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Banach spaces with finite dimensional expansions of identity and universal bases of finite dimensional subspaces

by

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Abstract. Separable Banach spaces with the property (BAP) that the identity operator on the space is the pointwise limit of finite dimensional operators are considered. It is shown that (BAP) is equivalent to the fact that the space is isomorphic to a complemented subspace of a Banach space with a basis of finite dimensional subspaces. This leads to an alternative proof of Kadec's result [6] that there exists a separable Banach space complementably universal for all Banach spaces with (BAP). Other applications to a linear extension theorem and to the existence of various universal spaces are obtained.

Introduction. The present paper deals with separable Banach spaces with the property that the identity operator on the space is the pointwise limit of finite dimensional operators. This property — called by Kadec [6] "The Banach Approximation Property" — is a stronger requirement than the Approximation Property of Grothendieck [4] but weaker than the Metric Approximation Property [4].

In Section 1 we observe that a Banach space has the Banach Approximation Property iff it is isomorphic to a complemented subspace of a Banach space with a basis of finite dimensional subspaces. A similar result for unconditional basis of finite dimensional subspaces is also established.

In Section 2 we used the technique developped in Section 1 to prove a result, due to C. Ryll-Nardzewski (unpublished), which generalizes a linear extension theorem of E. Michael and the first named author [11].

Section 3 is devoted to the construction of a complementably universal basis of finite dimensional subspaces for the class of all bases of finite dimensional subspaces. This is done by a modification of the construction of the universal basis (cf. [13]). Combining this result with the results of Section 1 we obtain an alternative proof of Kadec's theorem (cf. [6]) on the existence of separable Banach space complementably universal for the class of all separable Banach spaces with the Banach Approximation Property.

Section 4 is closely related to Kadec's paper [6]. We show that the universal space constructed in [6] has a basis of finite dimensional subspaces and therefore is isomorphic to the universal space constructed in Section 3. Using Kadec's method we also construct a separable Banach space, whose dual is an L_1 space, which is complementably universal for the class of all separable Banach spaces whose duals are L_1 spaces. By an L_1 space we mean a space $L_1(\mu)$ for suitable measure μ. The above result is related to the recent investigation by Lazar and Lindenstrauss [7] and Lindenstrauss and Wulbert [9].

The terminology and notation used in the present paper is standard. We only mention here that if $f\colon X\to Y$ is a function and if Z is a subset of X, then $f|Z$ denotes the restriction of f to Z.

1. Finite dimensional expansions of identity and bases of finite dimensional subspaces. In the sequel "f. d." stands for "finite dimensional" "operator" stands for "bounded linear operator" and "subspace" stands for "closed linear subspace". For any sequence (A_n) of operators acting between Banach spaces X and Y let

$$k\big((A_n)\big) = \sup_n \Big\|\sum_{j=1}^{n} A_j\Big\|,$$

$$k_u\big((A_n)\big) = \sup_n \sup_{|\varepsilon_j|=1} \Big\|\sum_{j=1}^{n} \varepsilon_j A_j\Big\|.$$

DEFINITION 1.1. A sequence of non zero f. d. operators (A_n) from a Banach space X into itself is called an *(unconditional) f. d. expansion of identity* of X if

$$x = \sum_n A_n(x) \quad \text{for} \quad x\in X$$

(and the series converges unconditionally). Moreover if $A_nA_m = 0$ for $n\neq m$ $(n, m = 1, 2, ..)$ then (A_n) is called an *(unconditional) orthogonal expansion of identity* of X.

The Banach–Steinhaus Principle implies that if a sequence (A_n) of f. d. operators is an (unconditional) f. d. expansion of identity of X then span $\bigcup_n A_n(X)$ is dense in X and $k\big((A_n)\big)<\infty$ $\big($resp. $k_u\big((A_n)\big)<\infty\big)$. Hence if a Banach space X has an f. d. expansion of identity, then X is separable. The following observation is also well known

PROPOSITION 1.1. *For any separable Banach space X and any $k\geqslant 1$ the following conditions are equivalent*:

(a) *For any compact $K\subset X$ and $\varepsilon>0$ there exists an f. d. operator $B\colon X\to X$ such that 1) $\|B(x)-x\|<\varepsilon$ for $x\in K$, 2) $\|B\|\leqslant k$.*

(b) *There exists a sequence (B_n) of f. d. operators in X such that*

$$\lim_n \|B_n(x)-x\| = 0 \quad \textit{for } x\in X \textit{ and } \|B_n\|\leqslant k \textit{ for } n = 1, 2, \ldots$$

(c) *There exists an f. d. expansion of identity of X, say (A_n) with $k\big((A_n)\big)\leqslant k$.*

For the proof use: for (c) → (a) — that a pointwise convergent sequence of operators in a Banach space converges uniformly on any compact set; for (a) → (b) — that any separable Banach space has a dense linear set which is the union of an increasing sequence of compacta; for (b) → (c) assume without loss of generality that $B_{m+1}\neq B_m$ for $m = 1, 2, \ldots$ and put $A_1 = B_1$, $A_{2m} = A_{2m+1} = \frac{1}{2}(B_{m+1}-B_m)$ for $m = 1, 2, \ldots$

The condition (a) for $k=1$ is usually called the *metric approximation property* (cf. [4]). Kadec [6] uses the phrase "*X has* BAP" for "X satisfies (b) for some $k<\infty$". Concluding X has an f. d. expansion of identity iff X has BAP in the sens of Kadec [6]. This property for separable Banach spaces is weaker than the metric approximation property but stronger that the approximation property of Grothendieck [4] which means that X satisfies (a) without requirement 2).

If (A_n) is an f. d. orthogonal expansion of identity of X, then A_n are projections because $A_n(x) = A_n\big(\sum_m A_m(x)\big) = \sum_m A_nA_m(x) = A_n^2(x)$ for $x\in X$ $(n = 1, 2, \ldots)$

A sequence (X_n) of f. d. subspaces of X is *an (unconditional) basis of finite dimensional subspaces for X*, shortly an *f. d. s. basis for X* if for any $x\in X$ there exists a unique sequence (x_n) such that $x_n\in X_n$ for $n = 1, 2, \ldots$ and $x=\sum_n x_n$ (and this series converges unconditionally). Recall the following well known

PROPOSITION 1.2. (cf. e. g. [14]) *If (A_n) is an (unconditional) orthogonal f. d. expansion of identity of X, then $\big(A_n(X)\big)$ is an (unconditional) f. d. s. basis for X.*

Conversely, if (X_n) is an (unconditional) f. d. s. basis for X, then for any n there exists a projection from X onto X_n annihilating all X_m for $m\neq n$. The sequence of these projections is an (unconditional) orthogonal f. d. expansion of identity of X.

Observe that the classical concept of a basis for a Banach space corresponds to the f. d. s. basis (X_n) with $\dim X_n = 1$.

