Since, in view of (3.38),

$$\frac{x}{t_1}-\frac{x}{t_2}=x\frac{t_2-t_1}{t_1t_2}>N\frac{t_2-t_1}{t_1t_2}\geqslant 1,$$

there is an integer n_0 such that

$$\frac{x}{t_2}\leqslant n_0\leqslant\frac{x}{t_1}. \tag{3.39}$$

Thus, by (3.38), we have

$$n_0\geqslant\frac{x}{t_2}>\frac{N}{t_2}\geqslant N_5. \tag{3.40}$$

It follows from (3.39) that $t_0 = x/n_0 \in [t_1, t_2]$. Consequently (3.37) holds for $n = n_0$ and $t = t_0$, that is (*) holds for all $x > N$. The theorem is proved.

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Sequence spaces and interpolation problems for analytic functions

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§ 1. INTRODUCTION

1.1. DEFINITION. Let $w = \{z_n\}$ be a sequence of distinct points in the disk $D = \{z: |z| < 1\}$ with $|z_n| \to 1$. For $1 \leqslant p \leqslant \infty$ let H^p be the usual Hardy class of analytic functions on D with boundary values in L^p. Let $H^p(w) = \{\{f(z_n)\}: f \in H^p\}$.

The purpose of the present work is three-fold. First, an examination of the sequence space structure of $H^p(w)$ is given. Then in the context of general FK spaces some results, many of which were suggested by properties of $H^p(w)$, are considered. In particular the conull property of FK spaces is examined. (See [6] and [10] for previous work on the conull property. J. Sember in [4] studied the conull property in its relation to variation matrices.) Finally, it is shown that there exists a sequence w such that $H^2(w)$ contains all bounded sequences and $H^\infty(w)$ does not, answering a natural question on interpolation by analytic functions.

In § 3 it is shown that $H^p(w)$ is a BK space. If $p < \infty$, then $H^p(w)$ has the AD property. If $1 < p < \infty$, then the coordinate projections are fundamental in $H^p(w)^*$, but $H^1(w)^*$ is not separable.

In § 4 $H^p(w)$ is considered in the context of the conull, conservative, coercive, and wedge properties, and in terms of three new sequence space properties. In particular, it is shown that $H^p(w)$, for $1 < p < \infty$, is conull if and only if $H^p(w)$ contains every sequence of bounded variation. The fact that $H^p(w)$ may be coregular for $p < \infty$ shows that Theorem 6 of [6] fails in the context of non-conservative spaces. (Recently, J. Sember has announced an essentially different example of this failure. See 4.12 for an outline of some of his results.)

Let δ^n denote the sequence $\delta_k^n = 0$ for $k \neq n$, $\delta_n^n = 1$. It is shown in § 5 that $\{\delta^n\}$ is a basis for $H^p(w)$, $p < \infty$, if and only if $w = \{z_n\}$ is an interpolating sequence.

In § 6 it is proved that $H^2(w)$ may be coercive without $H^\infty(w)$ being coercive.

Some questions for further research are given in § 7, along with a discussion of the possibility of extending the consideration of $H^p(w)$ to the context of abstract Hardy classes on compact abelian groups.

The present work was motivated by certain interpolation questions for analytic functions, for instance the existence of interpolating sequences. The latter led to the study of summability properties of $H^1(w)$. In particular, in [7] it is shown that if $H^1(w)$ is contained in the convergence domain of a positive regular matrix, then $\sum(1-|z_n|) = \infty$. The latter result would be immediate if $H^1(w)$ is conull whenever $\sum(1-|z_n|) < \infty$. However, $H^1(w)$ may fail to be conull.

Some of the results in this work were presented in a seminar at Lehigh University during the academic year 1968–69. The author acknowledges, with thanks, helpful conversations with G. Bennett, W. Ruckle, and A. Wilansky during the seminar and during the preparation of the manuscript.

§ 2. PRELIMINARIES

Let D denote the open unit disk in the plane. For $1 \leqslant p \leqslant \infty$ let L^p denote the usual Lebesgue class of complex-valued functions on the interval $[-\pi, \pi]$. The Banach space L^p may be identified in the usual way with a class of functions defined on the unit circle $\{z: |z| = 1\}$.

