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On Liouville F-Algebras*

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Abstract. This paper investigates the spectra of elements from a Liouville F-algebra. A generalization of the notion of a Shilov boundary for a Banach algebra is defined and the principle result concerns the presence of algebraically principal closed maximal ideals on this boundary.

1. Introduction. A commutative F-algebra A with an identity element is called a Liouville F-algebra if the spectrum of each non-constant element in A is an unbounded subset of the complex plane C [2]. The entire functions E in the topology of uniform convergence on the compact subsets of C is an example of a singly generated Liouville F-algebra. Birtel [2] was interested in characterizing E when he defined the Liouville property. The first example of a singly generated Liouville F-algebra which properly contains E was constructed in [3].

In Section 2 we investigate conditions which guarantee that the spectrum of an element from a Liouville F-algebra is identifiable with C. We introduce and investigate a generalization of the Shilov boundary for a Banach algebra. The major result of this study is the existence of algebraically principal closed maximal ideals at non-isolated points on our boundary. The reader is referred to [8] for the basic information on F-algebras.

2. Spectra in Liouville F-algebras.

DEFINITION 2.1. An F-algebra A with identity element e is called a *Liouville F-algebra* provided the following condition is satisfied:

If $a \in A$ and there exists an $M > 0$ such that $|h(a)| \leqslant M$ for each $h \in M_A$ then $a = \lambda e$ for some $\lambda \in C$.

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From this definition it follows immediately that a Liouville F-algebra is a semisimple algebra.

PROPOSITION 2.2. *If A is a Liouville F-algebra, $x \in A$ and $x \neq \lambda e$ for any $\lambda \in C$, then the spectrum of x, $\sigma(x)$, is a dense connected subset of C. Moreover, $\sigma(x)^c$ (complement of $\sigma(x)$ in C) contains no closed connected subsets other than single points.*

Proof. Assume that $\sigma(x) \cap N(\lambda, r) = \emptyset$, where $N(\lambda, r) = \{z \in C: |z-\lambda| < r\}$. Now $f(z) = (z-\lambda)^{-1}$ is analytic on $\overline{N(\lambda, r/2)}^c$, an open subset of C containing $\sigma(x)$. A well known result for F-algebras implies that there exists a unique $y \in A$ such that $h(y) = f(h(x))$ for each $h \in M_A$ [9, Theorem 10.1]. However, $y \in A$ has a bounded spectrum, which contradicts the Liouville hypothesis since $y \neq \lambda e$ for all $\lambda \in C$. Hence, $\sigma(x)$ is a dense subset of C.

Now assume that $\sigma(x) \subset V_1 \cup V_2$, where $\{V_i\}_{i=1}^2$ are separating non-empty open subsets of C with $\sigma(x) \cap V_i \neq \emptyset$, $i = 1, 2$. The $\{V_i\}_{i=1}^2$ may be chosen to be disjoint, for if $V_1 \cap V_2 \neq \emptyset$ then $\sigma(x) \cap (V_1 \cap V_2) \neq \emptyset$ since $\sigma(x)$ is dense in C. This contradicts the fact that V_1 and V_2 separate $\sigma(x)$. For n sufficiently large, $V_i \cap \sigma(x_n) \neq \emptyset$, $i = 1, 2$. Shilov has shown that there exists an idempotent element $u_n \in \bar{A}_n$ such that $\hat{u}_n(h) = f(\hat{x}_n(h))$, $h \in M_{\bar{A}_n}$, where f is the analytic function defined on $V_1 \cup V_2$ satisfying $f|V_1 \equiv 1$ and $f|V_2 \equiv 0$ [9]. Moreover, if $j > i$, then $\pi_i^j(u_j)\hat{} \equiv u_i$ and, $\pi_i^j(u_j) = \pi_i^j(u_j^2) = \pi_i^j(u_j)^2$. The uniqueness of $u_i \in \bar{A}_i$ modulo the radical implies $\pi_i^j(u_j) = u_i$ [8]. Hence, A contains a proper idempotent element [9, Theorem 5.1]. This contradicts the Liouville hypothesis. Therefore, $\sigma(x)$ is a connected subset of C.

Next let K be a connected closed subset of C, and assume $K \subset \sigma(x)^c$. We assume that K is a subset of S, the extended complex plane. If K is an unbounded subset of C, we adjoin the point at infinity to K. Let $U = S-K$. Now, $\sigma(x) \subseteq U$ and $\sigma(x)$ is a dense subset of C. Hence $\sigma(x)$ is dense in U. If U is not a connected subset of C, then $\sigma(x)$ would not be a connected subset of C, which contradicts the previous result. Hence, U is a connected subset of C, and by our assumption on K, U is also an open subset of S. Moreover, $S-U = K$ implies that U is a simply connected region in S. If K contains more than one point, then by the Riemann Mapping Theorem, there exists $f \in \mathrm{Hol}(U)$ such that f maps U onto the open unit disc [11]. Since $\sigma(x) \subset U$ and A is closed under the application of analytic functions, $f \circ \hat{x}$ defines a unique element of A. The transform $f \circ \hat{x}$ is not constant since $\sigma(x)$ is dense in U and f is not constant on $\sigma(x)$. Furthermore, $f \circ \hat{x}$ has a bounded spectrum, which contradicts the Liouville property. Hence, K must consist of at most one point.

If A is a commutative F-algebra, then M_A may be topologized in at least two natural ways.

(1) The weak topology on M_A which M_A inherits as a subset of the dual space A' equipped with the weak topology induced on A' by A [9, p. 6].

(2) The direct limit topology on M_A: a set $U \subset M_A$ is open if and only if $U \cap M_{\bar{A}_n}$ is open relative to the weak topology on $M_{\bar{A}_n}$ induced by $\bar{A}_n$ for each $n = 1, 2, \ldots$

The weak topology on M_A is the weak* topology determined by the subbasis of sets of the form $V_{h_0,\varepsilon,a} = \{h \in M_A: |\hat{a}(h) - \hat{a}(h_0)| < \varepsilon\}$ where $a \in A$. The direct limit topology on M_A is stronger than the weak topology on M_A. The technique used in the previous proposition yields the following:

COROLLARY 2.3. *If A is a Liouville F-algebra, then M_A is connected in the direct limit topology. Moreover, if $h \in M_A$, then h is not isolated with respect to infinitely many $M_{\bar{A}_n}$, $n = 1, 2, \ldots$*

We note at this point that the known examples of singly generated Liouville F-algebras all have $\sigma(a) = C$ [2, 3]. However, the Liouville hypothesis is not sufficient to guarantee that $\sigma(a)$ is equal to C for all choices of a generator, as Example 2.4 will illustrate. We precede the example with a discussion of a method for forming singly generated Liouville F-algebras which will be referred to later.