An orthogonal f. d. expansion of identity (A_n) is said to be *monotone* (resp. the correspondent *f. d. s. basis $\big(A_n(X)\big)$ is monotone*) if $k\big((A_n)\big) = 1$. For a monotone f. d. s. basis we have

PROPOSITION 1.3. *For any Banach space X the following conditions re equivalent*

(i) *X has a monotone f. d. s. basis;*

(ii) *X is a π_1-space (cf. [11]), i. e. there is a sequence of f. d. projections (P_n) from X into itself such that $\|P_n\| = 1$ for $n = 1, 2, \ldots$; $P_1(X) \subset P_2(X) \subset \ldots$; $\bigcup_n P_n(X)$ is dense in X;*

(iii) *there exists a sequence of non-zero f. d. subspaces (X_n) such that $\operatorname{span}\bigcup_n X_n$ is dense in X and*

$$\Big\|\sum_{j=1}^{n} x_j\Big\| \leqslant \Big\|\sum_{j=1}^{n+1} x_j\Big\|$$

for any $x_j \in X_j$ $(j = 1, 2, \ldots, n+1;\ n = 1, 2, \ldots)$.

Proof. (i) $\to$ (ii). Put $P_1 = A_1$, $P_n = \sum_{j=1}^{n} A_j$ for $n = 2, 3, \ldots$ where (A_n) is a monotone orthogonal f. d. expansion of identity of X.

(ii) $\to$ (iii). Put $X_1 = P_1(X)$; $X_n = P_n(X) \cap \ker P_{n-1}$ for $n = 1, 2, \ldots$ Then for any $x_j \in X_j$ $(j = 1, 2, \ldots, n+1)$ we have

$$\Big\|\sum_{j=1}^{n+1} x_j\Big\| \geqslant \Big\|P_n\Big(\sum_{j=1}^{n+1} x_j\Big)\Big\| = \Big\|\sum_{j=1}^{n} x_j\Big\|.$$

(iii) $\to$ (i). See e. g. [14].

Now we are ready for the main result of the present section.

THEOREM 1.1. *A Banach space X has an (unconditional) f. d. expansion of identity iff X is isomorphic to a complemented subspace of a Banach space with an (unconditional) f. d. s. basis.*

The proof of Theorem 1.1. is an immediate consequence of the next two lemmas.

LEMMA 1.1. *Let (A_n) be an (unconditional) f. d. expansion of identity for a Banach space Y and let $P\colon Y \to Y$ be a projection. Let $X = P(Y)$. Then $(PA_n|X)$ is an (unconditional) f. d. expansion of identity for X. Moreover $k\big((PA_n|X)\big) \leqslant \|P\| k\big((A_n)\big)$* (resp. $k_u\big((PA_n|X)\big) \leqslant \|P\| k_u\big((A_n)\big)$).

Proof. Obvious.

DEFINITION 1.2. The *(unconditional) envelope of an (unconditional) f. d. expansion of identity of X, (A_n)* is the Banach space $\sum\big(A_n(X)\big)$ (resp. $\sum_u\big(A_n(X)\big)$) whose elements are sequences (x_j) such that $x_j \in A_j(X)$ for $j = 1, 2, \ldots$ and the series $\sum_j x_j$ is (unconditionally) convergent. The operation of addition and multiplication by scalars are defined coordinatwise and the norm in $\sum\big(A_n(X)\big)$ (resp. in $\sum_u\big(A_n(X)\big)$) is defined by

$$\|(x_j)\| = \sup_n \Big\|\sum_{j=1}^{n} x_j\Big\| \qquad \Big(\text{resp. } \|(x_j)\|_u = \sup_n \sup_{|\varepsilon_j|=1} \Big\|\sum_{j=1}^{n} \varepsilon_j x_j\Big\|\Big).$$

LEMMA 1.2. *If (A_n) is an (unconditional) f. d. expansion of identity of X, then the space $Y = \sum\big(A_n(X)\big)$ (resp. $Y = \sum_u\big(A_n(X)\big)$) has the following properties:*

(α) *The sequence (Y_n) is a monotone (unconditional) f. d. s. basis for Y where*

$$Y_n = \{(x_j) \in Y\colon x_j = 0 \quad \text{for} \quad j \neq n,\ j = 1, 2, \ldots\}.$$

(β) *The operator $U\colon X \to Y$ defined by $U(x) = \big(A_n(x)\big)$ for $x \in X$ is an isomorphic embedding such that $\|x\| \leqslant \|U(x)\| \leqslant k\big((A_n)\big)\|x\|$ (resp. $\|x\| \leqslant \|U(x)\|_u \leqslant k_u\big((A_n)\big)\|x\|$) for $x \in X$.*

(γ) *The operator $P\colon Y \underset{onto}{\to} U(X)$ defined by*

$$P\big((x_j)\big) = \Big(A_n\Big(\sum_j x_j\Big)\Big) \qquad \text{for } (x_j) \in Y$$

is a projection with $\|P\| \leqslant k\big((A_n)\big)$ (resp. $\|P\|_u \leqslant k_u\big((A_n)\big)$).

Proof. Routine.

A moment of reflection gives:

COROLLARY 1.1. *A Banach space X is isomorphic to a complemented subspace of a Banach space Y with a monotone (unconditional) basis iff there exists a sequence of one dimensional operators which is an (unconditional) f. d. expansion of identity of X.*

By a nonessential modification of the proofs (cf. e. g. [15]) one can check that all known examples of Banach spaces which are not isomorphic to any subspace of a Banach space with an unconditonal basis also are not isomorphic to any subspace of a Banach space with an unconditional f. d. s. basis and therefore to any subspace of a space with an unconditional f. d. expansion of identity. However there exists a Banach space which does not have any unconditional f. d. expansion of identity but which is isomorphic to a subspace of a Banach space with an unconditional basis.

EXAMPLE 1. 1. *The Lindenstrauss space Λ defined to be a kernel of any surjection of l_1 onto L_1 (cf. [8]) does not have any unconditional f. d. expansion of identity.*

Proof. We shall identify a Banach space with its canonical image in the second dual. Then Λ is not complemented in Λ^{**} (cf. [8]). Hence Λ is not isomorphic to any complemented subspace of a Banach space Y with the property that Y is complemented in Y^{**}. In particular Λ is not isomorphic to a dual of a Banach space (cf. [2], Th. 15). Since l_1 (and therefore Λ) does not have subspaces isomorphic to c_0, the desired conclusion is an immediate consequence of Lemma 1.2. and the following.

LEMMA 1.3. *If a Banach space X contains no subspace isomorphic to c_0 and if (A_n) is an unconditional f. d. expansion of identity of X, then $\sum_u(A_n(X))$ is isomorphic to a dual of a Banach space.*

Proof. By [14], Theorem II 10, it is enough to check that the sequence (Y_n) defined in Lemma 1.2. (α) is boundedly complete f. d. s. basis for $\sum_u(A_j(X))$. Precisely we have to check that if $y_j = (0, \ldots, x_j, 0, \ldots) \in Y_j$ are choosen so that

$$\sup_n \Big\| \sum_{j=1}^{n} y_j \Big\|_u < \infty \tag{1}$$

then the series $\sum_j y_j$ converges in $\sum_u(A_n(X))$. Clearly (1) is equivalent to the inequality

$$\sup_n \sup_{|\varepsilon_j|=1} \Big\| \sum_{j=1}^{n} \varepsilon_j x_j \Big\| < \infty. \tag{2}$$

Since X does not contain any subspace isomorphic to c_0, the inequality (2) implies that the series $\sum_j x_j$ is unconditionally convergent (cf. [1], Theorem 5) but this is equivalent to the fact that the series $\sum_j y_j$ converges in $\sum_u(A_n(X))$.

