{B_1})$ so that we arrive to the inclusion (*), and the Proposition follows.

THEOREM 6.1. *Given a locally convex space (X,τ) and its p-components $\xi_0\leqslant\xi_1\leqslant\xi_2$. If ξ_2 is reflexive, then every linear subset of X is strongly well located in X with respect to $\xi_0\leqslant\xi_1\leqslant\xi_2$ iff it is well located in X with respect to $\xi_0\leqslant\xi_1\leqslant\xi_2$.*

Proof. This amounts to putting together the two previously proved propositions.

7. Epimorphisms of spaces with bases of p-components. Given a linear space Y and a family Λ of $(\mathscr{F})$-classes such that for every $(\varrho)\in\Lambda$ the elements from $\boldsymbol{L}_\varrho$ are linear subsets of Y. We say that Λ overwhelms in Y if to every second category metric space (M,d) and a transformation T of a linear subspace Z of Y onto M there corresponds $(\varrho)\in\Lambda$ such that $T(Z\cap L)$ are of the second category in (M,d) for all $L\in\boldsymbol{L}_\varrho$.

In [5] and [8] the notion of overwhelming families has been thoroughly discussed and numerous examples have been provided.

A family Ξ of semi pre-$(\mathscr{DF})$-spaces is said to be unified if all the spaces L_ξ, $\xi\in\Xi$, are subspaces of the same linear space.

Given a barreled space (X,τ). A unified family Ξ of p-components of (X,τ) is said to be a basis of p-components for (X,τ) if to every ξ_1, $\xi_2\in\Xi$ there corresponds $\xi_3\in\Xi$ with $\xi_1,\xi_2\leqslant\xi_3$ (i.e. Ξ is directed by the relation $\leqslant$ and if every continuous seminorm in (X,τ) admits the continuous extension to (L_ξ,ι_ξ) for some $\xi\in\Xi$; a basis Ξ is called a strong basis if the family

$$\Xi^{\bigcirc}=\{(\varrho_\xi^{\bigcirc})\colon\ \xi\in\Xi\}$$

of the polar $(\mathscr{F})$-classes of the components from Ξ overwhelms in the adjoint X' to (X,τ) and if all $\xi\in\Xi$ admit (D).

Every $(\mathscr{LF})$-space in the sense of [1] admits a strong basis of p-components.

Consider two barreled spaces (X_i,τ_i), $i=1,2$, (X_1,τ_1) being Hausdorff, and a densely defined linear mapping P from X_1 to X_2. Let P' denote the adjoint of P.

THEOREM 7.1. *If (X_2,τ_2) admits a strong basis of p-components Ξ_2 and P' is an epimorphism, i.e.*

$$P'D_{P'}=X_1',$$

then P is one-to-one, to every p-component ξ_1 of (X_1,τ_2) there corresponds a p-component $\xi_2\in\Xi_2$ such that P' is open from $(\varrho_{\xi_2}^{\bigcirc})$ to $(\varrho_{\xi_1}^{\bigcirc})$ and P^{-1} is continuous from ξ_2 to ξ_1.

Proof. If $Px=0$ for some $x\in X_1$, then for every $x_1'\in X_1'$ we have $x_1'x=(P'x_2')x=x_2'(Px)=0$ so that $x=0$. Take an arbitrary ξ_1. Since $\Xi_2^{\bigcirc}$ overwhelms, we can find $\xi_2\in\Xi_2$ such that $P'(L\cap D_{P'})$ is of the second category in $(X_1',\varrho_{\xi_1}^{\bigcirc})$ for every $L\in L_{\varrho_{\xi_2}^{\bigcirc}}$. The mapping P' is easily verified to be complete-closed from $(X_2',\varrho_{\xi_2}^{\bigcirc})$ in $(X_1',\varrho_{\xi_1}^{\bigcirc})$ and thus applying Theorem 2 of [7], we find that it must be open. Hence, given $B_2\in\xi_2$, there exists $B_1\in\xi_1$ such that $P'\big(D_{P'}\cap(B_2\cap X_2)^{\bigcirc}\big)\supset(B_1\cap X_1)^{\bigcirc}$. Taking $\{y_n\}\subset D_P$ with $\{Py_n\}$ bounded in $(L_{B_2},\|\cdot\|_{B_2})$, we find that for every $x_1'\in L_1'$ we have $x_1'y_n=(P'x_2')y_n=x_2'(Py_n)$ bounded so that $\{y_n\}$ must be bounded in $(L_{B_1},\|\cdot\|_{B_1})$. This concludes the proof of the Theorem.