The Cauchy kernel is the family of functions $\{C_r\}$, $0 \leqslant r < 1$, defined by

$$C_r(t) = \frac{1}{1-re^{it}}, \qquad -\pi \leqslant t \leqslant \pi.$$

Let $z = re^{i\theta} \in D$. Let

$$C_z(t) = \overline{C_r(\theta - t)}, \qquad -\pi \leqslant t \leqslant \pi.$$

Hence, if $f \in L^1$ and the Fourier coefficients of f vanish on the negative integers, then the formula

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{C_z(t)}\,dt$$

yields the analytic function on D with the assigned boundary values f.

For $1 \leqslant p < \infty$, the Hardy class H^p consists of those analytic functions f on D for which

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^p\,d\theta$$

is bounded as $r \to 1$; H^∞ is the class of bounded analytic functions on D. As in [2], p. 39, H^p may be identified, via the Cauchy kernel, with the class of all functions in L^p whose Fourier coefficients vanish on the negative integers. This identification will be used liberally in what follows. Note that H^p is a closed subspace of L^p, hence a Banach space under the L^p norm $\|\cdot\|_p$.

For each complex sequence $x = \{x_k\}$ and each positive integer n let $\pi_n(x) = x_n$. An *FK space* (*BK space*) E is a linear subspace of the space of all complex sequences which is a locally convex Fréchet space (Banach space) such that each functional π_n is continuous. In the FK space context let E' denote the set of all continuous linear functionals on E. In the context of BK space let E^* denote the conjugate space of E. In an appropriate setting, the symbol 1 will denote the constant sequence $\{1, 1, 1, \ldots\}$. Let δ^n denote the sequence $\delta^n_k = 0$ for $k \neq n$, $\delta^n_n = 1$. Let $\psi^n = 1 - \sum_{k=1}^{n} \delta^k$.

Define the following frequently occurring FK spaces with corresponding norms or families of seminorms:

(i) s = the space of all complex sequences; $\{|\pi_n|\}$;

(ii) $c_0 = \{x \in s: \lim_n x_n = 0\}$;

$c = \{x \in s: \lim_n x_n \text{ exists}\}$, $\|x\| = \sup_n |x_n|$;

$m = \{x \in s: \{x_n\} \text{ is bounded}\}$;

(iii) bv = the space of sequences of bounded variation $= \{x \in s: \sum_{k=1}^{\infty} |x_{k+1} - x_k| < \infty\}$;

$$\|x\| = |x_1| + \sum_{k=1}^{\infty} |x_{k+1} - x_k|.$$

Let E^∞ denote the linear span of the sequences $\{\delta^n\}$.

Make the blanket assumption, unless indicated otherwise, that every FK space considered contains 1 and E^∞.

Let $A = (a_{nk})$ be an infinite matrix. The *A-transform* Ax of a sequence $x = \{x_n\}$ is the sequence

$$\Big\{\sum_{k=1}^{\infty} a_{nk} x_k\Big\}.$$

For any FK space E let $E_A = \{x \in s: Ax \in E\}$. Then E_A is also an FK space with appropriate seminorms. The matrix A is called *conservative* if $c \subset c_A$; *regular* if in addition $\lim Ax = \lim x$ for all $x \in c$. Let $\lim_A x = \lim Ax$

for all $x \in c_A$. Call a conservative matrix A *conull* if

$$\lim_A 1 = \sum_{k=1}^{\infty} \lim_A \delta^k;$$

otherwise *coregular*. Call A *coercive* if $m \subset c_A$.

See, for example, [9] for a discussion of the elementary properties of FK spaces and matrix transformations.

An FK space E is called *conull* if $\varphi^n \to 0$ weakly in E; otherwise E is called *coregular*. Some properties of conull and coregular spaces are developed in [6] and [10]. A space E is called *conservative* if $c \subset E$; *coercive* if $m \subset E$. It is well-known that a conservative matrix A is conull if and only if the FK space c_A is conull. E is said to have the AD *property* if E^∞ is dense in E. A sequence $\{x^n\}$ in a Fréchet space E is called a *basis* (*unconditional basis*) if for each $x \in E$ there exists a unique sequence $\{t_n\}$ of scalars such that

$$x = \sum_{k=1}^{\infty} t_k x^k,$$

with the series converging (converging unconditionally) in E. In the terminology of [1], a *wedge space* is a BK space in which $\|\delta^n\| \to 0$.