Remark. Let us observe that to the contrary of [10], Theorem 6.1 and Corollary 8 to Theorem 6.1, there exists in l_1 an unconditional f. d. expansion of identity, say (A_n), which is not absolute, i.e. $\sum_n \|A_n(x)\| = +\infty$ for some $x \in l_1$. To construct such (A_n) pick any unconditionally summable sequence (x_j) in l_1 satisfying the conditions $\sum_{j=1}^{\infty} x_j = 0$ and $\sum_{j=1}^{\infty} \|x_j\| = +\infty$, and put

$$A_n(x) = x(n)e_n + \sum_{k=1}^{\infty} x\big(2^{k-1}(2j-1)\big)x_i$$

for $n = 2^{i-1}(2j-1)$ and for $x = \big(x(n)\big)_{n=1}^{\infty} \in l_1$ $(i, j = 1, 2, \ldots)$. (Here e_n denotes the n-th unit vector in l_1 $(n = 1, 2, \ldots)$). We omit an easy verification that the sequence (A_n) defined above is the desired unconditional f. d. expansion of identity.

CONJECTURE. If X is an infinite-dimensional $\mathscr{L}_1$-space (in the sense of [10]) with an unconditional f. d. expansion of identity, then X is isomorphic to l_1.

2. A linear extension theorem. Let T be a topological space, S a closed subset of T, and $C(S)$ and $C(T)$ the Banach spaces of bounded continuous scalar-valued functions on S and T respectively. Let E and H be subspaces

of $C(S)$ and $C(T)$ respectively. An operator $u\colon E \to H$ is called a linear extension if $u(f)$ is an extension of f for any $f \in E$.

DEFINITION 2.1. The pair (E, H) has the *bounded extension property* if, given any $\varepsilon > 0$, every $f \in E$ has a bounded family of extensions

$$\Phi(f, \varepsilon) = \{f_{\varepsilon,W}\colon W \supset S,\ W \text{ open in } T\} \subset H$$

such that $|f_{\varepsilon,W}(x)| \leqslant \varepsilon$ whenever $x \in T \setminus W$.

The following result was proved in [11].

THEOREM 2.1. *Let S be a closed subset of a topological space T. If $E \subset C(S)$ is a separable π_1-space and if (E, H) has the bounded extension property, then there exists a linear extension $u\colon E \to H$ of norm one.*

The concept of the envelope of an f. d. expansion of identity allows us to prove the following improvement of Theorem 2.1. (due to Ryll-Nardzewski, unpublished).

THEOREM 2.2. *Let S be a closed subset of a topological space T. If $E \subset C(S)$ has an f. d. expansion of identity, say (A_n), and if (E, H) has the bounded extension property, then there exists a linear extension $u\colon E \to H$ such that $\|u\| \leqslant k\big((A_n)\big)$.*

Before passing to the proof of this result we shall need some lemmas.

LEMMA 2.1. *If the pair (E, H) has the bounded extension property, then given any $\varepsilon > 0$ and open $W \supset S$, every $f \in E$ has an extension $\tilde{f}_{\varepsilon,W} \in H$ such that $|\tilde{f}_{\varepsilon,W}(t)| \leqslant \varepsilon$ for $t \in T \setminus W$ and $\|\tilde{f}_{\varepsilon,W}\| \leqslant 3\|f\|$.*

Proof. By homogenity argument, it is enough consider the case where $f \in E$ and $\|f\| = 1$. Let us put

$$M(f, \varepsilon) = \sup\{\|f_{\varepsilon,W}\|\colon f_{\varepsilon,W} \in \Phi(f, \varepsilon)\},$$

where $\Phi(f, \varepsilon)$ is that of Definition 2.1. Using the bounded extension property of (E, H) we define inductively two sequences: $(\tilde{f}_n)$ in $\Phi(f, \varepsilon) \subset H$ and (W_n) of open subsets of T such that $W_n \supset S$; $\tilde{f}_n = f_{\varepsilon,W_n}$; $W_{n+1} = W_n \cap \big(T \setminus \{t \in T\colon |\tilde{f}_n(t)| \geqslant 2\}\big)$ $(n = 1, 2, \ldots)$.

Having done this we put $\tilde{f}_{\varepsilon,W} = N^{-1}\sum_{j=1}^{N} \tilde{f}_j$ where N is any integer greater than $M(f, \varepsilon)$. One can easily check that $\|\tilde{f}_{\varepsilon,W}\| \leqslant 3$. This completes the proof of the Lemma.

If Z is a Banach space, then $c(Z)$ denote the space of all convergent sequences of elements of Z with the coordinatwise operations of addition and multiplication by scalars and with the norm $\|(z_n)\| = \sup_n \|z_n\|$.

Let $[\omega]$ denote the one point compactification of the positive integers. Denote the "limit point" of $[\omega]$ by "∞". For any topological space T there is a natural isometric isomorphism

$$j_T\colon c\big(C(T)\big) \underset{\text{onto}}{\to} C(T\times[\omega])$$

defined by

$$j_T\big((f_k)\big)(t,n) = \begin{cases} f_n(t) & \text{for } t\in T,\ n=1,2,\ldots, \\ \lim\limits_n f_n(t) & \text{for } t\in T,\ n=\infty. \end{cases}$$

LEMMA 2.2. *If (E,H) has the bounded extension property, then*

$$\big(j_S(c(E)),\, j_T(c(H))\big)$$

has the same property.

Proof. Given $\varphi = j_S\big((f_n)\big)$ for $(f_n)\in c(E)$, $\varepsilon>0$ and open subset W of $T\times[\omega]$ such that $W\supset S\times[\omega]$. Let $W_\infty = \{t\in T\colon (t,n)\in W \text{ for all } n=1,2,\ldots\infty\}$. Since the natural projection $T\times[\omega]\to T$ is a closed map (see e. g. [3], Chap. III, Theorem 8), W_∞ is an open subset of T such that $S\times[\omega]\subset W_\infty\times[\omega]\subset W$.

Let $f_\infty = \lim\limits_n f_n$. By Lemma 2.1 there exists an $\tilde f_\infty\in H$ such that $\|\tilde f_\infty\|\leqslant 3\|f_\infty\|\leqslant 3\|\varphi\|$, $\tilde f_\infty$ extends f_∞ and $|\tilde f_\infty(t)|\leqslant\varepsilon/3$ for $t\in T\setminus W_\infty$. Let $g_n = f_n - f_\infty$. Again, by Lemma 2.1, there exists in H function $\tilde g_n$ which extends g_n and $\|\tilde g_n\|\leqslant 3\|g_n\|$; $|\tilde g_n(t)|\leqslant\varepsilon/3$ for $t\in T\setminus W_\infty$ $(n=1,2,\ldots)$. Observe that $\lim\limits_n\|g_n\|=0$. Hence $\lim\limits_n\|\tilde g_n\|=0$. This implies that $\lim\limits_n(\tilde f_\infty + \tilde g_n) = \tilde f_\infty$. Let us put $\tilde\varphi = j_T\big((\tilde f_\infty+\tilde g_n)\big)$. Clearly $\tilde\varphi$ extends φ and $\|\tilde\varphi\| = \sup\limits_n\|\tilde f_\infty+\tilde g_n\|\leqslant 3\|\varphi\| + 3\sup\limits_n\|g_n\|\leqslant 9\|\varphi\|$.