THEOREM 7.2. *Consider a barreled Hausdorff space (X_1,τ_1) with a basis Ξ_1 of p-components, a barreled space (X_2,τ_2) with a strong basis Ξ_2 of reflexive p-components and a linear continuous mapping P of (X_1,τ_1) into (X_2,τ_2).*

In order that $P'X_2'=X_1'$, it is necessary and sufficient that the following conditions be satisfied.

a. P is one-to-one and to every $\zeta\in\Xi_1$ there corresponds $\xi\in\Xi_2$ such that P^{-1} is continuous from ξ to ζ and that P' is nearly surjective from $(\varrho_\xi^{\bigcirc})$ to $(\varrho_\zeta^{\bigcirc})$.

b. To every $\xi_1\in\Xi_2$ there corresponds $\xi_2\in\Xi_2$, $\xi_1\leqslant\xi_2$, such that PX_1 is well located in X_2 for every $\xi_0\leqslant\xi_1$.

Proof. If a. and b. hold, then to sufficiently fine $\zeta \in \Xi_1$ we can assign $\xi_0 \leqslant \xi_1 \leqslant \xi_2$ from Ξ_2 in such a way that Theorem 5.1 and Proposition 5.1 can be applied. Since it holds for every $\zeta \in \Xi_1$ and Ξ_1 form a basis, it must be $P'X_2' = X_1'$.

The necessity of a. follows directly from Theorem 7.1. To verify the necessity of b. fix an arbitrary $\xi_1 \in \Xi_2$ and consider $U = PX_1$. Given $u' \in U'$, we find that $u'P \in X_1'$ and since P' is an epimorphism, there exists $v' \in X_2'$ such that $u'P = P'v'$. Hence, the operation I' of the restriction of functionals from X_2 to U constitutes an epimorphism of X_2' onto U'. Applying Theorem 7.1 for $P = I$, we find $\xi_2 \in \Xi_2$ such that I' is open from $(\varrho^{\circ}_{\xi_2})$ to $(\varrho^{\circ}_{U\cap\xi_1})$, and then from Theorems 5.2 and 6.1 we obtain the good location of PX_1 and the Theorem holds.

THEOREM 7.3. *Consider a densely defined linear mapping P from a locally convex space (Y, σ) to a barreled space (X, τ) and let (ϱ') be an $(\mathscr{F})$-class such that $(L', \varrho') \geqslant (Y', \sigma(Y', Y))$ for some $L' \in \boldsymbol{L}_{\varrho'}$. Assume that (X, τ) admits a strong basis of p-components Ξ. If for some $L' \in \boldsymbol{L}_{\varrho'}$,*

$$P' D_{P'} \supset L',$$

then there exists $\xi \in \Xi$ such that P' is open from (ϱ°_{ξ}) to (ϱ').

Proof. The Theorem is a trivial consequence of Theorem 1 of [8] with T defined as the restriction of P' to $P'^{-1}L'$. The assumed continuity of the imbedding of (L', ϱ') into $(Y', \sigma(Y', Y))$ guarantees that T is closed in the sense required in Theorem 1 of [8].

COROLLARY 7.1. *Consider a barreled space (X, τ) with a strong basis of p-components Ξ. If (ϱ') is an $(\mathscr{F})$-class such that $(L', \varrho') \geqslant (X', \sigma(X', X))$, then there exists $\xi \in \Xi$ such that the identical injection from (ϱ') to (ϱ°_{ξ}) is continuous.*

Proof. In Theorem 7.3, put $(X, \tau) = (Y, \sigma)$ and let I be the identical mapping of Y onto X. Then P' is the identical mapping of X' onto X' so that the condition $P'X' = Y' \supset L'$ of Theorem 7.3 is satisfied. Hence we can find $\xi \in \Xi$ such that P' is open from (ϱ°_{ξ}) to (ϱ') and thus P'^{-1} is continuous from (ϱ') to (ϱ°_{ξ}). And since P'^{-1} is also the identical mapping of X' onto X', the Corollary follows.