A sequence $w = \{z_n\}$ in D is called an *interpolating sequence* if $H^\infty(w) = m$. See [2] or [5] for a discussion of interpolating sequences. L. Carleson has given the following characterization: $\{z_n\}$ is interpolating if and only if

$$\inf_n \prod_{k \neq n} \left| \frac{z_k - z_n}{1 - \bar{z}_k z_n} \right| > 0.$$

If $|z_n| \to 1$ exponentially, i. e. if there is a constant $d < 1$ such that

$$\frac{1 - |z_{n+1}|}{1 - |z_n|} < d$$

for all n, then $\{z_n\}$ is interpolating. In the case of sequences $\{z_n\}$ satisfying $0 < z_n < z_{n+1} < 1$ for all n, the latter is a characterization of interpolating sequences.

§ 3. THE SEQUENCE SPACES $H^p(w)$

3.1. THEOREM. *Let $w = \{z_n\}$ be a sequence of distinct points in D and let $S = \{f \in H^p: f(z_n) = 0$ for all $n\}$. There exists a norm on $H^p(w)$ with the following properties:*

(i) *$H^p(w)$ is a BK space congruent to the quotient space H^p/S;*

(ii) *if $1 < p < \infty$, then $H^p(w)$ is reflexive;*

(iii) *$H^2(w)$ is a Hilbert space congruent to the orthogonal complement of S in H^2.*

Proof. Let $f \in H^p$ and $z \in D$. Then

$$|f(z)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \overline{C_z(t)}\, dt \right| \leqslant \|f\|_p \|C_z\|_q,$$

where $1/p + 1/q = 1$. But for each z the function $C_z \in L_q$, so "evaluation at z" is a member of $(H^p)^*$. It follows that S is a closed subspace of H^p.

Consider the map $f \to \{f(z_n)\}$ of H^p onto $H^p(w)$. The kernel is precisely S, so the map $f + S \to \{f(z_n)\}$ is an isomorphism of H^p/S onto $H^p(w)$. Let $H^p(w)$ have the norm of H^p/S induced by $f + S \to \{f(z_n)\}$. Then $H^p(w)$ is a Banach space. Since the mappings $f \to f(z_n)$ vanish on S, it follows that $\{x_n\} \to x_n$ is continuous on $H^p(w)$ for each n. Therefore, $H^p(w)$ is a BK space.

If $1 < p < \infty$, then H^p, hence H^p/S, is reflexive. Then property (ii) follows from (i). Property (iii) also follows immediately from (i).

From now on assume that $\sum_{k=1}^{\infty} (1 - |z_k|) < \infty$. Among other things this assumption guarantees that $E^\infty \subset H^p(w)$.

For the purposes of 3.2 below define the following Blaschke product:

$$B(z) = \prod_{k=1}^{\infty} \frac{\bar{z}_k}{|z_k|} \frac{z_k - z}{1 - \bar{z}_k z}.$$

Further, let B_n denote the product obtained by deleting the factor corresponding to $k = n$ in B. Let P_n be the product of the first n factors, and let $P^{(n)}$ be the product of the factors for $k > n$.

3.2. THEOREM. (i) *if $p < \infty$, then $H^p(w)$ has the* AD *property;*

(ii) *if $1 < p < \infty$, then $\{\pi_n\}$ is fundamental in $H^p(w)^*$;*

(iii) *if $p = 2$, then*

$$\|\delta^n\| = \frac{(1 - |z_n|^2)^{1/2}}{|B_n(z_n)|} \quad \text{and} \quad \|\pi_n\| = \frac{1}{(1 - |z_n|^2)^{1/2}};$$

(iv) *if $1 \leqslant p < \infty$, then*

$$\|\delta^n\| \geqslant \frac{1}{\|\pi_n\| \, |B_n(z_n)|}.$$

Proof. It is not hard to see that $P_n \to B$ in H^2 on the circle. (See [2], p. 65.) Hence, $\|P_n(1 - P^{(n)})\|_2 \to 0$, so $P^{(n)} \to 1$ in H^2. Therefore, there exists a subsequence $\{P^{(n_k)}\}$ such that $P^{(n_k)} \to 1$ pointwise almost everywhere on the circle.