Finally if $(t,n)\in T\times[\omega]\setminus W$, then $(t,n)\in T\times[\omega]\setminus W_\infty\times[\omega]$. Hence, by definition of j_T, $|\tilde\varphi(t,n)|\leqslant\frac{2}{3}\varepsilon$. This completes the proof of the Lemma.

Proof of Theorem 2.2. The envelope $\sum\big(A_n(E)\big)$ can be obviously regarded as a subspace of $c(E)$ (To any $(e_n)\in\sum\big(A_n(E)\big)$ we assign the sequence $\big(\sum\limits_{j=1}^{n} e_j\big)\in c(E)$). Hence, Lemma 2.2 and the definition of the bounded extension property implies that the pair

$$\Big(j_S\big(\textstyle\sum(A_n(E))\big),\, j_T\big(c(H)\big)\Big)$$

has the bounded extension property. By Lemma 1.2 and Proposition 1.3 the envelope $\sum\big(A_n(E)\big)$ and therefore $j_S\big(\sum(A_n(E))\big)$ are π_1-spaces.

Hence by Theorem 2.1, there exists a linear extension $v\colon j_S\big(\sum(A_n(E))\big)\to j_T\big(c(H)\big)$ with $\|v\|=1$. Let us consider the following diagram

$$E \xrightarrow{U_E} \sum\big(A_n(E)\big) \xrightarrow{j_T^{-1}vj_S} c(H) \xrightarrow{Q_H} H$$

where $U_E(e) = \big(A_n(e)\big)$ for $e\in E$ and $Q_H\big((h_n)\big) = \lim\limits_n h_n$ for $(h_n)\in c(H)$. Clearly $\|U_E\|\leqslant k\big((A_n)\big)$ and $\|Q_H\|=1$. One can easily check that

$$u = Q_H j_T^{-1} v j_S U_E\colon E\to H$$

is a linear extension with $\|u\|\leqslant k\big((A_n)\big)$. This completes the proof.

3. Universal f. d. s. bases. Two f. d. s. bases (X_n) and (Y_n) in spaces X and Y, respectively, are *equivalent* (*isometrically equivalent*) if there exists an isomorphism (isometry) $T\colon X\xrightarrow{\text{onto}} Y$ such that $T(X_n) = Y_n$ for $n=1,2,\ldots$ By an isomorphism and an isometry we always mean the linear isomorphism and the linear isometry. A (complemented) f. d. s. subbasis of an f. d. s. basis (X_n) is any subsequence (X_{n_k}) (such that $P\colon X\to X$ defined by $P(\sum\limits_{n=1}^{\infty} x_n) = \sum\limits_{k=1}^{\infty} x_{n_k}$ is a continuous projection).

DEFINITION 3.1. An f. d. s. basis (X_n) is (*complementably*) *universal* for a class $\mathscr{B}$ of f. d. s. bases if any $(Y_n)\in\mathscr{B}$ is equivalent to a (complemented) f. d. s. subbasis (X_{n_k}) of the f. d. s. basis (X_n).

If we have an f. d. s. basis $(X_i)_{i=1}^{k}$ $(k=1,2,\ldots\infty)$, then the Banach space in which (X_i) is a basis will be denoted by $\bigoplus\limits_{i=1}^{k} X_i$, or sometime by $\bigoplus X_i$. The basis (X_i) will be denoted by $\mathbf{X}$. For any f. d. s. basis (X_i) by P_n we will always mean the projection $P_n\colon \bigoplus\limits_{i=1}^{k} X_i\to\bigoplus\limits_{i=1}^{k} X_i$ defined by $P_n(\sum\limits_i x_i) = \sum\limits_{i=1}^{n} x_i$. Observe that (X_i) is a monotone f. d. s. basis iff $\|P_n\|=1$ $(n=1,2,\ldots)$.

Let us consider the set $\mathscr{A}_n$ of all monotone f. d. s. bases $(E_i)_{i=1}^{k}$ where $\dim\bigoplus\limits_{i=1}^{k} E_i\leqslant n$. By the *index* of an f. d. s. basis $(E_i)_{i=1}^{k}$ we mean the sequence $(\dim E_1, \dim E_2, \ldots, \dim E_k)$ and by the *dimension of this basis*, the number $\dim\bigoplus\limits_{i=1}^{k} E_i$. In the set $\mathscr{A}_n$ we introduce a metric ϱ by the formula

$$\varrho\big((E_i),(F_i)\big) = \begin{cases} n & \text{if indices are different,} \\ \ln\inf\|T\|\,\|T^{-1}\| & \text{otherwise,} \end{cases}$$

where the infimum is extended on all isomorphisms $T\colon \bigoplus_{i=1}^{k} E_i \to \bigoplus_{i=1}^{k} F_i$ such that $T(E_i) = F_i$ $(i = 1, 2, \ldots k)$.

Remark. As a topological space $\mathscr{A}_n$ is a disjoint sum of sets of bases with the same index, topologized by the metric ϱ.

LEMMA 3.1. *The metric space $(\mathscr{A}_n, \varrho)$ is compact.*

Proof. Let us consider the sequence of positive integers $k_1 < k_2 < \ldots < k_r = n$. The set $E_n(k_1, \ldots, k_r)$ of all sequences of Banach spaces $E_1 \subset E_2 \subset \ldots \subset E_r$, $\dim E_i = k_i$ with the metric

$$\theta\big((E_1 \subset \ldots \subset E_r), (F_1 \subset \ldots \subset F_r)\big)$$
$$= \ln\inf\{\|T\|\ \|T^{-1}\|\colon T \text{ is an isomorphism from } E_r \text{ onto } F_r \text{ and } T(E_i) = F_i,\ i = 1, 2, \ldots, r\}$$

is a compact metric space. This result is implicite contained in Gurarij [5], Lemma 1b. It is easy to see that the closed and open subset of $\mathscr{A}_n$ consisting of all f. d. s. bases of the index $(k_1, \ldots, k_r)$ is a closed subset of $E_{k_1+k_2+\ldots+k_r}(k_1, k_1+k_2, \ldots, k_1+\ldots+k_r)$. This completes the proof.

Define $J_n\colon \mathscr{A}_{n+1} \to \mathscr{A}_n$ by

$$J_n\big((E_i)_{i=1}^{k}\big) = \begin{cases} (E_i)_{i=1}^{k-1} & \text{if dimension of } (E_i)_{i=1}^{k} \text{ is } n+1, \\ (E_i)_{i=1}^{k} & \text{if dimension of } (E_i)_{i=1}^{k} \text{ is } < n+1. \end{cases}$$

LEMMA 3.2. *Let $\mathscr{S}$ be a finite ε-net for $\mathscr{A}_n$, $0 < \varepsilon < 1$. Then for any $\eta > 0$, such that $\varepsilon + \eta < 1$, there exists a finite $(\varepsilon+\eta)$-net for $\mathscr{A}_{n+1}$, say $\mathscr{S}_1$ such that $J_n(\mathscr{S}_1) = \mathscr{S}$ and if $\varrho\big(J_n((E_i)), (E_i^1)\big) \leqslant \varepsilon$ for $(E_i) \in \mathscr{A}_{n+1}$ and $(E_i^1) \in \mathscr{S}$, then there exists an $(E_i^2) \in \mathscr{S}_1$ such that $J_n\big((E_i^2)\big) = (E_i^1)$ and $\varrho\big((E_i^2), (E_i)\big) \leqslant \varepsilon + \eta$.*

Proof. Let $\mathscr{S} = \{\boldsymbol{F}^1, \boldsymbol{F}^2, \ldots, \boldsymbol{F}^r\}$. Pick an $\eta/2$-net $\mathscr{S}^j$ for every $J_n^{-1}(\boldsymbol{F}^j)$. We will show that $\mathscr{S}_1 = \bigcup_{j=1}^{r} \mathscr{S}^j$ is an $(\varepsilon+\eta)$-net satisfying conditions of the Lemma.