If D is a compact subset of C, then $\mathrm{Hol}(D)$ denotes the algebra of functions which are analytic in $\mathring{D}$ (the interior of D), and have a continuous extension to D. It is well known that $\mathrm{Hol}(D)$ is a Banach algebra under the norm, $\|f\|_D = \sup_{z \in D} |f(z)|$ for $f \in \mathrm{Hol}(D)$. A theorem of Mergelyan [11], states that $\mathrm{Hol}(D)$ is the uniform closure on D of the algebra of polynomials in z, if and only if D is a non-separating subset of C. Let $\{D(n)\}_{n=1}^{\infty}$ denote an increasing sequence of compact non-separating subsets of C, and $\sigma = \bigcup_{n=1}^{\infty} D(n)$. Assume $\pi_n^{n+1}: \mathrm{Hol}(D(n+1)) \to \mathrm{Hol}(D(n))$ is the natural homomorphism defined by $\pi_n^{n+1}(f) = f \mid D(n)$ for each $f \in \mathrm{Hol}(D(n+1))$, $n = 1, 2, \ldots$ Since the sequence $\{D(n)\}_{n=1}^{\infty}$ is an increasing sequence, π_n^{n+1} is continuous, and $\pi_n^{n+1}(\mathrm{Hol}(D(n)))$ is clearly dense in $\mathrm{Hol}(D(n))$. Using Arens' terminology [1], let $\mathrm{Hol}(\sigma)$ denote the strong dense inverse limit of $\{\mathrm{Hol}(D(n)), \pi_n^{n+1}\}_{n=1}^{\infty}$. In Michael's terminology, $\mathrm{Hol}(\sigma)$ is the projective limit of the Banach algebras $\{\mathrm{Hol}(D(n)), \|\ \|_{D(n)}\}_{n=1}^{\infty}$. $\mathrm{Hol}(\sigma)$ is clearly a singly generated semisimple F-algebra and $M_{\mathrm{Hol}(\sigma)}$ is identifiable pointwise with $\sigma(a) = \bigcup_{n=1}^{\infty} D(n)$.

EXAMPLE 2.4. Let

$$D(n) = \{z \in C: 0 \leqslant \arg z \leqslant 2\pi - 1/n: 1/n \leqslant |z| \leqslant n\}$$
$$\cup \{z \in C: \arg z = 2\pi - 1/k, k \geqslant n+1, 1/n \leqslant |z| \leqslant n\}.$$

LEMMA 2.5. *If* $\sigma = \bigcup_{n=1}^{\infty} D(n)$ *then* $A = \mathrm{Hol}(\sigma)$ *is a Liouville F-algebra. Furthermore,* $\bigcup_{n=1}^{\infty} D(n)$ *is a proper subset of* C.

Proof. We note that $D(n) \subseteq D(n+1)$ for each $n = 1, 2, \ldots$, and $\sigma = \bigcup_{n=1}^{\infty} D(n) = C - \{0\}$. $\mathrm{Hol}(\sigma)$ is a singly generated semisimple F-algebra with an identity element, and $\sigma(a)$ is identifiable with $\sigma(a) = C - \{0\}$. The proof that A is a Liouville F-algebra follows from the argument used in the proof of Lemma 3 of [3].

The example may easily be modified to exclude infinitely many points from $\sigma(a)$. We now discuss what conditions must be placed on A, which will guarantee that $\sigma(a) = C$ where a generates A.

DEFINITION 2.6. An algebra A is said to admit square roots, if for each $f \in A$ with $f^{-1} \in A$, $x^2 - f = 0$ has a solution in A.

The algebra A in Example 2.4 does not admit square roots, as is demonstrated by the following argument. Since $0 \notin \sigma(a)$, $a^{-1} \in A$ [9, Theorem 5.2]. If $s \in A$ were a solution to $x^2 - a = 0$, then $s^2 = a$, and $s(z) = \pm \sqrt{z}$ for $z \in \sigma(a)$. Without loss of generality, assume that $s(1) = 1$. Now, $s(z)$ is analytic off the positive real axis, and so $s(z)$ defines a single branch of the square root function away from the positive real axis. The choice of $D(n)$ implies that $s(z)$ has a continuous extension to a circle Γ about the origin in the relative euclidean topology on Γ. This clearly is a contradiction. Hence, $x^2 - a = 0$ has no solution in A.

Example 2.4 provides a counterexample to Theorem 3.1 of Birtel [2]. We now give the correct formulation of his proposition. We let $\mathring{\sigma}(a)$ denote the Euclidean interior of $\sigma(a) \subseteq C$.

PROPOSITION 2.7. *If A is a singly generated Liouville F-algebra, with generator a, and A admits square roots then* $\sigma(a) = C$ *provided* $\mathring{\sigma}(a) \neq \emptyset$.

Proof. Without loss of generality we assume the m-base for A is chosen such that $\sigma(a_n) \subseteq \sigma(a_{n+1})$ for each $n = 1, 2, \ldots$ Now $\sigma(a) = \bigcup_{n=1}^{\infty} \sigma(a_n)$. If $\mathring{\sigma}(a) \neq \emptyset$, then an elementary application of the Baire Category Theorem implies that $\mathring{\sigma}(a_n) \neq \emptyset$ for all n sufficiently large.

Let $\sigma(a) \nsubseteq C$ and without loss of generality we assume that $0 \notin \sigma(a)$. Then a is regular in A, i. e., $a^{-1} \in A$ [9, Theorem 5.2]. Let $s \in A$ denote a solution to $x^2 - a = 0$. Now $\hat{s}\colon M_A \to \sigma(s)$ is a one-one mapping since $\hat{s}(h)^2 = \hat{a}(h)$ for each $h \in M_A$ and $\hat{a}\colon M_A \to \sigma(a)$ is a one-one mapping. Moreover, if $\hat{s}(h) = -\hat{s}(k)$ for some $h \in M_A$ and $k \in M_A$ then $\hat{a}(h) = \hat{a}(k)$ and $h = k$. Thus $\sigma(s) \cap \sigma(-s) = \emptyset$ since $0 \notin \sigma(a)$. But then $\sigma(s) \cap \cap (-\hat{s} \mid \mathring{\sigma}(a_n)) = \emptyset$ for n sufficiently large i. e., $\sigma(s)$ must miss an open disk in C. This contradicts Proposition 2.2 and thus $\sigma(a) = C$.

3. A boundary for F-algebras.

DEFINITION 3.1. Let A denote an F-algebra. If $\underline{P}$ is an m-base of zero in A, then the boundary $\Gamma_{\underline{P}}$ of A is defined as follows: $\Gamma_{\underline{P}} = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \partial_{\bar{A}_n}$ where $\partial_{\bar{A}_n}$ denotes the Shilov boundary of $\bar{A}_n$.

PROPOSITION 3.2. *If A is an F-algebra, then $\Gamma_{\underline{P}}$ is independent of the choice of m-base for A.*

Proof. Let $\underline{P}$ and $\underline{P}'$ denote two choices of m-base for A, and $\{\bar{A}_n\}_{n=1}^{\infty}$ and $\{\bar{A}'_n\}_{n=1}^{\infty}$ the associated sequence of Banach algebras determined by $\underline{P}$ and $\underline{P}'$ respectively. Let $h \in \Gamma_{\underline{P}} = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \partial_{\bar{A}_n}$. Then there exists K such that $h \in \partial_{\bar{A}_n}$ for all $n \geqslant K$. The hemicompactness of M_A with respect to the sequence $\{M_{\bar{A}_n}\}_{n=1}^{\infty}$ implies that given n, there exists $m = m(n)$ such that $M_{\bar{A}'_n} \subseteq M_{\bar{A}_m}$. Without loss of generality we assume $m = m(n) \geqslant K$. If $h \in M_{\bar{A}'_n}$, it suffices to prove that $h \in \partial_{\bar{A}'_n}$. Let $r_n^m\colon \hat{\bar{A}}_m \to \hat{\bar{A}}'_n$ be the natural restriction homomorphism of $\hat{\bar{A}}_m$ into $\hat{\bar{A}}'_n$ defined by $r_n^m(\hat{f}) = \hat{f} \mid M_{\bar{A}'_n}$ for each $f \in \bar{A}_m$. Now $M_{\bar{A}'_n}$ is a compact subset of $M_{\bar{A}_m}$, and since for Banach algebras the Shilov boundary depends only on the transform algebras, we may apply Corollary 6.2 [10] to conclude that $M_{\bar{A}'_n} \cap \partial_{\bar{A}_m} \subseteq \partial_{\bar{A}'_n}$. Thus, $h \in M_{\bar{A}'_n}$ implies $h \in \partial_{\bar{A}'_n}$ and it follows that $\Gamma_{\underline{P}} \subseteq \Gamma_{\underline{P}'}$. Similarly, $\Gamma_{\underline{P}'} \subseteq \Gamma_{\underline{P}}$ and the proposition is proven.