8. Regularities assigned to p-components. Given a linear space X and a ball C in X. A functional $x' \in X^*$ is said to be C-regular if

$$\|x'\|_{C^{\circ}} \overset{\mathrm{df}}{=} \sup\{|x'x| : x \in C\} < \infty.$$

If (X, τ) is a locally convex space, then we denote by $L_{C^{\circ}}$ the subspace of X' consisting of all functionals of regularity C.

Given a p-component ξ of (X, τ), the class $\boldsymbol{L}_{\varrho^{\circ}_{\xi}}$ contains all spaces $L_{(B\cap X)^{\circ}}$, $B \in \xi$. We call $\boldsymbol{L}_{\varrho^{\circ}_{\xi}}$ the class of regularities assigned to ξ.

Functionals from the intersection

$$\bigcap \boldsymbol{L}_{\varrho^{\circ}_{\xi}} \overset{\mathrm{df}}{=} \bigcap_{L \in \boldsymbol{L}_{\varrho^{\circ}_{\xi}}} L$$

of all spaces from $\boldsymbol{L}_{\varrho^{\circ}_{\xi}}$ shall be called ξ-infinitely regular.

We know from Part 4 of this paper that while spaces from $\boldsymbol{L}_{\varrho^{\circ}_{\xi}}$ can be given only a group topology, the space $\cap \boldsymbol{L}_{\varrho^{\circ}_{\xi}}$ is always a Fréchet space.

THEOREM 8.1. *Consider a linear mapping T from the dual X' of a barreled space (X, τ) to a dual Y' of a barreled space (Y, τ). Suppose that T is weak* sequentially closed and that (X, τ) admits a strong basis of p-components Ξ. If T maps its domain onto Y', then to every p-component ζ of (Y, σ) there corresponds a p-component $\zeta \in \Xi$ such that to every $B \in \xi$ there corresponds $C \in \xi$ such that given C-regular $y' \in Y'$ one can find a B-regular $x' \in X'$ with $y' = P'x'$.*

It is clearly visible from all of the previous investigations that the main difficulty of the theory lies in the fact that in Theorem 8.1 given ζ it might be impossible to fix $\xi \in \Xi$ in such a way that for every ζ-infinitely regular y' there would exists a ξ-infinitely regular x' such that $y' = P'x'$. A counter-example can be found in [3]. It means that given ζ-infinitely regular y', we can provide B-regular solutions for every $B \in \xi$, while at the same time we might be unable to produce any ξ-infinitely regular solution.

THEOREM 8.2. *Consider a barreled space (X, τ) and an $(\mathscr{F})$-class (ϱ). Suppose that (X, τ) admits a strong basis of p-components Ξ. Let in the sequel T be a linear mapping from X' to some $L_1 \in \boldsymbol{L}_{\varrho}$ which is closed as a mapping from (ϱ°_{ξ}) to (ϱ) for every $\xi \in \Xi$. If the image of the domain D_T of T contains at least one $L \in \boldsymbol{L}_{\varrho}$, i.e. if $TD_T \supset L$ for some $L \in \boldsymbol{L}_{\varrho}$, $L \subset L_1$, then there exists $\xi \in \Xi$ such that T is open from (ϱ°_{ξ}) to (ϱ).*

Proof. It is sufficient to notice that the family $\{(\varrho^{\circ}_{\xi}) : \xi \in \Xi\}$ overwhelms in X' and then there must exist a $\xi \in \Xi$ such that T turns every $K \cap D_T$, $K \in \boldsymbol{L}_{\varrho^{\circ}_{\xi}}$, onto a second category set in (L, ϱ). Then we apply Theorem 2 of [7] and the Theorem follows.

Theorem 8.1 is an easy consequence of Theorem 8.2.