Let $f \in H^p$ be arbitrary. Then $fP^{(n_k)} \in H^p$ and $fP^{(n_k)} \to f$ pointwise almost everywhere. By dominated convergence, $fP^{(n_k)} \to f$ in H^p. By

3.1 (i), it follows that $\{(fP^{(n_k)})(z_n)\}\to\{f(z_n)\}$ in $H^2(w)$ as $k\to\infty$. But $\{(fP^{(n_k)})(z_n)\}\in E^\infty$, so E^∞ is dense in $H^p(w)$, i. e. $H^p(w)$ has AD.

Conclusion (ii) follows immediately from 3.1 (ii).

Let $S=\{f\in H^2\colon f(z_n)=0 \text{ for all } n\}$, and let $(f,g)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\overline{g(t)}\,dt$, the usual inner product for H^2. By 3.1 (iii), $H^2(w)$ is congruent to $S^\perp$, the orthogonal complement of S in H^2. The congruence is the map $f\to\{f(z_n)\}$ on $S^\perp$. Note that

$$B_n=|z_n|B+(B_n-|z_n|B)$$

and

$$(|z_n|B,\,B_n-|z_n|B)=|z_n|(B,B_n)-|z_n|^2(B,B)$$

$$=|z_n|\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\bar z_n}{|z_n|}\,\frac{z_n-e^{it}}{1-\bar z_ne^{it}}\,dt-|z_n|^2=|z_n|^2-|z_n|^2=0$$

But $BH^2=S$, so $B_n-|z_n|B\in S^\perp$. Now $B_n/B_n(z_n)$ interpolates the sequence δ^n, so

$$f_n=\frac{B_n-|z_n|B}{B_n(z_n)}\in S^\perp$$

and f_n interpolates δ^n as well. Hence

$$\|\delta^n\|^2=\|f_n\|_2^2=(f_n,f_n)=\frac{(B_n,B_n-|z_n|B)}{|B_n(z_n)|^2}$$

$$=\frac{(B_n,B_n)-|z_n|(B_n,B)}{|B_n(z_n)|^2}=\frac{1-|z_n|^2}{|B_n(z_n)|^2}.$$

Next, for $f\in H^2$ and fixed $z\in D$,

$$f(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\overline{C_z(t)}\,dt$$

and $C_z\in H^2$. Therefore, the map $f\to f(z)$ has norm $\|C_z\|_2$. But

$$C_z(t)=\sum_{k=0}^{\infty}\bar z^ke^{ikt},$$

so

$$\|C_z\|_2^2=\sum_{k=0}^{\infty}|z|^{2k}=\frac{1}{1-|z|^2}.$$

Also, for each n the map $f\to f(z_n)$ vanishes on S, so

$$\|\pi_n\|=\frac{1}{(1-|z_n|^2)^{1/2}},$$

computed in $H^2(w)^*$.

To prove (iv), note that, as above, $\|\pi_n\|=\|C_{z_n}\|_q$ where $1/p+1/q=1$. If $f\in H^p$ and $f(z_n)=1$, then

$$1=|f(z_n)|\leqslant\|f\|_p\|C_{z_n}\|_q=\|f\|_p\|\pi_n\|.$$

Therefore,

$$\left\|\frac{fB_n}{|B_n(z_n)|}\right\|_p=\frac{\|f\|_p}{|B_n(z_n)|}\geqslant\frac{1}{|B_n(z_n)|\,\|\pi_n\|}.$$

But by 3.1 (i),

$$\|\delta^n\|=\inf\{\|g\|_p\colon g\in H^p \text{ and } g \text{ interpolates } \delta^n\}.$$

Hence,

$$\|\delta^n\|\geqslant\frac{1}{|B_n(z_n)|\,\|\pi_n\|},$$

since any $g\in H^p$ which interpolates δ^n may be written in the form

$$\frac{fB_n}{B_n(z_n)}$$

with $f\in H^p$ and $f(z_n)=1$.