If A is now assumed to be a singly generated F-algebra and a denotes a generator for A, then $\sigma(a)$ may be identified pointwise with the carrier space M_A of A. Moreover, $\hat{a}(\Gamma) = \Gamma_a$ implies by Proposition 3.2 that Γ_a is independent of the choice of m-base for A. Since the topological boundary of $\sigma(a_n)$, $\mathrm{bd}\,\sigma(a_n)$, can be identified with $\partial_{\bar{A}_n}$ whenever $\bar{A}_n$ is singly generated [9], we have $\Gamma_a = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \mathrm{bd}\,\sigma(a_n)$. Also, $\Gamma_a = \sigma(a) - \bigcup_{n=1}^{\infty} \mathring{\sigma}(a_n)$ where $\mathring{\sigma}(a_n)$ denotes the Euclidean interior of $\sigma(a_n)$.

PROPOSITION 3.3. *If A is a singly generated Liouville F-algebra with generator a, then* $\bar{\Gamma}_a = C - \bigcup_{n=1}^{\infty} \mathring{\sigma}(a_n)$ (closure in the Euclidean topology on C).

Proof. Without loss of generality we choose an m-base $\underline{P}$ for A such that $\sigma(a_n) \subseteq \sigma(a_{n+1})$ for each $n = 1, 2, \ldots$ If $\mathring{\sigma}(a_n) = \emptyset$ for each $n = 1, 2, \ldots$, then $\Gamma_a = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \mathrm{bd}\,\sigma(a_n) = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \sigma(a_n) = \bigcup_{k=1}^{\infty} \sigma(a_k) = \sigma(a)$. Since $\sigma(a)$ is dense in C (Proposition 2.2) we have $\bar{\Gamma}_a = \overline{\sigma(a)} = C$.

Without loss of generality we may assume $\sigma(a_1) \neq \emptyset$. Clearly $\overline{\Gamma}_a \subseteq C - \bigcup_{n=1}^{\infty} \mathring{\sigma}(a_n)$. Let $\lambda \in C - \bigcup_{n=1}^{\infty} \mathring{\sigma}(a_n)$. It suffices to prove that if $N_r(\lambda) = \{z \in C: |z-\lambda| < r\}$ for any $r > 0$, then $\overline{N_r(\lambda)} \cap \overline{\Gamma}_a \neq \emptyset$. Assume $\overline{N_r(\lambda)} \cap \overline{\Gamma}_a = \emptyset$ for some $r > 0$. Now $[\mathring{\sigma}(a_n) \cap N_r(\lambda)]^c$ where the complement is taken with respect to $\overline{N_r(\lambda)}$, is a compact subset of $\overline{N_r(\lambda)}$. Let D_n denote the component of $[\mathring{\sigma}(a_n) \cap N_r(\lambda)]^c$ containing λ and let $D = \bigcap_{n=1}^{\infty} D_n$. We prove that D must contain a point $x_0 \notin \sigma(a)$ where $x_0 \neq \lambda$. Let γ_r denote the circle of radius r about λ. Now $\gamma_r \subset \sigma(a_n)$ for any $n = 1, 2, \ldots$, since $\sigma(a_n)$ is a non-separating subset of C and $\lambda \notin \sigma(a)$. Moreover, since $\sigma(a)$ is a dense connected subset of C, we know that $\sigma(a) \cap \gamma_r \neq \emptyset$. Without loss of generality we may assume that $\sigma(a_1) \cap \gamma_r \neq \emptyset$. But $\sigma(a_n)$ being a non-separating subset of C, implies that there exists an arc joining λ to ∞ which misses $\sigma(a_n)$, and intersects γ_r. Thus, there exists $x_n \in D_n \cap \gamma_r$ for each $n = 1, 2, \ldots$ Because γ_r is compact, $\{x_n\}_{n=1}^{\infty}$ has a limit point $x_0 \in D$ since $x_j \in D_k$ for each $j \geqslant k$. Moreover, $x_0 \neq \lambda$ since $x_0 \in \gamma_r$.

Now $D = \bigcap_{n=1}^{\infty} D_n$. Since $\{D_n\}_{n=1}^{\infty}$ is a decreasing sequence of compact connected subsets of C, D is a compact, connected subset of C. Also, $D \cap \sigma(a) \subseteq D \cap (\bigcup_{n=1}^{\infty} \mathring{\sigma}(a_n)) \cup (D \cap \overline{\Gamma}_a) = \emptyset$ since $D_m \cap \mathring{\sigma}(a_m) = \emptyset$ for each $m \geqslant n$ and $D \cap \overline{\Gamma}_a \subseteq \overline{N_r(\lambda)} \cap \overline{\Gamma}_a = \emptyset$. Thus, D is a closed connected subset of $\sigma(a)^c$ which contains more than one point. This contradicts Proposition 2.2. Thus, $\overline{N_r(\lambda)} \cap \Gamma_a \neq \emptyset$ for any $r > 0$ and the proposition is proven.

Even if A is a Liouville F-algebra with generator a, Γ_a is in general not a connected subset of C. However, we do have the following proposition.

PROPOSITION 3.4. *If A is a singly generated Liouville F-algebra with generator a, then $\overline{\Gamma}_a$ (closure now taken in the extended complex plane S) is a connected subset of S.*

Proof. We again assume without loss of generality that $\sigma(a_n) \subseteq \sigma(a_{n+1})$ for each $n = 1, 2, \ldots$ Now $\sigma(a_n)^c$ (complement in S) is an open connected subset of S and hence $\overline{\sigma(a_n)^c}$ (closure in S) is a compact, connected subset of S. Moreover, $\overline{\sigma(a_n)^c} \supseteq \overline{\sigma(a_{n+1})^c}$ implies that $D = \bigcap_{n=1}^{\infty} \overline{\sigma(a_{n+1})^c}$ is a compact, connected subset of S. If $\lambda \in \overline{\Gamma}_a$ and $\lambda \neq \infty$ then $\lambda \in \mathrm{bd}\,\sigma(a_n)$ whenever $\lambda \in \sigma(a_n)$. Thus $\lambda \in \overline{\sigma(a_n)^c}$ for each $n = 1, 2, \ldots$, and $\lambda \in D$. If $\lambda = \infty$ then clearly $\lambda \in D$ and thus $\overline{\Gamma}_a \subseteq D$. Conversely, if $\lambda \in D$ and $\lambda \neq \infty$ then $\lambda \in \mathring{\sigma}(a_n)$ for each $n = 1, 2, \ldots$ By Proposition 3.3, $\overline{\Gamma}_a = C - \bigcup_{n=1}^{\infty} \mathring{\sigma}(a_n)$ implies that $\lambda \in \overline{\Gamma}_a$. Thus $D = \overline{\Gamma}_a$ and the proposition is proven.