9. Two examples. Given a vector bundle (E, p, M) over a non-compact C^{∞} manifold M we write $C_0^{\infty}(M, E)$ for the linear space of all C^{∞} sections of (E, p, M) which has compact supports. Using the definition of Atiyah and Bott (Annals of Math. 86 (1967), p. 391), we introduce the space of distributional sections $\mathscr{D}'(M, E)$. Consider a trivilization

$$E'_{|U} \xrightarrow{\tilde{h}} V \times C^{q},$$

where V is an open set in the appropriate Euclidean space. For $f \in [C_0^\infty(V)]^q$ we write

$$(\tilde{h}^* f)(t) = \tilde{h}^{-1}(t, f(t)), \quad t \in V,$$

and for $g \in C_0^\infty(V)$ we write

$$g_{(p)} = (g_1, \ldots, g_q), \quad g_i = \begin{cases} 0 & i \neq p, \\ g & i = p, \end{cases}$$

so that for $u \in \mathscr{D}'(M, E)$ we can write

$$(\tilde{h}^* u)_p(g) = u(\tilde{h}^* g_{(p)})$$

and

$$\tilde{h}^* u = ((\tilde{h}^* u)_1, \ldots, (\tilde{h}^* u)_q).$$

Thus we defined the transformation

$$\tilde{h}^*: \mathscr{D}'(M, E) \to [\mathscr{D}'(V)]^q$$

and then for every $r \in M$ and $u \in \mathscr{D}'(E, M)$ we define

$$(\operatorname{ord} u)(r) \overset{\mathrm{df}}{=} \max_{1 \leqslant i \leqslant q} \operatorname{ord}(\tilde{h}^* u)_i.$$

Here the definition is independent of the choice of $\tilde{h}$ only that $r \in p\tilde{h}^{-1}(V \times C^q)$.

It is easy to see that for $u \in \mathscr{D}'(M, E)$ the function $\operatorname{ord} u$ assumes only a finite number of values on every compact in M.

Denote by $\mathfrak{N}$ the set of all natural valued functions defined on M and assuming only a finite number of values on every compact in M.

Suppose that we are given two vector bundles (E_1, p_1, M_1) and (E_2, p_2, M_2) and consider the distribution spaces $\mathscr{D}'_1 = \mathscr{D}'(M_1, E_1)$ and $\mathscr{D}'_2 = \mathscr{D}'(M_2, E_2)$.

Directly from Theorem 8.1 we obtain the following

THEOREM 9.1. *Given a linear sequentially closed transformation T of a subspace $Y \subset \mathscr{D}'_2$ onto $\mathscr{D}'_1$. To every $\boldsymbol{n}_1 \in \mathfrak{N}_1$ there corresponds $\boldsymbol{n}_2 \in \mathfrak{N}_2$ such that to every compact K_2 in M_2 there coresponds a compact K_1 in M_1 such that to every $v \in \mathscr{D}'_1$ with $\operatorname{ord} v \leqslant \boldsymbol{n}_1$ pointwise on K_1 there corresponds $u \in \mathscr{D}'_2$, $Tu = v$, with $\operatorname{ord} u \leqslant \boldsymbol{n}_2$ pointwise on K_2.*

Loosely speaking, taking the right sides of the equation $Tu = v$ with $\operatorname{ord} v \leqslant \boldsymbol{n}_1$ pointwise on the whole M_1, there might not exist $\boldsymbol{n}_2$ such that there would always exist a solution u with $\operatorname{ord} u \leqslant \boldsymbol{n}_2$ pointwise on the whole M_2. One can find a counter-example in [3].

However, if we only require the existence of $\boldsymbol{n}_2$ such that for an arbitrary compact $K_2 \subset M_2$ and any $v \in \mathscr{D}'_1$, $\operatorname{ord} u \leqslant \boldsymbol{n}_1$ pointwise on M_1, there exists a solution $u \in Y$ such that $\operatorname{ord} u \leqslant \boldsymbol{n}_2$ pointwise only on K_2, then Theorem 9.1 answers it in the affirmative. Even more, the function $\boldsymbol{n}_2$ assigned to $\boldsymbol{n}_1$ can be chosen in such a way that to every compact $K_2 \subset M_2$ one can find a compact $K_1 \subset M_1$ such that for every $v \in \mathscr{D}'_1$ with $\operatorname{ord} v \leqslant \boldsymbol{n}_1$ pointwise on K_1 there exists a solution u with $\operatorname{ord} u \leqslant \boldsymbol{n}_2$ pointwise on K_2.