Let us consider any $\boldsymbol{E} = \{E_i\}_{i=1}^{k+1} \in \mathscr{A}_{n+1}$ and $\boldsymbol{E}^1 = \boldsymbol{F}^s \in \mathscr{S}$ such that $\varrho\big(J_n(\boldsymbol{E}), \boldsymbol{F}^s\big) \leqslant \varepsilon$. There are two possibilites:

1° dimension of $\boldsymbol{E} < (n+1)$. Then $J_n(\boldsymbol{E}) = \boldsymbol{E}$ so it is enough to take $\boldsymbol{E}^2 = \boldsymbol{E}^1$. Let us remark that $\boldsymbol{F}^s \in \mathscr{S}^s$ because $\boldsymbol{F}^s \in J_n^{-1}(\boldsymbol{F}^s)$ and is an unique element of $J_n^{-1}(\boldsymbol{F}^s)$ whose dimension is $< n+1$.

2° dimension of $\boldsymbol{E}$ is $(n+1)$. Then $J_n(\boldsymbol{E}) = (E_i)_{i=1}^{k}$. Put $\tilde{E} = \bigoplus_{i=1}^{k} E_i$, $E^1 = \bigoplus_{i=1}^{k} E_i^1$ and $E = \bigoplus_{i=1}^{k+1} E_i$. We can choose an isomorphism $T\colon E^1 \to \tilde{E}$ such that $\ln\|T\|\ \|T^{-1}\| \leqslant \varepsilon + \eta/2$ and $T(E_i^1) = E_i$ $(i = 1, 2, \ldots, k)$.

Thus we have the following inclusion

$$aT(K_{E^1}) \subset K_{\tilde{E}} \subset bT(K_{E^1}),$$

where $\ln(ba^{-1}) \leqslant \varepsilon + \eta/2$ and K_X denote the closed unite ball in the Banach space X. Let Q be the smallest convex body in E containing K_E and $bT(K_{E^1})$. By $\tilde{\boldsymbol{E}}$ we denote the basis $(E_i)_{i=1}^{k+1}$ in the space E equipped with the norm $|||\cdot|||$ defined by the Minkowski functional of the set Q.

Observe that $\tilde{\boldsymbol{E}} \in J_n^{-1}(F^s)$. Indeed, for $P_k\colon \tilde{E} \to \tilde{E}$ we have $|||P_k||| = 1$ because $P_k\big(bT(K_{E^1})\big) = bT(K_{E^1}) \subset Q$, and $P_k(K_E) \subset K_{\tilde{E}} \subset bT(K_{E^1}) \subset Q$, and sets Q and $bT(K_{E^1})$ are convex, so we get $P_k(Q) \subset Q$ which is equivalent to $|||P_k||| = 1$. Moreover $(E_i)_{i=1}^{k}$ with the norm $|||\cdot|||$ is isometrically equivalent to $\boldsymbol{E}^1$ (T is a desired isometry). Thus $\tilde{\boldsymbol{E}} \in J_n^{-1}(\boldsymbol{F}^s)$. Obviously $\varrho(\tilde{\boldsymbol{E}}, \boldsymbol{E}) \leqslant \varepsilon + \eta/2$.

$\boldsymbol{E}^2$ we choose as an element of $\mathscr{S}^s$ such that $\varrho(\tilde{\boldsymbol{E}}, \boldsymbol{E}^2) \leqslant \eta/2$. Hence we get $\varrho(\boldsymbol{E}, \boldsymbol{E}^2) \leqslant \varepsilon + \eta$ and $J_n(\boldsymbol{E}^2) = \boldsymbol{F}^s = \boldsymbol{E}^1$. This completes the proof.

From Lemma 3.2 by an easy induction we obtain the following

PROPOSITION 3.1. *Let $1 > \varepsilon > 0$. Then there exists a sequence $(\mathscr{B}_n)_{n=1}^{\infty}$ such that*

(i) $\mathscr{B}_n$ is a finite $\varepsilon(1-2^{-n})$-net for $\mathscr{A}_n$;

(ii) if $\varrho\big(J_n((E_i)), (E_i^1)\big) \leqslant \varepsilon(1-2^{-n})$ for $(E_i) \in \mathscr{A}_{n+1}$ and $(E_i^1) \in \mathscr{B}_n$, then there exists $(E_i^2) \in \mathscr{B}_{n+1}$ such that $J_n\big((E_i^2)\big) = (E_i^1)$ and $\varrho\big((E_i), (E_i^2)\big) \leqslant \varepsilon(1-2^{-n-1})$.

LEMMA 3.3. *Let $(Z_i)_{i=1}^{k+1}$ and $(Y_i)_{i=1}^{m}$ be monotone f. d. s. bases. Let $(Z_i)_{i=1}^{k}$ be isometrically equivalent to $(Y_{i_r})_{r=1}^{k}$ and the projection $\pi\colon \bigoplus Y_i \to \bigoplus Y_i$ defined by $\pi\big(\sum_{i=1}^{n} y_i\big) = \sum_{r=1}^{k} y_{i_r}$ be of norm one. Then there exists a monotone f. d. s. basis $(X_i)_{i=1}^{m+1}$ such that*

(a) $(Z_i)_{i=1}^{k+1}$ is isometrically equivalent to $(X_{i_1}, \ldots, X_{i_r}, X_{m+1})$,

(b) $(Y_i)_{i=1}^{m}$ is isometrically equivalent to $(X_i)_{i=1}^{m}$,

(c) the projection $\pi_1\colon \bigoplus_{i=1}^{m+1} X_i \to \bigoplus_{i=1}^{m+1} X_i$ defined by $\pi_1\big(\sum_{i=1}^{m+1} x_i\big) = \sum_{r=1}^{k} x_{i_r} + x_{m+1}$ is of norm one.

Proof. $(X_i)_{i=1}^{m+1}$ is the f. d. s. basis $(Y_1, Y_2, \ldots, Y_m, Z_{k+1})$ in the space $\bigoplus_{i=1}^{m} Y_i \times Z_{k+1}$ equipped with the norm

$$\|(y, z)\| = \inf_{v \in \bigoplus_{i=1}^{k} Z_i} \big(\|y - U(v)\| + \|z + v\|\big),$$

where U is an isometry from $\bigoplus_{i=1}^{k} Z_i$ into $\bigoplus_{i=1}^{m} Y_i$ such that $U(Z_l) = (Y_{i_l})$ $(l = 1, 2, \ldots, k)$.