It follows from the above proposition that the closure of Γ_a in C is a Euclidean perfect set without any bounded components.

In [3], Birtel stated that if for some choice of m-base, $\bigcap_{n=1}^{\infty} \mathrm{bd}\,\sigma(a_n) = \emptyset$, then $\sigma(a) = C$. [3, Theorem 3.2]. His proof assumed that A admits square roots as previously noted. This assumption is not needed, nor is the theorem dependent on the choice of m-base for A.

PROPOSITION 3.5. *If A is a singly generated Liouville F-algebra with generator a and $\Gamma = \emptyset$, then $\sigma(a) = C$. Moreover, $\hat{a}: M_A \to C$ is a homeomorphism of M_A with the direct limit topology onto C with the Euclidean topology.*

Proof. Proposition 3.3 implies that $C = \bigcup_{n=1}^{\infty} \mathring{\sigma}(a_n)$. The remainder of the proof follows as in [2, Theorem 3.2].

In fact, when $\Gamma = \emptyset$, Birtel's proof in Theorem 3.3 together with Proposition 3.5 now yields the following characterization of the algebra E of entire functions on C.

PROPOSITION 3.6. (Birtel) *If A is a singly generated Liouville F-algebra with identity then $\Gamma = \emptyset$ if and only if A is topologically isomorphic to the algebra E of entire functions on the complex plane with the compact-open topology.*

If A is a singly generated F-algebra, then the existence of Liouville F-algebras with $\Gamma \neq \emptyset$ shows that our Γ is not a maximizing set for any $\hat{a} \in \hat{A}$ where $a \neq \lambda e$. In Section 4 we show that the two known concepts of topological divisors of zero for F-algebras differ. For one of these concepts the non-topological divisors of zero may have a zero on Γ. We now construct an example which shows that Γ is in general not a determining set for a singly generated F-algebra.

EXAMPLE 3.7. For each positive integer n and k, define:

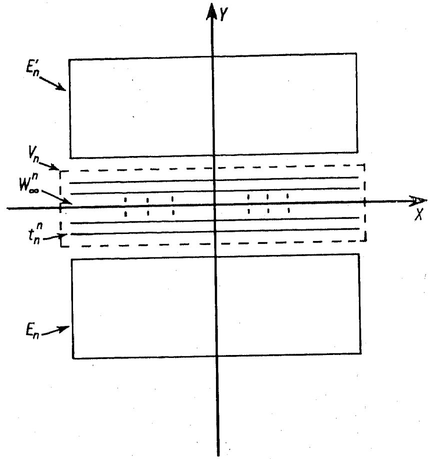
$$E'_n = \{z = x+iy:\ 1/n \leqslant y \leqslant n,\ -n+1/n \leqslant x \leqslant n-1/n\},$$

$$E_n = \{z = x+iy:\ -n \leqslant y \leqslant -1/n,\ -n+1/n \leqslant x \leqslant n-1/n\},$$

$$w_k^n = \{z = x+iy:\ y = \frac{2k+1}{2k(k+1)},\ -n+1/n \leqslant x \leqslant n-1/n\},$$

$$t_k^n = \{z = x+iy:\ y = -\frac{2k+1}{2k(k+1)},\ -n+1/n \leqslant x \leqslant n-1/n\},$$

$$w_\infty^n = \{z = x+iy:\ y = 0,\ -n+1/n \leqslant x \leqslant n-1/n\},$$



Let $D_n = E_n \cup E'_n \cup w^n_\infty \cup \bigcup\limits_{k=1}^{\infty} w^n_k \cup \bigcup\limits_{k=1}^{\infty} t^n_k$. Now D_n is a compact non-separating subset of C, $t^n_k \subseteq \mathring{E}_n$, $w^n_k \subseteq \mathring{E}'_n$ for $k = 1, 2, \ldots, n-1$, and $C = \bigcup\limits_{n=1}^{\infty} D_n$. Let z_0 be a fixed point in $\mathring{E}_1$ and set

$$V_n = \{z = x+iy \mid -\frac{4n+3}{4n(n+1)} < y < \frac{4n+3}{4n(n+1)}, -n < x < n\}.$$

Note that $\bigcup\limits_{k=n}^{\infty} t^n_k \subseteq V_n$ and $\bigcup\limits_{k=n}^{\infty} w^n_k \subseteq V_n$. Moreover, $\overline{V}_n \cap E_n = \overline{V}_n \cap E'_n = \emptyset$ and if $m \geqslant n$ then $V_m \cap D_n \subseteq V_n$. Also $E_n \subseteq E_{n+1}$ and $E'_n \subseteq E'_{n+1}$ for each $n = 1, 2, \ldots$

We now define a sequence of polynomials inductively as follows: Let $p_1(z)$ be a polynomial in z such that $p_1(z_0) = 1$, and $|p_1|_{\overline{V}_1 \cup E'_1} < 1$ ($|p|_D = \sup\limits_{z \in D} |p(z)|$). Such a polynomial exists by Runge's theorem. Assume $p_1(z), \ldots, p_{n-1}(z)$ have been constructed where $p_i(z_0) = 1$, $|p_i|_{\overline{V}_i \cup E'_i} < 1/2^i$ and $|p_i - p_{i-1}|_{E_i} < 1/2^i$ for $i = 1, 2, \ldots, n-1$. We now show how to pick $p_n(z)$. Since $\overline{V}_n$, E_n and E'_n are disjoint, non-separating compact sets in C we can find a polynomial $q(z)$ such that $|q|_{\overline{V}_n \cup E'_n} < 1/2^{n+1}$ and $|q - p_{n-1}|_{E_n} < 1/2^{n+1}$. Let $p_n(z) = q(z) + (1 - q(z_0))$. Then $p_n(z_0) = 1$, and

$$|p_n|_{\overline{V}_n \cup E'_n} = |q + (1 - q(z_0))|_{\overline{V}_n \cup E'_n} \leqslant |q|_{\overline{V}_n \cup E'_n} + |p_{n-1}(z_0) - q(z_0)|$$
$$< 1/2^{n+1} + 1/2^{n+1} = 1/2^n$$

since $z_0 \in E_n$. Furthermore,

$$|p_n - p_{n-1}|_{E_n} = |q - p_{n-1} + 1 - q(z_0)|_{E_n} \leqslant |q - p_{n-1}|_{E_n} + |p_{n-1}(z_0) - q(z_0)|$$
$$< 1/2^{n+1} + 1/2^{n+1} = 1/2^n.$$

Thus, there exists a sequence of polynomials $\{p_n(z)\}_{n=1}^{\infty}$ satisfying $p_n(z_0) = 1$, $|p_n|_{\overline{V}_n \cup E'_n} < 1/2^n$ and $|p_n - p_{n-1}|_{E_n} < 1/2^n$.

Let A denote the projective limit of $\{\mathrm{Hol}(D_n)\}_{n=1}^{\infty}$ using our previous construction. We view A as an algebra of functions on $C = \bigcup\limits_{n=1}^{\infty} D_n$. Then the boundary Γ is $\{z \in C\colon z \text{ real}\}$.