Take an open subset Ω of the N-dimensional Euclidean space and a Hilbert space $(H, \|\cdot\|)$. Denote by $\mathscr{D}(\Omega, H)$ the space of all infinitely often differentiable H-valued functions with compact supports. To every non-negative integer n we assign a norm

$$\|f\|_n = \left(\Sigma_{|p| \leqslant n} \int \|D^p f(t)\|^2 dt\right)^{\frac{1}{2}},$$

where $p = (p_1, \ldots, p_N)$ is an N-tuple of non-negative integers, $|p| = p_1 + \ldots + p_N$, and D^p denotes the differentiation $|p|$ times, p_i on the i-th variable. For each compact K, the subspace

$$\mathscr{D}(K, H) = \{f \in \mathscr{D}(\Omega, H): \operatorname{supp} f \subset K\}$$

provided with the topology induced by seminorms $\|\cdot\|_k$, $k = 1, 2, \ldots$ is a Fréchet space. The space

$$\mathscr{D}'(\Omega, H)$$

consisting of all linear functionals defined on $\mathscr{D}(\Omega, H)$ which are continuous in every $\mathscr{D}(K, H)$ we shall call the space of H-valued distributions.

Denote by $\mathscr{K}$ the set of all compact subsets of Ω. Furthermore, denote by $\mathfrak{M}$ the set of all natural valued functions $\boldsymbol{n}$ defined on $\mathscr{K}$ such that for every $K \in \mathscr{K}$ the set

$$\{n: n = \boldsymbol{n}(L), K \supset L \in \mathscr{K}\}$$

is finite. For $u \in \mathscr{D}'(\Omega, H)$ and $K \in \mathscr{K}$ write

$$\|u\|_{n,K}^* = \sup\{|uf|: \|f\|_n \leqslant 1, \operatorname{supp} f \subset K\}.$$

Define for $\boldsymbol{n} \in \mathfrak{M}$ and $K \in \mathscr{K}$

$$\|u\|_{\boldsymbol{n},K}^* = \sup\{\|u\|_{\boldsymbol{n}(L),L}^*: K \supset L \in \mathscr{K}\}.$$

Due to the definition of the set $\mathfrak{M}$ we can always find $L_1, \ldots, L_n$ such that

$$\|u\|_{\boldsymbol{n},K}^* = \sup\{\|u\|_{\boldsymbol{n}(L_i),L_i}^*: i = 1, 2, \ldots, n\}.$$

Consider now two open subsets Ω_1 and Ω_2 contained in Euclidean spaces E^{N_1} and E^{N_2} respectively, and two Hilbert spaces $(H_1, \|\cdot\|_1)$, $(H_2, \|\cdot\|_2)$. Then we introduce $\mathscr{K}_i$ and $\mathfrak{M}_i$, $i = 1, 2$, accordingly.

A linear transformation

$$T: Y \to \mathscr{D}'(\Omega_1, H_1),$$

where Y is a linear subspace of $\mathscr{D}'(\Omega_2, H_2)$ is said to be sequentially closed if for any $\boldsymbol{n}_i \in \mathfrak{M}_i$, $i = 1, 2$, and any $\{u_n\} \subset Y$, $u \in \mathscr{D}'(\Omega_2, H_2)$, $v \in \mathscr{D}'(\Omega_1, H_1)$

$$\|u_n - u\|^*_{\boldsymbol{n}_2, K} \quad \text{tends to zero for every} \quad K \in \mathscr{K}_2$$

and

$$\|Tu - v\|^*_{\boldsymbol{n}_1, K} \quad \text{tends to zero for every} \quad K \in \mathscr{K}_1$$

implies

$$v \in Y \quad \text{and} \quad Tu = v.$$

Directly from Theorem 8.2 we obtain the following

THEOREM 9.2. *If T is sequentially closed and if $TY = \mathscr{D}'(\Omega_1, H_1)$, then to every $\boldsymbol{n}_1 \in \mathfrak{M}_1$ there corresponds an $\boldsymbol{n}_2 \in \mathfrak{M}_2$ such that to every $K_2 \in \mathscr{K}_2$ there corresponds an $K_1 \in \mathscr{K}_1$ and $C > 0$ such that*

$$T\{u \in Y: \|u\|^*_{\boldsymbol{n}_2, K_2} \leqslant C\} \supset \{v \in \mathscr{D}'(\Omega_1, H_1): \|v\|^*_{\boldsymbol{n}_1, K_1} \leqslant 1\}.$$

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