LEMMA 3.4. *Let $(X_i)_{i=1}^\infty$ and $(Y_i)_{i=1}^\infty$ be monotone f. d. s. bases. Suppose that there exists an increasing sequence (i_k) of indices and constants M and N such that*

$$\varrho\big((X_i)_{i=1}^n, (Y_{i_k})_{k=1}^n\big) \leqslant M \quad \textit{for} \quad n = 1, 2, \ldots$$

and $\|Q_n\| \leqslant N$ where $Q_n\big(\sum_{i=1}^\infty y_i\big) = \sum_{k=1}^n y_{i_k}$. Then $Q\big(\sum_{i=1}^\infty y_i\big) = \sum_{k=1}^\infty y_{i_k}$ is a projection of norm $\leqslant N$ and the basis (X_i) is equivalent to a subbasis (Y_{i_k}).

Proof. The first statement is obvious. To prove the second observe that our hypothesis implies that there exists a sequence of operators $T_n\colon \bigoplus_{i=1}^n X_i \to \bigoplus_{k=1}^n Y_{i_k}$ such that $T_n(X_k) = Y_{i_k}$ $(k = 1, 2, \ldots)$ and $\max\|T_n\|\,\|T_n^{-1}\| \leqslant \exp 2M$ for $n = 1, 2, \ldots$ Moreover for any two finite dimensional Banach spaces B_1 and B_2 and the number $C \geqslant 1$ the set $\mathscr{F}(B_1, B_2, C)$ of all isomorphisms $U\colon B_1 \to B_2$ such that $\max\|U\|\,\|U^{-1}\| \leqslant C$ equipped with the metric $\nu(U_1, U_2) = \max(\|U_1 - U_2\|, \|U_1^{-1} - U_2^{-1}\|)$ is compact (cf. [5], Lemma 1. c.) Hence by the standard diagonal procedure we can choose a sequence of operators $S_n = T_{k_n}$ such that:

(α) For each n and $k > n$, $S_k|\bigoplus_{i=1}^n X_i\colon \bigoplus_{i=1}^n X_i \to \bigoplus_{k=1}^n Y_{i_k}$ and $S_k(X_r) = Y_{i_r}$ $(r = 1, 2, \ldots, n)$.

(β) For each n the sequence $(S_k|\bigoplus_{i=1}^n X_i)_{k>n}$ is a Cauchy sequence in $\mathscr{F}(\bigoplus_{i=1}^n X_i, \bigoplus_{k=1}^n Y_{i_k}, \exp 2M)$.

Let $F_n = S_n P_n$ and $T(x) = \lim F_n(x)$. T is an isomorphical embedding because for $x = \sum_{k=1}^n x_k$ we have

$$\frac{1}{\exp 2M}\|x\| \leqslant \|\lim S_k(x)\| = \|T(x)\| \leqslant \|x\| \exp 2M.$$

Moreover $T(X_k) = Y_{i_k}$. This concludes the proof.

THEOREM 3.1. *There exists an f. d. s. basis (E_n) which is complementable universal for all f. d. s. bases.*

Proof. We inductively construct a sequence of finite dimensional monotone f. d. s. bases $(Z_i^n)_{i=1}^{k_n}$ such that:

1° each $\boldsymbol{F} \in \mathscr{B}_n$ is isometrically equivalent to a subbasis $(Z_{i_k}^n)_{k=1}^{\nu_F}$ of the basis $(Z_i^n)_{i=1}^{k_n}$ and the projection $Q_{\boldsymbol{F}}\colon \oplus Z_i^n \to \oplus Z_i^n$ defined by $Q_{\boldsymbol{F}}(\sum_{i=1}^{k_n} z_i) = \sum_{k=1}^{\nu_F} z_{i_k}$ is of norm one;

2° $(Z_i^n)_{i=1}^{k_n}$ is isometrically equivalent to the subbasis $(Z_i^{n+1})_{i=1}^{k_n}$ of the f. d. s. basis $(Z_i^{n+1})_{i=1}^{k_{n+1}}$;

3° if $\boldsymbol{F} \in \mathscr{B}_{n+1}$, $J_n(\boldsymbol{F}) \neq \boldsymbol{F}$ and $J_n(\boldsymbol{F})$ is isometrically equivalent to $(Z_{i_k}^n)_{k=1}^r$, then $\boldsymbol{F}$ is isometrically equivalent to $(Z_{i_1}^{n+1}, \ldots, Z_{i_r}^{n+1}, Z_{i_{r+1}}^{n+1})$ for some $i_{r+1} > k_n$.

Let $\boldsymbol{Z}^1$ be the one dimensional space. Suppose that $\boldsymbol{Z}^n$ have been choosen to satisfy the conditions 1°–3°. Let $\big((F_i^1)_{i=1}^{l_1}, \ldots, (F_i^r)_{i=1}^{l_r}\big)$ be a sequence of all elements of $\mathscr{B}_{n+1}$ which dimension is $n+1$.

From the inductive hypothesis $(F_i^1)_{i=1}^{l_1-1}$ is isometrically equivalent to $(Z_{i_s}^n)_{s=1}^{l_1-1}$ and the canonical projection onto it is of norm one. Thus in view of Lemma 3.3 there exists a monotone f. d. s. basis $(Z_1^{n,1}, Z_2^{n,1}, \ldots \ldots, Z_{k_n}^{n,1}, Z_{k_n+1}^{n,1})$ such that $(Z_i^n)_{i=1}^{k_n}$ is isometrically equivalent to $(Z_i^{n,1})_{i=1}^{k_n}$ and $(F_i^1)_{i=1}^{l_1}$ is isometrically equivalent to $(Z_{i_1}^{n,1}, \ldots, Z_{i_{l_1-1}}^{n,1}, Z_{k_n+1}^{n,1})$ and the projection $Q_{\boldsymbol{F}^1}\big(\sum_{i=1}^{k_n+1} z_i\big) = \sum_{k=1}^{l_1-1} z_{i_k} + z_{k_n+1}$ is of norm one.

If we repeat this procedure replacing $\boldsymbol{Z}^n$ by $\boldsymbol{Z}^{n,1}$ and $\boldsymbol{F}^1$ by $\boldsymbol{F}^2$ and, so on, we obtain after r steps a f. d. s. basis $\boldsymbol{Z}^{n,r}$ which is our basis $\boldsymbol{Z}^{n+1}$. It is easily seen that $\boldsymbol{Z}^{n+1}$ satisfies the conditions 1°–3°.

If we identify $\oplus Z_i^n$ with its image in $\oplus Z_i^{n+1}$ (see 2°) we obtain the Banach space $E = \overline{\bigcup_{n=1}^\infty \bigoplus_{i=1}^{k_n} Z_i^n}$ with the monotone f. d. s. basis $(E_n)_{n=1}^\infty$ having the property: for any n the basis $(Z_i^n)_{i=1}^{k_n}$ is isometrically equivalent to $(E_i)_{i=1}^{k_n}$. It is the desired complementably universal f. d. s. basis. To this end let us consider any f. d. s. basis $(X_i)_{i=1}^\infty$. Since any f. d. s. basis is equivalent to the monotone one, we can assume that $(X_i)_{i=1}^\infty$ is monotone. It is a consequence of properties 1°–3° and the definition of $\mathscr{B}_n$ that there exists an increasing sequence (i_k) of positive integers such that

$$\varrho\big((X_l)_{l=1}^n, (E_{i_k})_{k=1}^n\big) \leqslant \varepsilon \quad \text{for} \quad n = 1, 2, \ldots$$

and $\|Q_n\| = 1$ where $Q_n\big(\sum_{k=1}^\infty e_k\big) = \sum_{k=1}^n e_{i_k}$. The desired conclusion follows from Lemma 3.4.