PROPOSITION 3.8. *The algebra A is a singly generated Liouville F-algebra. Moreover, the sequence $\{p_n(z)\}_{n=1}^{\infty}$ determines a non-zero $f \in A$ satisfying $f \mid \Gamma \equiv 0$. Furthermore, $f(z) \equiv 0$ whenever* $\mathrm{im}(z) \geqq 0$.

Proof. A is clearly a single generated semisimple F-algebra. The proof that A satisfies the Liouville property is the same as in [3].

It suffices to prove that $\lim\limits_{n\to\infty} p_n(z) = f(z)$ exists for each $z \in C$ and $f \mid D_1 \in \mathrm{Hol}(D_1)$ for each $1 = 1, 2, \ldots$ Let 1 denote a fixed positive integer. We prove that $\{p_n(z)\}_{n=1}^{\infty}$ is a Cauchy sequence on D_1. Let $\varepsilon > 0$ be given. Choose $j > 0$ such that $j > 1$ and $1/2^{j-2} < \varepsilon$. Let $m > j$. We will show that $|p_m(z) - p_j(z)| < \varepsilon$ for each $z \in D_1$. Let z be a fixed point in D_1.

If $z \in D_1 \cap V_m$ then

$$|p_m(z) - p_j(z)| \leqslant |p_m(z)| + |p_j(z)| \leqq |p_m|_{V_m} + |p_j|_{V_j} < 1/2^m + 1/2^j < 1/2^{j-1}$$

since $z \in V_m \cap D_1 \subseteq V_j$ for $m > j$.

If $z \notin D_1 \cap V_m$ then there exists a $k > 0$ such that $z \notin E_k \cup E'_k$ but $z \in E_{k+1} \cup E'_{k+1}$. Now $m > k$ since $z \in E_m \cup E'_m$.

Case I. Let $j > k$. Then $z \in E_j \cup E'_j \subseteq E_m \cup E'_m$ since $j > 1$. If $z \in E_j$ then

$$|p_m(z) - p_j(z)| \leqq \sum_{n=j}^{m-1} |p_{n+1}(z) - p_n(z)| \leqq \sum_{n=j}^{m-1} |p_{n+1} - p_n|_{E_{n+1}}$$
$$\leqslant \sum_{n=j}^{m-1} 1/2^{n+1} < 1/2^j.$$

If $z \in E'_j$ then

$$|p_m(z) - p_j(z)| \leqq |p_m(z)| + |p_j(z)| \leqq |p_m|_{E'_m} + |p_j|_{E'_j} \leqslant 1/2^m + 1/2^j < 1/2^{j-1}.$$

Case II. Let $k \geqslant j$. Recall that $z \in E_{k+1} \cup E'_{k+1}$. Let $z \in E_{k+1}$. Since $z \in E_n$ for $n > k$ and since $z \in V_k \cap D_1 \subset V_j$ for $k \geqslant j$ we have:

$$|p_m(z)-p_j(z)| \leqslant \sum_{n=k}^{m-1} |p_{n+1}(z)-p_n(z)|+|p_k(z)|+|p_j(z)|$$
$$\leqslant \sum_{n=k}^{m-1} |p_{n+1}-p_n|_{E_{n+1}}+|p_k|_{V_k}+|p_j|_{V_j}$$
$$\leqslant \sum_{n=k}^{m-1} 1/2^{n+1}+1/2^k+1/2^j < 1/2^{j-2}.$$

Now let $z \in E'_{k+1}$. Then $z \in E'_n$ for $n > k$ and

$$|p_m(z)-p_j(z)| \leqq |p_m(z)|+|p_j(z)|$$
$$\leqq |p_m|_{E'_m \cup V_n}+|p_j|_{V_j} < 1/2^m+1/2^j < 1/2^{j-1}$$

since $z \in V_k \cap D_1 \subset V_j$.

Thus in all cases, if $z \in D_1$, then $|p_m(z)-p_j(z)| < 1/2^{j-2} < \varepsilon$. Then $\{p_n(z)\}_{n=1}^{\infty}$ converges uniformly on D_1 to a function f_1 in $\mathrm{Hol}(D_1)$. Since $f_{1+1} \mid D_1 = f_1$ then $f(z) = \lim_{n\to\infty} p_n(z)$ exists for each $z \in C$ and defines an element $f \in A$. We note that $f(z_0) = \lim_{n\to\infty} p_n(z_0) = 1$. Also, if z is a real number then $z \in \bigcap_{n=k} V_n$ for some $k \geqslant 1$ and $|p_n(z)| < 1/2^n$ for $n \geqslant k$. Thus $f(z) = \lim_{n\to\infty} p_n(z) = 0$ for each $z \in \Gamma$ i. e., $f \mid \Gamma = 0$. Similarly, $|p_n|_{E'_n} < 1/2^n$ implies $f(z) = 0$ whenever $\mathrm{im}(z) \geqslant 0$. This completes the proof of the proposition.

The algebra of entire functions and the published examples of Liouville F-algebras are all integral domains. Example 3.7 illustrates that this is not a consequent of the Liouville property. By reversing the roles of E_n and E'_n we may construct a $g \in A$ with $g \neq 0$ and $g(z) = 0$ whenever $\mathrm{im}(z) \leqq 0$. Then $f \cdot g = 0$ where $f \neq 0$ and $g \neq 0$ and thus A has divisors of zero.

4. Algebraically principal closed maximal ideals in singly generated F-algebras. For Banach algebras we have the following characterization of principal maximal ideals. This answers a question posed in [12].

PROPOSITION 4.1. *Let A be a commutative semisimple Banach algebra with an identity element. If $h \in M_A$ and $h^{-1}(0) = xA$ for some $x \in A$ then $h \in \partial_A$ if and only if h is isolated in M_A.*

Proof. If $h \in M_A$ and $h^{-1}(0) = xA$ where h is isolated in M_A then we may assume that x is an idempotent element in A with $\hat{x}(h) = 0$ and $\hat{x} \mid M_A - \{h\} = 1$. It follows that $h \in \partial_A$. Conversely, assume that h is not

isolated in M_A. Now $\psi(y) = xy$ for $y \in A$ is continuous and one-one since h is not isolated in M_A. Moreover, ψ^{-1} is bounded by the Inverse Mapping Theorem. Now $\|\psi^{-1}\| = \sup_{xy\neq 0} \dfrac{\|\psi^{-1}(xy)\|}{\|xy\|} = \dfrac{1}{\inf_{y\neq 0} \frac{\|xy\|}{\|y\|}}$ since $xy \neq 0$ if and only if $y \neq 0$ using the fact that h is not isolated in M_A and $\hat{x}(h') \neq 0$ for each $h' \in M_A - \{h\}$. Thus $\inf_{y\neq 0} \dfrac{\|xy\|}{\|y\|} = M \neq 0$. By a result of Arens' [0] there exists an extension B of A in which $x \in A$ has an inverse. Thus $h \in M_A$ does not extend to a multiplicative linear functional on B and it follows that $h \notin \partial_A$.