3.3. THEOREM. *$H^1(w)^*$ is not separable.*

Proof. Identify L^∞ with $(L^1)^*$ by the formula

$$\varphi(f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)g(t)\,dt$$

for $f\in L^1$, $g\in L^\infty$.

First note that since H^1 is a closed subspace of L^1,

$$(H^1)^*=L^\infty/(H^1)^\perp,$$

where $(H^1)^\perp$ is the annihilator of H^1 in L^∞. Since $H^1(w)$ is identified with H^1/S where $S=\{f\in H^1\colon f(z_n)=0 \text{ for all } n\}$, it follows that $H^1(w)^*$ is the annihilator of S in $(H^1)^*$. Now $S=BH^1=\{Bf\colon f\in H^1\}$, where

$$B(z)=\prod_{k=1}^{\infty}\frac{\bar z_k}{|z_k|}\,\frac{z_k-z}{1-\bar z_kz}.$$

It is easy to see that for $f\in L^\infty$,

$$\int_{-\pi}^{\pi}Bgf\,dt=0$$

for all $g\in H^1$ if and only if $Bf\in H_0^\infty=\{h\colon h\in H^\infty \text{ and } \int_{-\pi}^{\pi}h\,dt=0\}$. But $f=\bar BBf$ almost everywhere on the circle, so $Bf\in H_0^\infty$ if and only if $f\in\bar BH_0^\infty$.

Therefore, the annihilator of S in $(H^1)^*$ is $(\bar{B}H_0^\infty)/(H^1)^\perp$. Also, $(H^1)^\perp = H_0^\infty$, so $H^1(w)^* = (\bar{B}H_0^\infty)/H_0^\infty$.

It suffices to show that $(\bar{B}H^\infty)/H_0^\infty$ is not separable. It is well-known that this is equivalent to showing that the unit ball of $(\bar{B}H^\infty/H_0^\infty)^*$ is not weak*-metrizable. Now $(\bar{B}H^\infty/H_0^\infty)^*$ is the annihilator of H_0^∞ in $(\bar{B}H^\infty)^*$. Also, $(\bar{B}H^\infty)^*$ may be identified with $(H^\infty)^*$ by the formula

$$\tilde{\varphi}(\bar{B}f) = \varphi(f)$$

for all $f \in H^\infty$ and $\varphi \in (H^\infty)^*$. The proof is completed by showing that the unit ball of $(BH_0^\infty)^\perp$ in $(H^\infty)^*$ is not weak*-metrizable.

Let $\{z_{n_k}\}$ be a subsequence of $w = \{z_k\}$ such that $\{z_{n_k}\}$ is an interpolating sequence. Embedding D in the maximal ideal space M of H^∞, it follows that the closure of $\{z_{n_k}\}$ in M is homeomorphic to the Čech compactification of the integers βN. (See [2], p. 205.) Also, if $\varphi \in M$ is a limit point of $\{z_{n_k}\}$ and if $f \in H_0^\infty$, then $\varphi(Bf) = \varphi(B)\varphi(f) = 0$, since $B(z_n) = 0$ for all n. Therefore, the unit ball of $(BH_0^\infty)^\perp$ in the weak*-topology contains a copy of βN, and βN is not metrizable.

§ 4. SOME THEOREMS ON FK SPACES WITH APPLICATIONS TO $H^p(w)$

4.1. Definition. An FK space E is a *C-space* if there exists $\{x^n\} \subset E^\infty$ and $M > 0$ such that $x^n \to 1$ in E and $|x_k^n| < M$ for all n, k. Say that E has *property* G, if for each proper subset T of the positive integers the set $\{\delta^k\colon k \notin T\}$ is fundamental in

$$\bigcap\{\pi_k^\perp\colon k \in T\} = \{x \in E\colon x_k = 0 \text{ for all } k \in T\}.$$

Also, say that E has *property* A, if for each $x \in E$ there exists $\{x^n\} \subset E^\infty$ such that $x^n \to x$ in E and $|x_k^n| \leqslant |x_k|$ for all n, k.

The following theorem is part of [6], Theorem 6.