Combining Theorem 3.1. and Theorem 1.1. we get

THEOREM 3.2 (Kadec [6]). *There exists a Banach space E with the f. d. s. basis which is complementably universal for all Banach spaces with f. d. expansion of identity.*

Moreover it is true the following:

THEOREM 3.3. *If X is a Banach space with the f. d. expansion of identity which is complementably universal for all Banach spaces with the f. d. expansion of identity, then X is isomorphic to E.*

For the proof use the standard decomposition method (cf. [13], Corollary 4).

THEOREM 3.4. *There exists an unconditional f. d. s. basis which is complementably universal for all unconditional f. d. s. bases.*

Proof. Let (E_n) be a complementably universal f. d. s. basis. We set

$$\tilde{E} = \{e = \sum_n e_n \in E : \sum_n e_n \text{ is unconditionally convergent}\}.$$

In $\tilde{E}$ we introduce the norm by $|||\sum_n e_n||| = \sup_n \sup_{|\varepsilon_k|=1} \|\sum_{k=1}^{n} \varepsilon_k e_k\|$. It is easy to see that $(\tilde{E}, |||\cdot|||)$ is a Banach space, has an unconditional f. d. s. basis (E_n) and this basis is complementably universal for all unconditional f. d. s. bases.

The method of the above proof is essentially due to Zippin [17].

THEOREM 3.5. *There exists a Banach space with an unconditional f. d. s. basis which is complementably universal for all spaces with an unconditional f. d. expansion of identity. This space is unique up to isomorphism among spaces with an unconditional f. d. expansion of identity.*

The first statement follows from Theorems 1.1 and 3.4. The proof of the second uses the standard decomposition method (cf. [13], Corollary 4).

Remark. a) A f. d. s. basis (X_i) is said to be *Besslian* if for any $x = \sum_i x_i$, $x_i \in X_i$ we have $\sum_i \|x_i\|^2 < \infty$ and is said to be *Hilbertian* if for any sequence (x_i), $x_i \in X_i$, such that $\sum_i \|x_i\|^2 < \infty$ the series $\sum_i x_i$ converges.

THEOREM 3.6. *The following classes of f. d. s. bases contain an universal element:*

(i) *the class of all Besslian f. d. s. bases,*

(ii) *the class of all Hilbertian f. d. s. bases.*

b) A f. d. s. basis (X_i) is said to be *boundedly complete* if for any sequence (x_i), $x_i \in X_i$, such that $\sup_n \|\sum_{i=1}^{n} x_i\| < \infty$ the series $\sum_i x_i$ converges and is said to be *shrinking* if for any $f \in (\oplus X_i)^*$ *we have*

$$\lim_n \left(\sup\left\{|f(x)|: \|x\| = 1;\ x = \sum_{i=n}^{\infty} x_i\right\}\right) = 0.$$

THEOREM 3.7. *The following classes of f. d. s. bases do not contain an universal member:*

(i) *the class of all boundedly complete f. d. s. bases,*

(ii) *the class of all shrinking f. d. s. bases,*

(iii) *the class of all f. d. s. bases of weakly sequentially complete spaces,*

(iv) *the class of all f. d. s. bases of spaces which do not contain a subspace isomorphic to c_0.*

The proofs of the above results are nonessential modifications of proofs of Zippin [17] (cf. also [13], Theorem 4, and [16]).

4. Some remarks on Kadec's paper [6]. In this section we will preserve Kadec's notation of [6]. The symbol R^n is reserved for Euclidean n-space equipped with the sup norm. We want to show

PROPOSITION 4.1. *The Kadec's space E (cf. [6]) has an f. d. s. basis.*

Proof. It is sufficient to prove (see [14], Th. II. 2.) that there exists a sequence of finite dimensional projections P_n fulfiling the following condition

(*) $P_n(e) \to e$ for any $e \in E$ and $P_n P_m = P_{\min(n,m)}$.

To do this we need the sequence (C_n) with the properties:

(1) C_n is a finite subset of J,

(2) for each component $J_n^{(\nu_1\ldots\nu_{n-1})}$ there is exactly one point in $C_n \cap J_n^{(\nu_1\ldots\nu_{n-1})}$,

(3) for each $t = \{\nu_i\} \in C_n$ there exists a sequence of projections $(\pi_k^t)_{k>n}$, $\pi_k^t\colon X(t) \xrightarrow{\text{onto}} X_k^{(\nu_1\ldots\nu_{k-1})}$, $\pi_k^t \cdot \pi_r^t = \pi_{\min(k,r)}^t$ and $\|\pi_k^t\| \leqslant e^{\varepsilon}$,

(4) $C_{n+1} \supset C_n$ $(n = 1, 2, 3, \ldots)$.

We will construct such a sequence inductively.

$C_1 = \{t\}$ where $t = \{\nu_n\}$ is an element of J such that the following diagram commutes

$$\begin{array}{ccccccc}
R^1 & \xrightarrow{\text{id}} & R^2 & \xrightarrow{\text{id}} & R^3 & \xrightarrow{\text{id}} & \ldots \\
\downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} & & \downarrow{\scriptstyle \tau_3} & & \\
X_1 & \xrightarrow[u_1]{} & X_2^{(\nu_1)} & \xrightarrow[u_2]{} & X_3^{(\nu_1\nu_2)} & \xrightarrow[u_3]{} & \ldots
\end{array}$$

and $\lim_n \|\tau_n\| \cdot \|\tau_n^{-1}\| \leqslant e^{\varepsilon}$.

Suppose that $C_1, C_2, \ldots, C_n$ have been chosen to satisfy the above four conditions. Consider all sets $J_{n+1}^{(\nu_1\ldots\nu_n)}$ such that

$$C_n \cap J_{n+1}^{(\nu_1\ldots\nu_n)} = \emptyset.$$

From each such $J_{n+1}^{(\nu_1\ldots\nu_n)}$ we choose an element $t = \{\mu_k\}$ such that the following diagram commutes

$$\begin{array}{ccccccccc}
X_1 \xrightarrow[u_1]{} & \ldots & X_{n+1}^{(\nu_1\ldots\nu_n)} & \xrightarrow{\text{id}} & (X_{n+1}^{(\nu_1\ldots\nu_n)} + R^1)_0 & \xrightarrow{\text{id}} & (X_{n+1}^{(\nu_1\ldots\nu_n)} + R^2)_0 & \xrightarrow{\text{id}} & \ldots \\
\downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \tau_{n+2}} & & \downarrow{\scriptstyle \tau_{n+3}} & & \\
X_1 \xrightarrow[u_1]{} & \ldots & X_{n+1}^{(\mu_1\ldots\mu_n)} & \xrightarrow[u_{n+1}]{} & X_{n+2}^{(\mu_1\ldots\mu_{n+1})} & \xrightarrow[u_{n+2}]{} & X_{n+3}^{(\mu_1\ldots\mu_{n+2})} & \xrightarrow[u_{n+3}]{} & \ldots
\end{array}$$

and $\lim_k \|\tau_k\| \cdot \|\tau_k^{-1}\| \leqslant e^{\varepsilon}$.