PROPOSITION 4.2. *If A is a singly generated semisimple F-algebra without proper idempotent elements then A contains a dense subalgebra B such that:*

(1) *B is a singly generated semisimple F-algebra without proper idempotent elements in a stronger topology.*

(2) *M_A and M_B are homeomorphic with respect to their direct limit topologies.*

(3) *Γ_A is homeomorphic with Γ_B with respect to the direct limit topologies.*

(4) *Each closed maximal ideal of B is algebraically principal.*

Proof. Let a denote a choice of generator for A. A is the inverse limit of $\{\bar{A}_n\}_{n=1}^{\infty}$. We assume without loss of generality that the seminorms are chosen on A such that $\|e\|_n = 1$ and $\|a\|_n \leqslant \|a\|_{n+1}$ for each $a \in A$ and $n \geqslant 1$. We identify M_A with $\sigma(a)$ (pointwise) and $M_{\bar{A}_n}$ with $\sigma(a_n)$. As noted in Section 2, $M_{\bar{A}_n}$ is homeomorphic with $\sigma_{\bar{A}_n}(a_n)$. Since A is a semisimple algebra, we may obtain a concrete realization for elements in A as functions on $\sigma(a)$ by defining $a(\lambda) = \hat{a}(h)$ where $\hat{a}(h) = \lambda \in \sigma(a)$ for each $h \in M_A$. In particular, $a(\lambda) = \lambda$ for each $\lambda \in \sigma(a)$.

If $p(z)$ is a polynomial with complex coefficients in the complex variable z then $p(z) = p(\lambda)+(z-\lambda)q_\lambda(z)$ where $q_\lambda(z)$ is a polynomial in z. We note that $q_\lambda(z)$ is uniquely determined and when λ is allowed to vary, $q_\lambda(z)$ becomes a polynomial in the two variables λ and z. Now $p(a) \in A$ and $p(a) = p(\lambda)e+(a-\lambda e)q_\lambda(a)$ where $q_\lambda(a) \in A$ is a similar decomposition in A for each $\lambda \in \sigma_A(a)$. The semisimplicity of A and the fact that M_A has no isolated points guarantees the uniqueness of the representation.

We define a new sequence of seminorms on the algebra of polynomials, $c[a] \subset A$, inductively as follows: Set $\sigma_{\bar{A}_n}(a_n) = \sigma(a_n)$, $\sigma(a_0) = \sigma(a_1)$ and write p for $p(a)$. Let $|p|_0 = \|p\|_1$; $|p|_1 = \|p\|_1 + \sup_{\lambda\in\sigma(a_0)} |a-\lambda e|_0 \cdot \sup_{\lambda\in\sigma(a_0)} |q_\lambda|_0$. If $|\ |_i$ has been defined for $i = 1, 2, \ldots, n-1$ then set $|p|_n = \|p\|_n +$

$+\sup_{\lambda\in\sigma(a_{n-1})}|a-\lambda e|_{n-1}\cdot\sup_{\lambda\in\sigma(a_{n-1})}|q_\lambda|_{n-1}$. Clearly $|\cdot|_n$ is an additive seminorm and the submultiplicative property is easily proven by induction.

If $N_n=\{p\in C[a]\colon \|p\|_n=0\}$ and $N_n'=\{p\in C[a]\colon |p|_n=0\}$ then $\|p\|_n\leqslant |p|_n$ for each $n\geqslant 1$ and $p\in C[a]$ implies that $\bigcap_{n=1}^{\infty}N_n'\subseteq\bigcap_{n=1}^{\infty}N_n=(0)$ since A is an F-algebra. Let $B_n=C[a]/N_n'$ and let $\bar{B}_n$ denote the completion of B_n with respect to the quotient norm $|\ |_n'$ determined by $|\ |_n$. $\bar{B}_n$ is a singly generated Banach algebra. If e_n' denotes the identity element in $\bar{B}_n$ then we note that $|e_n'|_n=\|e_n\|_n=1$. We use the notation a_n' to denote the generator $a+N_n'$ of $\bar{B}_n$ and $\sigma(a_n')$ to denote the spectrum of $a_n'\in\bar{B}_n$.

We next prove that $\sigma(a_n')=\sigma(a_n)$ pointwise and thus $M_{\bar{A}_n}$ is homeomorphic with $M_{\bar{B}_n}$ for each $n=1,2,\ldots$ If $\lambda\in\sigma(a_n)$ then for any polynomial $p\in C[a]$, $|p(\lambda)|\leqslant\|p\|_n\leqslant|p|_n$ for each $n\geqslant 1$. Since polynomials are dense in $\bar{B}_n$, $\lambda\in\sigma(a_n')$ and thus $\sigma(a_n)\subseteq\sigma(a_n')$ for each $n\geqslant 1$. We now prove by induction on n that $\sigma(a_n')=\sigma(a_n)$ for each $n\geqslant 1$.

Let $u\in\sigma(a_1')$ where $u\notin\sigma(a_1)$. Let γ denote a simple closed rectifiable curve such that $\sigma(a_1)\subseteq$ (interior of γ) and u lies outside of γ. Using Runge's theorem, we may choose a sequence of polynomials $\{p_k(z)\}_{k=1}^{\infty}$ in the complex variable z converging uniformly on γ to $1/z-u$ and satisfying $|p_k(u)|\geqslant k$ for each $k\geqslant 1$. It follows from Theorem 5.1 [6] that $\{p_k(a_1)\}_{k=1}^{\infty}$ is a Cauchy sequence in $\|\ \|_1$. Indeed, there is a constant $k_1>0$ such that $\|p(a)\|_1\leqslant K_1\cdot\sup_{z\in\gamma}|p(z)|$ for any polynomial $p(z)$. Since $p_k(z)=p_k(\lambda)+(z-\lambda)q_{k,\lambda}(z)$, we have for $z\in\gamma$, $\lambda\in\sigma(a_1)$:

$$\big|(z-\lambda)\big(q_{k,\lambda}(z)-q_{m,\lambda}(z)\big)\big|\leqslant\big|p_k(z)-p_k(\lambda)+\big(p_m(z)-p_m(\lambda)\big)\big|$$
$$\leqslant 2\sup_{z\in\gamma}|p_k(z)-p_m(z)|\underset{k,m}{\to}0.$$

Now

$$\sup_{\substack{z\in\gamma\\ \lambda\in\sigma(a_1)}}\big|(z-\lambda)\big(q_{k,\lambda}(z)-q_{m,\lambda}(z)\big)\big|\geqslant\inf_{\substack{z\in\gamma\\ \lambda\in\sigma(a_1)}}|z-\lambda|\cdot\sup_{\substack{z\in\gamma\\ \lambda\in\sigma(a_1)}}|q_{k,\lambda}(z)-q_{m,\lambda}(z)|.$$

Since $\sigma(a_1)$ is a compact set contained in the interior of γ, $\inf_{\substack{z\in\gamma\\ \lambda\in\sigma(a_1)}}|z-\lambda|=\delta_1>0$. Thus

$$\sup_{\lambda\in\sigma(a_1)}\|q_{k,\lambda}-q_{m,\lambda}\|_1\leqslant K_1\cdot\sup_{\substack{z\in\gamma\\ \lambda\in\sigma(a_1)}}|q_{k,\lambda}(z)-q_{m,\lambda}(z)|$$
$$\leqslant 2K_1/\delta_1\cdot\sup_{\lambda\in\sigma(a_1)}|p_k(z)-p_m(z)|\underset{k,m}{\to}0.$$

Then $\{q_{k,\lambda}\}_{k=1}^{\infty}$ converges uniformly in $\lambda\in\sigma(a_1)$ and

$$|p_k-p_m|_1=\|p_k-p_m\|_1+\sup_{\lambda\in\sigma(a_1)}\|a_1-\lambda e_1\|_1\sup_{\lambda\in\sigma(a_1)}\|q_{k,\lambda}-q_{m,\lambda}\|_1\underset{k,m}{\to}0.$$

Thus $p_n(a_1')\underset{n}{\to}b_1\in\bar{B}_1$. Now if $u\in\sigma(a_1')$, then there exists $h\in M_{\bar{B}_1}$ such that $h(a_1')=u$. But $h(b_1)=\lim_{n\to\infty}h\big(p_n(a_1')\big)=\lim_{n\to\infty}p_n(u)$ which is a contradiction since $|p_n(u)|\geqslant n$ for each $n\geqslant 1$. It follows that $\sigma(a_1)=\sigma(a_1')$.