4.2. Theorem. *Let E be a conservative FK space. Then E is conull if and only if E is a C-space.*

4.3. Theorem. *If an FK space E has property* A, *then it is a C-space and has property* G.

Proof. Let T be a proper subset of the positive integers, and let $x \in E$ with $x_k = 0$ for $k \in T$. Assume that $\{x^n\} \subset E^\infty$ and $x^n \to x$ in E with $|x_k^n| \leqslant |x_k|$ for all n, k. Then each $x^n \in \operatorname{span}\{\delta^k\colon k \notin T\}$. Also, $x^n \to x$ in $\bigcap\{\pi_k^\perp\colon k \in T\}$, since the latter is closed in E, hence has the relative topology of E. Therefore, E has property G.

Taking $x = 1$ and T empty it follows that E is also a C-space.

4.4. Theorem. *If $p < \infty$, then $H^p(w)$ has property* A. *Hence, for $p < \infty$, $H^p(w)$ is a C-space and has property* G.

Proof. Using the notation of the proof of 3.2, the Blaschke products $P^{(n_k)} \to 1$ in H^p as $k \to \infty$ and for every $f \in H^p$, $fP^{(n_k)} \to f$ in H^p. Going to the quotient $H^p(w)$, $\{(fP^{(n_k)})(z_n)\} \to \{f(z_n)\}$ in $H^p(w)$ as $k \to \infty$. For each k, $\{(fP^{(n_k)})(z_n)\} \in E^\infty$ and

$$|(fP^{(n_k)})(z_n)| \leqslant |f(z_n)|.$$

Therefore, $H^p(w)$ has property A. The second conclusions follow from 4.3.

4.5. Example. Not every BK space with the AD property is a C-space or has property G.

Let $A = (a_{nk})$ be the infinite matrix defined as follows:

$$a_{n,2k-1} = \begin{cases} 0 & \text{for } n < 2k-1, \\ 2^{-k} & \text{for } n \geqslant 2k-1; \end{cases}$$

$$a_{n,2k} = \begin{cases} 1 & \text{for } n = 2k \text{ or } 2k+1, \\ 0 & \text{for all other } n. \end{cases}$$

It is shown in [8] that the convergence domain c_A of A has the AD property.

Let T be the odd integers. Then

$$\bigcap\{\pi_k^\perp\colon k \in T\} = \{x\colon x_{2k-1} = 0 \text{ for all } k \text{ and } \{x_{2k}\} \in c\}.$$

It is clear that $\{\delta^k\colon k \notin T\}$ is not fundamental in the latter.

Note that c_A is coregular, so by 4.2 c_A is not a C-space either.

4.6. Theorem. *Consider the following conditions on an FK space E:*

(i) *E is conull;*

(ii) *$\{\psi^n\}$ is bounded in E (in the sense of locally convex space);*

(iii) *$bv \subset E$.*

Then (i) *implies* (ii), *and* (ii) *is equivalent to* (iii). *If E is a BK space and $\{\pi_n\}$ is fundamental in E^*, then* (ii) *implies* (i). *Hence, for $H^p(w)$, $1 < p < \infty$, conditions* (i), (ii), *and* (iii) *are equivalent. For $p = 1$, $c \subset H^1(w)$ implies condition* (i).

Proof. If E is conull, then $\psi^n \to 0$ weakly, so $\{\psi^n\}$ is bounded accordingly to the uniform boundedness principle.

Assume $\{\psi^n\}$ is bounded and $x \in bv$. Then

$$x = x_1 \cdot 1 + \sum_{n=1}^{\infty} (x_{n+1} - x_n)\psi^n.$$

But the latter series converges in E, since $\{\psi^n\}$ is bounded and $\sum_{n=1}^{\infty} |x_{n+1} - x_n| < \infty$. Hence, $x \in E$, so condition (iii) holds.

Assume bv $\subset E$. Now for each n the norm of ψ^n is 1 computed in bv. Hence, $\{\psi^n\}$ is bounded in E.

Assume $\{\pi_n\}$ is fundamental in E^* and condition (ii) holds. For each m, $\pi_m(\psi^n) \to 0$ as $n \to \infty$. Going to the second conjugate space E^{**}, $\hat{\psi}^n(\varphi) \to 0$ as $n \to \infty$ for all $\varphi \in E^*$. But the latter is just the condition $\psi^n \to 0$ weakly in E, i. e. E is conull.