Put C_{n+1} be a union of C_n and the set of all elements t constructed in the above way. Conditions (1)–(4) are obviously fulfiled.

The projection P_n we define by the formula:

$$P_n(f)(t) = \pi_n^{t_1}(f(t_1)) \quad \text{for } t \text{ such that} \quad \varphi(t, t_1) \leqslant \frac{1}{n} \text{ and } t_1 \in C_n.$$

It is a consequence of (2) and (3) that P_n is a projection onto E_n. We omit the standard verification that P_n satisfies (*). This completes the proof.

Remark 4.1. If we combine Proposition 4.1 and Theorem 3.3 we obtain that the space E of Theorem 3.2 is isomorphic to the space E construced by Kadec [6].

Remark 4.2. Observe that the Proposition 4.1 and Kadec's result [6], Theorem 1 give us an alternative proof of Theorem 3.2 and thus of Theorem 1.1.

Now we want to show the application of Kadec's method to the theory of separable Banach spaces whose duals are L_1 spaces. This important class of spaces, containing spaces of continuous functions on compact metric spaces and spaces of affine functions on metrisable Choquet simplexes, was extensively studied in [7], [9].

We will use the following:

THEOREM 4.1. (*Lazar–Lindenstrauss* [7]). *For a separable Banach space X the following statements are equivalent:*

a) *X^* is isometric to $L_1(\mu)$ for some measure μ,*

b) *there is a sequence $X_1 \subset X_2 \subset X_3 \subset \ldots$ of subspaces of X such that $X = \overline{\bigcup_{n=1}^{\infty} X_n}$ and X_n is isometric to R^n.*

PROPOSITION 4.2. *Let $\varepsilon > 0$. Then there exists a separable Banach space Y_ε whose dual is an L_1 space such that for any separable Banach space X whose dual is an L_1 space there exists an isomorphic embedding $T: X \to Y_\varepsilon$ with $\|T\|\,\|T^{-1}\| \leqslant e^\varepsilon$ and the projection $P: Y_\varepsilon \overset{\text{onto}}{\to} T(X)$ of norm one.*

Proof. Let us consider the space R^m and by $\mathscr{G}_{n,m}$ for $n > m$, denote the set of all pairs (Y, u) where Y is a Banach space isometric with R^n and u is an isometric embedding from R^m into Y. In $\mathscr{G}_{n,m}$ we introduce the metric by the formula

$$\varrho_1\big((Y, u), (Y_1, u_1)\big) = \ln\inf\|\tau\|\;\|\tau^{-1}\|$$

where the infimum is extended on all isomorphisms τ from Y onto Y_1 such that the following diagram commutes

$$\begin{array}{ccc} & & Y \\ & \overset{u}{\nearrow} & \big\downarrow \tau \\ R^m & & \\ & \underset{u_1}{\searrow} & \\ & & Y_1 \end{array}$$

It is easy to see that $\mathscr{G}_{n,m}$ is a compact metric space.

LEMMA 4.1. *Let $\tau: R^m \to R^m$ be an isomorphism. For any $(Y, u) \in \mathscr{G}_{n,m}$ there exist $(Y_1, u_1) \in \mathscr{G}_{n,m}$ and the isomorphism $T: Y \to Y_1$ such that $\|\tau\|\,\|\tau^{-1}\| = \|T\|\,\|T^{-1}\|$ and the following diagram commutes*

$$\begin{array}{ccc} R^m & \xrightarrow{\;u\;} & Y \\ \downarrow \tau & & \downarrow T \\ R^m & \xrightarrow[\;u_1\;]{} & Y_1 \end{array}$$

Proof. It is enough to consider only the case $n = m+1$ and $\|\tau\| \geqslant 1$, $\|\tau^{-1}\| \geqslant 1$. In R^m there are functionals e_k^* $(k = 1, 2, \ldots, m)$ such that

$$\|x\| = \sup|e_k^*(x)| \quad \text{for} \quad x \in R^m.$$

It was proved in [12] that there exist $y_1^*, y_2^*, \ldots, y_n^* \in Y^*$ such that

$$\|y\| = \sup|y_k^*(y)| \quad \text{for} \quad y \in Y$$

and $e_k^* = u^*(y_k^*)$ for $k = 1, 2, \ldots, m$.

Put $\tau^* e_i^* = g_i^*$ and choose $y_n \in Y$ such that $y_i^*(y_n) = \delta_{n,i}$. Observe that $y_n \notin u(R^m)$. We define the functionals $f_i \in Y^*$ by

$$f_i(y_n) = 0 \quad \text{and} \quad u^* f_i = g_i^* \quad (i = 1, 2, \ldots, m),$$

$$f_n = \xi^{-1} y_n^* \quad \text{where} \quad \begin{cases} \xi = \max\big(1, \sup\limits_{x \in Q} |y_n^*(x)|\big), \\ Q = u\{x \in R^m\colon |g_i^*(x)| \leqslant 1 \ (i = 1, 2, \ldots, m)\}. \end{cases}$$

Then Y_1 is the space Y equipped with the norm $|||y||| = \sup|f_i(y)|$, T is an identity from Y onto Y_1 and $u_1 = u\tau^{-1}$. We omit the verification.

Using $\mathscr{G}_{n,m}$ instead of $\mathfrak{m}_n(X)$ (cf. [6]) and the Lemma 4.1 instead of the Lemma 2 of [6] we construct the space Y_ε in exactly the same manner as E in the Kadec's proof. This completes the proof.

THEOREM 4.2. *There exists a separable Banach space Y whose dual is an L_1 space such that for any $\varepsilon > 0$ and any separable Banach space X whose dual is an L_1 space there exist an isomorphic embedding $T: X \to Y$ with $\|T\|\,\|T^{-1}\| \leqslant 1+\varepsilon$ and the projection $P: Y \overset{\text{onto}}{\to} T(X)$ of norm one.*

Proof. The required properties has the space of all sequences (y_n) such that $y_n \in Y_{\frac{1}{n}}$ and $\lim\limits_n \|y_n\| = 0$ with the sup norm, $\|(y_n)\| = \sup\limits_n \|y_n\|$.

An analogous argument gives (for definition of $\mathscr{L}_{\infty,\lambda}$-space see [10]).

THEOREM 4.3. *For any $\lambda > 1$ there exists a separable $\mathscr{L}_{\infty,\lambda}$-space Y such that for any separable $\mathscr{L}_{\infty,\lambda}$-space X and for any $\varepsilon > 0$ there exists an isomorphic embedding $T: X \to Y$ with $\|T\|\,\|T^{-1}\| < 1+\varepsilon$ and a projection $P: Y \underset{\text{onto}}{\to} T(X)$ of norm $\leqslant \lambda$.*

Added in proof. The following theorem is true:

Any Banach space with f. d. expansion of identity is isomorphic to a complemented subspace of a Banach space with Schauder basis.

It was proved independently by W. B. Johnson, H. Rosenthal, M. Zippin "*On bases, finite dimensional decompositions, and weaker structures in Banach spaces*" (to appear in Israel J. Math. vol 9) and by the first named author of the present paper "*Any separable Banach space with the bounded approximation property is a complemented subspace of a Banach space with the basis*" (to appear in Studia Math.).

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