Now assume inductively that $\sigma(a_{n-1})=\sigma(a_{n-1}')$. It suffices to prove that $\sigma(a_n')\subseteq\sigma(a_n)$. Choose a simple closed rectifiable curve γ about $\sigma(a_n)\supseteq\sigma(a_{n-1})$ as before, where $u\in\sigma(a_n')\cap\sigma(a_n)^c$ lies outside of γ. Theorem 5.1 [6], with our induction hypothesis, guarantees constants K_{n-1} and K_n' such that $|p|_{n-1}\leqslant K_{n-1}\sup_{z\in\gamma}|p(z)|$ and $\|p\|_n\leqslant K_n'\sup_{z\in\gamma}|p(z)|$ for each polynomial p. Next choose a particular sequence $\{p_n(z)\}_{n=1}^{\infty}$ where $p_n(z)\underset{n}{\to}1/z-u$ on $\sigma(a_n)$ and $p_n(u)\geqslant n$ for each $n\geqslant 1$. Following the previous argument we have:

$$|p_k-p_m|_n=\|p_k-p_m\|_n+\sup_{\lambda\in\sigma(a_{n-1})}|a-\lambda e|_{n-1}\cdot$$
$$\sup_{\lambda\in\sigma(a_{n-1})}|q_{k,\lambda}-q_{m,\lambda}|_{n-1}$$
$$\leqslant K_n'\sup_{z\in\gamma}|p_k(z)-p_m(z)|++K_{n-1}^2\sup_{\substack{z\in\gamma\\ \lambda\in\sigma(a_{n-1})}}|z-\lambda|\sup_{\substack{z\in\gamma\\ \lambda\in\sigma(a_{n-1})}}|q_{k,\lambda}(z)-q_{m,\lambda}(z)|$$
$$\leqslant K\sup_{z\in\gamma}|p_k(z)-p_m(z)|\quad\text{where}\quad K=K(K_{n-1},K_n',\delta_{n-1}).$$

Again by our choice of $\{p_k\}_{k=1}^{\infty}$ we have $(a_n'-ue_n')^{-1}\in\bar{B}_n$, i. e., $u\notin\sigma(a_n')$. It follows that $\sigma(a_n)=\sigma(a_n')$ and thus $M_{\bar{A}_n}$ and $M_{\bar{B}_n}$ are homeomorphic with respect to their weak $*$ topologies.

Let $\pi_n^{n+1}\colon\bar{B}_{n+1}\to\bar{B}_n$ denote the natural homomorphism of $\bar{B}_{n+1}$ into $\bar{B}_n$ determined by setting $\pi_n^{n+1}\big(p(a_{n+1}')\big)=p(a_n')$. Since $|p(a_n')|_n\leqslant|p(a_{n+1}')|_{n+1}$ for each $n\geqslant 1$, π_n^{n+1} is well defined and continuous. The sequence $\{\bar{B}_n,\pi_n^{n+1}\}$ determines an F-algebra B, the strong dense inverse limit of $\{\bar{B}_n,\pi_n^{n+1}\}$ [1, Theorem 2.4]. Now $c[a]\subseteq A$ and $c[a]\subseteq B$. Let $\psi\colon B\to A$ denote the natural extension of the identity mapping to all of B. ψ is clearly a continuous homomorphism of B into A. We prove that ψ is actually an isomorphism of B into A.

We first obtain a representation for any $b\in B$. If $b\in B$, there exists $\{p_n(a)\}_{n=1}^{\infty}$ such that $p_n(a)\underset{n}{\to}b$ in B. Now

$$|p_n-p_m|_k=\|p_n-p_m\|_k+\sup_{\lambda\in\sigma(a_{k-1})}|a-\lambda e|_{k-1}\sup_{\lambda\in\sigma(a_{k-1})}|q_{n,\lambda}-q_{m,\lambda}|_{k-1}\to 0$$

for each $k\geqslant 1$ implies that $\{q_{n,\lambda}\}_{n=1}^{\infty}$ converges to $b_\lambda\in B$ for each $\lambda\in\sigma_B(a)=\bigcup_{n=1}^{\infty}\sigma(a_n')$. This yields a representation for $b\in B$, namely, $b=b(\lambda)e'+$ $+(a-\lambda e')b_\lambda$.

Applying ψ, we obtain $\psi(b) = b(\lambda)e + (a-\lambda e)\psi(b_\lambda) = \psi(b)(\lambda)e + (a-\lambda e)\psi(b_\lambda)$ since $b(\lambda) = \psi(b)(\lambda)$ for each $\lambda \epsilon \sigma_A(a) = \sigma_B(a)$. If $\psi(b) = 0$ then $0 = (a-\lambda e)\psi(b_\lambda)$. If $\lambda' \epsilon \sigma_A(a)$ and $\lambda' \neq \lambda$ then $[(a-\lambda e)\psi(b_\lambda)]\hat{}(\lambda') = 0$ implies $\psi(b_\lambda)\hat{}(\lambda') = 0$ since $(a-\lambda e)\hat{}(\lambda') = \lambda'-\lambda \neq 0$. The fact that A contains no proper idempotent elements implies that λ is not an isolated point in $\sigma_A(a)$ (c.f. the proof of Proposition 2.2). Thus $\psi(b_\lambda)\hat{}(\lambda') \equiv 0$ for each $\lambda' \epsilon \sigma_A(a)$, i. e., $\psi(b_\lambda)\hat{} \equiv 0$. The semisimplicity of A implies that $\psi(b_\lambda) = 0$.

By our definition of $\{|\ |_n\}_{n=1}^{\infty}$, we have:

(1) $$|b|_1 = \|\psi(b)\|_1 + \operatorname*{Sup}_{\lambda\epsilon\sigma(a_1)} \|a-\lambda e\|_1 \operatorname*{Sup}_{\lambda\epsilon\sigma(a_1)} \|\psi(b_\lambda)\|_1.$$

In general, for $n > 1$ we have:

(2) $$|b|_n = \|\psi(b)\|_n + \operatorname*{Sup}_{\lambda\epsilon\sigma(a_{n-1})} |a-\lambda e'|_{n-1} \operatorname*{Sup}_{\lambda\epsilon\sigma(a_{n-1})} |b_\lambda|_{n-1}.$$

By applying (1) we obtain $|b|_1 = 0$. Since $\psi(b_\lambda) = 0$, the previous argument applied to b_λ implies that $|b_\lambda|_1 = 0$. Thus $|b|_2 = 0$. In general, if $|b|_{n-1} = 0$ then the above argument implies $|b_\lambda|_{n-1} = 0$ for each $\lambda \epsilon \sigma(a_{n-1})$. By applying (2) we obtain $|b|_n = 0$. Since B is an F-algebra we have that $b = 0$. Thus ψ: $B \to A$ is an isomorphism.