By 3.2 if $1 < p < \infty$, then $\{\pi_n\}$ is fundamental in $H^p(w)^*$, so the three conditions are equivalent.

Finally, by 4.4 $H^1(w)$ is a C-space. By 4.2 it follows that $H^1(w)$ is conull if it is conservative.

4.7. THEOREM. *Let E be a BK space and $E_n = \pi_n^\perp$ for all n. Let $\|\cdot\|_n$ denote the norm of E_n^*.*

(i) *If $\{\delta^n\}$ is bounded in E, in particular if E is conull, then* $\inf\{\|\pi_m\|_n : n \neq m\} > 0$;

(ii) *If E coercive and has property* G, *then $\|\pi_m\|_n \to \infty$ as $n, m \to \infty$ with $n \neq m$.*

Proof. (i) if $n \neq m$, then

$$1 = \pi_m(\delta^m) \leqslant \|\pi_m\|_n \|\delta^m\|,$$

since $\delta^m \in E_n$. Thus, $\|\pi_m\|_n$ is bounded away from zero, if $\|\delta^m\|$ is bounded. If E is conull, then by 4.6 $\|\psi^m\|$ is bounded. But for $m > 1$, $\delta^m = \psi^{m-1} - \psi^m$, so $\{\delta^m\}$ is also bounded.

(ii) Suppose $\|\pi_{m_k}\|_{n_k} < M$ for all k, where all of the integers m_k and n_j are distinct. Let $F = \bigcap_{k=1}^{\infty} E_{n_k}$. Then $\{\pi_{m_k}\}$ is a bounded sequence in F^*. Also, $\{\delta^j : j \neq n_k$ for $k = 1, 2, 3, \ldots\}$ is fundamental in F. For each such δ^j, $\pi_{m_k}(\delta^j) \to 0$ as $k \to \infty$. Therefore, $x_{m_k} = \pi_{m_k}(x) \to 0$ for all $x \in F$, contradicting the fact that E is coercive.

Of course, the condition $\inf\{\|\pi_m\|_n : n \neq m\} > 0$ holds for many coregular spaces. For instance, $E = c$ is an example. However, 4.7 is a convenient device to show that certain spaces fail to have the conull or coercive properties.

4.8. THEOREM. *There exists $w = \{z_n\}$ such that $H^1(w)$ is coregular.*

Proof. Suppose $0 < r_0 < s < 1$. It is easy to see that $\|C_{r_0} - C_s\|_\infty \to 0$ as $s \to r_0$, where $\{C_r\}$ is the Cauchy kernel. Let $w = \{z_k\}$ satisfy the following conditions:

(i) $0 < z_k < z_{k+1} < 1$ for all k;

(ii) $\|C_{z_{2k}} - C_{z_{2k-1}}\|_\infty \to 0$ as $k \to \infty$;

(iii) $\sum_{k=1}^{\infty} (1 - z_k) < \infty$.

Then for any $f \in H^1$,

$$|f(z_{2k}) - f(z_{2k-1})| \leqslant \|f\|_1 \|C_{z_{2k}} - C_{z_{2k-1}}\|_\infty.$$

Therefore, for each $x \in H^1(w)$,

$$|x_{2k} - x_{2k-1}| \leqslant \|x\| \|C_{z_{2k}} - C_{z_{2k-1}}\|_\infty \to 0$$

as $k \to \infty$. In the notation of 4.7, it follows that $\|\pi_{2k}\|_{2k-1} \to 0$ as $k \to \infty$, so by 4.7 (i), $H^1(w)$ is coregular, in fact, $\|\delta^k\|$ is unbounded.

4.9. THEOREM. *Let E be a BK space which is separable, coercive, and satisfies $x = \sum_{k=1}^{\infty} x_k \delta^k$ weakly in E for all $x \in m$. Then E is a wedge space.*

Proof. Let $\{\varphi_n\}$ be a sequence in the unit ball of E^*. First it will be shown that $\varphi_n(\delta^n) \to 0$.

