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On complementably universal Banach spaces

by

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Abstract. A Banach space X is said to have BAP if the identity operator on X is the pointwise limit of a sequence of finite dimensional bounded linear operators. It is shown that there exists a separable Banach space E with the property that any Banach space X with BAP is isomorphic to a complemented subspace of E.

The classical example of a universal Banach space — the space $C([0;1])$ — has the following negative property which considerably depreciates the universality of $C([0;1])$: any infinite-dimensional subspace of the space $C([0;1])$ which does not contain a subspace isomorphic to the space c_0 of scalar valued sequences converging to zero is not complemented in $C([0;1])$, cf. [1].

DEFINITION 1. A Banach space Z is *complementably universal* for the class $\mathfrak{M}$ of Banach spaces if for every $X \in \mathfrak{M}$ there exists a complemented subspace of Z isomorphic to X.

It was recently proved by Pełczyński [2] that among all Banach spaces with a basis (unconditional basis) there exists a complementably universal space B (resp. U) unique up to isomorphisms. This suggests the question whether there exists a complementably universal Banach space in the class of all separable Banach spaces. The negative answer on this question combined with the result of [2] would imply the negative solution of the basis problem.

DEFINITION 2. A Banach space X has the *Banach Approximation Property* (shortly BAP) if there exists a sequence of finite dimensional bounded linear operators which converges pointwise to the identity operator on X.

Obviously any Banach space with BAP is separable. We do not know whether there exists a seperable Banach space without BAP.

The main result of the present paper is the following

THEOREM. *There exists a separable Banach space E which is complementably universal for all Banach spaces with* BAP.

LEMMA 1. *If X has* BAP, *then X contains an increasing sequence of subspaces*

$$X_1 \subset X_2 \subset X_3 \subset \ldots; \quad \dim X_n = n,$$

and there exists a sequence of finite dimensional operators (σ_n) such that

$$\sigma_n(X) \subset X_n \quad and \quad \lim_n \|\sigma_n(x) - x\| = 0 \quad for\ any \quad x \in X.$$

Proof. Let $(\pi_n)_{n=1}^{\infty}$ be any sequence of finite-dimensional bounded linear operators such that $\lim_n \pi_n(x) = x$ for $x \in X$. Let us set $Z_n = \pi_n(X)$; $Y_k = \operatorname{span}\{Z_i\}$ $i = 1, 2, \ldots, k$; $\dim Y_k = n_k$. The desired sequence $(X_i)_{i=1}^{\infty}$ is any sequence of subspaces with the property that $X_i \subset X_{i+1}$, $\dim X_i = i$ and $X_{n_k} = Y_k$. The operators σ_n are defined by

$$\sigma_n = \pi_k \quad \text{for} \quad n_k \leqslant n < n_{k+1} \quad (k = 0, 2, \ldots;\ n_0 = 1, \pi_0 = 0).$$

Let X be an m-dimensional Banach space. Let us consider the set $\mathfrak{M}_n(X)$ (for $n > m$) of all pairs (Y, u) where Y is an n-dimensional Banach space and u is a linear isometric embedding of X into Y. We introduce in $\mathfrak{M}_n(X)$ a metric

$$\varrho\big((Y_1, u_1), (Y_2, u_2)\big) = \ln \inf \|\tau\| \cdot \|\tau^{-1}\|,$$

where the infimum is extended over all isomorphisms $\tau\colon Y_1 \xrightarrow[\text{onto}]{} Y_2$ and such that $u_2 = \tau u_1$.

It can be easily seen that $\mathfrak{M}_n(X)$ equipped with the metric ϱ is compact.

LEMMA 2. *Let X and X_1 be m-dimensional Banach spaces and let τ be an isomorphisms from X onto X_1. Let $(Y, u) \in \mathfrak{M}_n(X)$. Then there exists $(Y_1, u_1) \in \mathfrak{M}_n(X)$ such that the isomorphism τ can be extended to an isomorphism T from Y onto Y_1 such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ \tau\big\downarrow & & \big\downarrow T \\ X_1 & \xrightarrow[u_1]{} & Y_1 \end{array}$$

commutes and $\|\tau\|\,\|\tau^{-1}\| = \|T\|\,\|T^{-1}\|$.