Now $b = b(\lambda)e' + (a-\lambda e')b_\lambda$ is a unique representation for $b \epsilon B$ since ψ is an isomorphism and A is semisimple. Thus each closed maximal ideal in B is algebraically principal. This completes the proof of the proposition.

Corollary 4.3. *If A is a singly generated Liouville F-algebra, then A contains a dense subalgebra B such that*:

(1) *B is a singly generated Liouville F-algebra in a stronger topology.*

(2) *A is isomorphic with E if and only if B is isomorphic with E.*

(3) *Each closed maximal ideal in B is algebraically principal.*

Proof. Let B be the algebra constructed in the previous proposition. We identify B with its image $\psi(B) \subseteq A$. If $b \epsilon B \subseteq A$ has a bounded spectrum with respect to B, then $M_A = M_B$ implies that $b \epsilon A$ has a bounded spectrum with respect to A. Hence B is a Liouville F-algebra. Moreover, A is isomorphic with E if and only if $\Gamma_A = \emptyset$ (Proposition 3.6). The corollary now follows from the fact that $\Gamma_A = \Gamma_B$.

Two notions of topological divisors of zero have been proposed for a locally m-convex algebra. We state these here. Let $R_{x+\lambda e}$ denote the natural mapping $R_{x+\lambda e}$: $A \to A$ where $R_{x+\lambda e}(y) = y(x+\lambda e)$.

Definition 4.4. (Arens) Let A be a topological algebra and $x \epsilon A$. Then $x+\lambda e$ is a strong topological divisor of zero if either $R_{x+\lambda e}$ or $L_{x+\lambda e}$ is not a topological isomorphism into [9, p. 46].

Definition 4.5. (Michael) Let A be a locally m-convex algebra and let $x \epsilon A$. Then $x+\lambda e$ is a topological divisor of zero in A if, whenever $\{V_i\}_{i=1}^{\infty}$ is an m-base for A, there exists an i such that $x_i + \lambda e_i$ is a topological divisor of zero in $\bar{A}_i$ [9, p. 47].

For Banach algebras, these two definitions are equivalent [6]. Michael [9] notes in Proposition 11.3 that Definition 4.5 is stronger than Definition 4.4, and raises the question of their equivalence. Kuczma [7] has recently given an example where there are topological divisors of zero that are not strong topological divisors of zero, but his algebra is not semisimple. Our Proposition 4.2 provides a class of semisimple algebras in which the two definitions are different. We note that the conditions of the following proposition are satisfied if A is the algebra obtained by applying Proposition 4.2 to Example 2.4.

Proposition 4.6. *Let A denote a singly generated semisimple F-algebra without proper idempotent elements, and $h_\lambda \epsilon M_A$. Then $a-\lambda e$ is not a strong topological divisor of zero in A if and only if $h_\lambda^{-1}(0)$ is algebraically principal. Moreover, if $h_\lambda \epsilon \Gamma_A$ then $a-\lambda e$ is a topological divisor of zero in A.*

Proof. Now $R_{a-\lambda e}$: $A \to (a-\lambda e)A$ is clearly a linear homomorphism. We prove $R_{a-\lambda e}$ is one-one. If $(a-\lambda e)f = 0$ for some $f \epsilon A$, then $\hat{f}(h'_\lambda) = 0$ for each $h'_\lambda \neq h_\lambda$, $h'_\lambda \epsilon M_A$. The algebra A has no proper idempotent elements and hence h_λ is not isolated in M_A. Thus $\hat{f} \equiv 0$ and the semisimplicity of A implies $f = 0$. Multiplication is continuous in A, so $R_{a-\lambda e}$ is continuous. If $(a-\lambda e)A = h_\lambda^{-1}(0)$ then $(a-\lambda e)A$ is closed in A and $R_{a-\lambda e}$ is a topological isomorphism by the Inverse Mapping Theorem. Conversely, if $R_{a-\lambda e}$ is a topological isomorphism, then $(a-\lambda e)A$ is complete and hence closed in A. But $(a-\lambda e)A$ is dense in $h_\lambda^{-1}(0)$ since A is singly generated by a. Thus, $h_\lambda^{-1}(0) = (a-\lambda e)A$.

Proposition 3.2 states that Γ_A is independent of the choice of m-base for A. Now fix an m-base for A. If $h_\lambda \epsilon \Gamma_A$, then $(a-\lambda e)\hat{}(h_\lambda) = \hat{a}(h_\lambda) - \lambda = 0$. But this implies that $a_n - \lambda e_n$ has a transform which vanishes on the Shilov boundary of $\bar{A}_n$ for some n. Thus $a-\lambda e$ is a topological divisor of zero according to Definition 4.5.

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On a class of operators on Orlicz spaces

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Abstract. Let L^Φ be an Orlicz space over a σ-finite measure space. If $\mathfrak{X}$ is a Banach space and $t\colon L^\Phi \to \mathfrak{X}$ is a linear operator, $|||t|||_\Phi = \sup \sum_{i=1}^n \|a_i t(\chi_{E_i})\|$ where the supremum is taken over all measurable simple functions $f = \sum_{i=1}^n a_i \chi_{E_i}$ $\{E_i\}$ disjoint and $\|f\|_\Phi \leqslant 1$. Under fairly general assumptions on $\mathfrak{X}$ and Φ it is shown that $|||t|||_\Phi < \infty$ if and only if $t(f) = \int_\Omega fg du$ where $g\colon \Omega \to \mathfrak{X}$ is measurable and the above Bochner integral exists for all $f \in L^\Phi$. Consequently it is shown that such operators are compact. Finally, under moderate assumptions on Φ, it is shown that $t\colon L^\Phi \to L^\Phi$ has $|||t|||_\Phi < \infty$ if and only if t's adjoint is of finite double norm, thus providing a new characterization of Hilbert-Schmidt operators.

1. Introduction. Let (Ω, Σ, μ) be a sigma-finite measure space, Φ and Ψ be complementary Young's functions and $L^\Phi(\Omega, \Sigma, \mu) (= L^\Phi)$ and $L^\Psi(\Omega, \Sigma, \mu) (= L^\Psi)$ be the corresponding Orlicz spaces of (equivalence classes of) measurable functions on Ω. L^Φ is a Banach space under each of the equivalent norms N_Φ and $\|\cdot\|_\Phi$ defined for $f \in L^\Phi$ by $N_\Phi(f) = \inf\{K > 0\colon \int_\Omega \Phi(|f|/K) d\mu \leqslant 1\}$ and $\|f\|_\Phi = \sup\{\int_\Omega fg d\mu\colon g \in L^\Psi \; N_\Psi(g) \leqslant 1\}$. If $\mathfrak{X}$ is a Banach space and t is a bounded linear operator mapping L^Φ into $\mathfrak{X}$, Dinculeanu has defined $|||t|||_\Phi$ by

$$|||t|||_\Phi = \sup \sum_{i=1}^n \|a_i t(\chi_{E_i})\|,$$

where the supremum is taken over all measurable simple functions, $f = \sum_{i=1}^n a_i \chi_{E_i}$, $\{E_i\} \subset \Sigma$ disjoint, such that $N_\Phi(f) \leqslant 1$. This norm for operators has been the subject of some study by Dinculeanu in [1], [2], and [3]. The purpose of this note centers around proving a Bochner integral representation theorem for these operators, examining their compactness properties and looking at their rather close relationship with operators of finite double norm [8].