Proof. We can assume without loss of generality that $\|\tau\| = 1$. Let V_1 be the unit ball of the space X_1. Let us consider the closed convex hull of the unit ball U of the space Y and the set $u\tau^{-1}(V_1)$. Denote this set by U_1. Then $d^{-1}U_1 \subset U \subset U_1$ where $d = \|\tau\|\,\|\tau^{-1}\|$. We define the desired space Y_1 to be the space Y equipped with the new norm whose unit ball is U_1. The desired isomorphism T is the formal identity map from Y onto Y_1.

Proof of the Theorem. The desired space E is constructed to be a space of functions $f(t)$ defined on some compact space J with values in a "variable" Banach space $X(t)$.

Construction of the compact J. Let $\varepsilon > 0$. Let us consider the one dimensional Banach space X_1 and choose in the compact space $(\mathfrak{M}_2(X_1), \varrho)$ a finite $\varepsilon/2$ — net whose first components are $X_2^1, X_2^2, \ldots, X_2^{N_2}$. Next we choose a finite $\varepsilon/4$ — net in each compact space $(\mathfrak{M}_3(X_2^{\nu}), \varrho)$ for $l \leqslant \nu \leqslant N_2$. We can also assume without loss of generality that each of these nets has the same number of elements. In this way we obtain the nets of the second stage with the first components

$$X_3^{(\nu,1)}, X_3^{(\nu,2)}, \ldots, X_3^{(\nu,N_3)} \quad (1 \leqslant \nu \leqslant N).$$

In that manner we define inductively for $n = 1, 2, \ldots$ a finite $2^{-n}\varepsilon$-net for each of the spaces $\mathfrak{M}_{n+1}(X_n^{(\nu_1,\ldots,\nu_{n-1})})$ such that each of those nets has the same number of elements, say N_n. Having done this, we consider all sequences

(1) $$X_1 \subset X_2^{(\nu_1)} \subset X_3^{(\nu_1,\nu_2)} \subset \ldots \quad (l \leqslant \nu_k \leqslant N_k),$$

where the inclusions can be regarded as isometrical embedding (the second components). Clearly any sequence of form (1) is uniquely determined by a sequence of the indices

(2) $$(\nu_1, \nu_2, \ldots), \quad \text{where} \quad 1 \leqslant \nu_k \leqslant N_k \ \text{for}\ k = 1, 2, \ldots$$

There is a natural one to one correspondence between the set J of all sequences (2) and the set of all sequences of Banach spaces (1). We introduce the metric d on J by

$$d\big((\nu_k), (\mu_k)\big) = n^{-1},$$

where n is the first index such that $\nu_n \neq \mu_n$. One can easily seen that the space (J, d) is homeomorphic to the Cantor discontinuum. We call a component of rank n any maximal subset of J whose diametr is $\leqslant n^{-1}$. Clearly J is the union of mutualy disjoint M components of rank n where $M = \prod_{k=1}^{n-1} N_k$. The component of rank n which is uniquely determined by a sequence $(\nu_1, \ldots, \nu_{n-1})$ where $\nu_k = 1, 2, \ldots, N_k$ $(k = 1, 2, \ldots, n-1)$ will be denoted by $J^{(\nu_1,\nu_2,\ldots,\nu_{n-1}}$.

Construction of the universal space E. A picewise constant function of degree n is a function f on J such that $f(J^{(\nu_1,\ldots,\nu_{n-1})}$ is a one-point subset of $X_n^{(\nu_1,\ldots,\nu_{n-1})})$ for each admissible sequence of the indices $(\nu_1, \nu_2, \ldots, \nu_{n-1})$. Denote by E_n the Banach space of all picewise constant functions of degree n with pointwise operations of addition and multiplication by scalars and with the norm

$$\|f\| = \max_{t \in J} \|f(t)\|.$$

Clearly $E_1 \subset E_2 \subset E_3 \subset \ldots$ and the inclusion are isometric embeddings. Therefore the union $\bigcup_{n=1}^{\infty} E_n = E_\infty$ is a separable normed linear space. Let E be the completion of E_∞. We shall verify that E is the desired complementable universal Banach space.

Embedding of $X(t_0)$ into E. Fix a point $t_0 = (\nu_n)_{n=1}^{\infty} \in J$ and consider a Banach space $X(t_0)$ defined to be the completion of the normed linear space $\bigcup_{n=2}^{\infty} X_n^{(\nu_1,\ldots,\nu_{n-1})}$. Suppose that in $X(t_0)$ there exists a sequence (σ_n) of finite dimensional bounded linear operators such that $\lim \sigma_n(x) = x$ for all $x \in X(t_0)$ and $\sigma_n(X(t_0)) \subset X_n^{(\nu_1,\ldots,\nu_{n-1})}$. By the Banach–Steinhaus Theorem, $\sup_n \|\sigma_n\| = p < +\infty$. Now to any $x \in X(t_0)$ we assign the function $\varphi(\cdot) = A(x)$ defined by

$$\varphi(t_0) = x; \quad \varphi(t) = \sigma_n(x)$$

for such $t \in J$ that $d(t, t_0) = n^{-1}$ $(n = 1, 2, \ldots)$. We shall show that φ belongs to E or equivalently that φ can be uniformly approximated by picewise constant functions. To this end we choose for a given $\varepsilon > 0$ an index $n = n(\varepsilon)$ so that $\|x - \sigma_m(x)\| \leqslant \varepsilon$ for all $m > n$. Let us set

$$\varphi^{(n)}(t) = \begin{cases} \varphi(t) & \text{whenever } d(t, t_0) > n^{-1}, \\ \sigma_n(x) & \text{whenever } d(t, t_0) \leqslant n^{-1}. \end{cases}$$

Evidently $\varphi^{(n)}$ is a picewise constant function and $\|\varphi - \varphi^{(n)}\| \leqslant \varepsilon$. Thus φ can be uniformly approximable by picewise constant functions. Clearly the operator $A\colon X(t_0) \to E$ is linear. Moreover

$$\|x\| \leqslant \sup_{t \in J} \|\varphi(t)\| = \|A(x)\| \leqslant \sup_n \|\sigma_n\| \, \|x\| = p\|x\| \quad (x \in X(t_0)).$$

Hence A is an isomorphism from $X(t_0)$ onto a subspace $E(t_0)$ of E.

Let f be any function in E. Obviously the value of f at the point t_0 is an element of $X(t_0)$, say $f(t_0) = x$. We put $P(f) = A(x)$. One can easily seen that P is a bounded linear projection from E onto $E(t_0)$. Therefore $E(t_0)$ is a complemented subspace of E.

Universality of the space E. Let Y be any space with BAP. According to Lemma 1 we can choose a sequence of subspaces of Y' say (Y_n), and a sequence of bounded linear operators $(_n)$ so that

$$Y_1 \subset Y_2 \subset Y_3 \ldots; \quad \sigma_n(Y) \subset Y_n (n = 1, 2, \ldots); \quad \lim_n \|y - \sigma_n(y)\| = 0$$

for any $y \in Y$.

According to Lemma 2 and the construction of the spaces $X_n^{(\mu_1,\ldots,\mu_{n-1})}$ there exists a sequence of finite dimensional spaces of form (1) and a sequence (τ_n) of isomorphisms such that the following diagram is commutative

$$\begin{array}{ccccccc} Y_1 & \xrightarrow{v_1} & Y_2 & \xrightarrow{v_2} & Y_3 & \xrightarrow{v_3} & \ldots \\ \downarrow{\scriptstyle\tau_1} & & \downarrow{\scriptstyle\tau_2} & & \downarrow{\scriptstyle\tau_3} & & \\ X_1 & \xrightarrow[u_1]{} & X_2^{(\nu_1)} & \xrightarrow[u_2]{} & X_3^{(\nu_1,\nu_2)} & \xrightarrow[u_3]{} & \ldots \end{array}$$

where (u_n) and (v_n) are natural isometric embeddings (inclusions) and

$$\lim_n \|\tau_n\| \, \|\tau_n^{-1}\| \leqslant \exp(2^{-1}\varepsilon + 2^{-2}\varepsilon + \ldots) \leqslant \varepsilon.$$

Clearly the sequence (τ_n) of isomorphisms can be extended to an isomorphism $T\colon Y \xrightarrow[\text{onto}]{} X(t_0)$ where $t_0 = (\nu_n)_{n=1}^{\infty}$. Since Y has BAP, $X(t_0)$ has the same property. Therefore as it has been proved above $X(t_0)$ (and therefore Y) is isomorphic to a complemented subspace of E. Hence E is complementably universal for all Banach spaces with BAP.

References

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