Since E is separable, the unit ball of E^* is weak*-metrizable. Let $\{\varphi_{p_n}\}$ be a weak*-convergent subsequence of $\{\varphi_n\}$. Define an infinite matrix $A = (a_{nk})$ by $a_{nk} = \varphi_{p_n}(\delta^{p_k})$. Let $x \in m$ with $x_k = 0$ for $k \neq p_n$, $n = 1, 2, 3, \ldots$ Then

$$\varphi_{p_n}(x) = \sum_{k=1}^{\infty} x_k \varphi_{p_n}(\delta^k) = \sum_{k=1}^{\infty} x_{p_k} \varphi_{p_n}(\delta^{p_k}) = \sum_{k=1}^{\infty} a_{nk} x_{p_k}.$$

It follows that A sums every bounded sequence, so certainly $a_{nn} \to 0$. But $a_{nn} = \varphi_{p_n}(\delta^{p_n})$. It follows easily that $\varphi_n(\delta^n) \to 0$.

By the Hahn–Banach theorem, for each n choose $\varphi_n \in E^*$ such that $\|\varphi_n\| \leqslant 1$ and $\varphi_n(\delta^n) = \|\delta^n\|$. By the previous paragraph, $\|\delta^n\| \to 0$ as $n \to \infty$.

4.10. THEOREM. *Let E be a weakly sequentially complete FK space which is conservative. Then E is coercive and $x = \sum_{k=1}^{\infty} x_k \delta^k$ weakly in E for all $x \in m$.*

Proof. Let $x \in m$ and $\varphi \in E'$. Since E is conservative, $\sum_{k=1}^{\infty} |\varphi(\delta^k)| < \infty$. Thus, the series $\sum_{k=1}^{\infty} x_k \varphi(\delta^k)$ converges, so $\sum_{k=1}^{\infty} x_k \delta^k$ is weakly Cauchy in E. It follows that $x \in E$ and $\sum_{k=1}^{\infty} x_k \delta^k = x$ weakly.

4.11. COROLLARY. *For $1 < p < \infty$, if $H^p(w)$ is conservative, then $H^p(w)$ is coercive and a wedge space.*

Proof. According to 3.1 $H^p(w)$ is reflexive, hence, weakly sequentially complete. By 3.2 $H^p(w)$ is separable. The result follows from 4.9 and 4.10.

4.12. Remark. According to 4.4 and 4.8, there exists a sequence w such that $H^1(w)$ is a coregular C-space. This shows that 4.2 fails in the context of non-conservative spaces.

J. Sember has announced some related results on the conull property in non-conservative spaces. He has provided a coregular C-space which contains bv. Of course, such an example could not be given using $H^p(w)$. In addition, Sember has observed that every conull space is a C-space.

§ 5. BASIS PROPERTIES OF $H^p(w)$

5.1. Theorem. *The following conditions are equivalent for $p<\infty$.*

(i) *$\{\delta^n\}$ is an unconditional basis for $H^p(w)$;*

(ii) *$\{\delta^n\}$ is a basis for $H^p(w)$;*

(iii) *$w=\{z_n\}$ is an interpolating sequence;*

(iv) *$\{(1-|z_k|)^{1/p}x_k\}: x\in H^p(w)\} = l^p$;*

(v) *$l^1\subset H^\infty(w)$.*

Proof. Assume $\{\delta^n\}$ is a basis. There exists a constant $M>0$ such that $\|\delta^n\|\leqslant M/\|\pi_n\|$ for each n. (See [10], p. 213, Problem 12.) According to 3.2 (iv),

$$\|\delta^n\|\geqslant\frac{1}{|B_n(z_n)|\,\|\pi_n\|},$$

where

$$B_n(z)=\prod_{k\neq n}\frac{\bar z_k}{|z_k|}\,\frac{z_k-z}{1-\bar z_k z}.$$

Thus,

$$|B_n(z_n)|\geqslant\frac{1}{M}$$

for all n, so by Carleson's theorem, condition (iii) holds.

The equivalence of (iii) and (iv) is known. See [6], Theorem 2.

Assume condition (v) holds. Since $\{\delta^n\}$ is bounded in l^1, the sequences δ^n may be interpolated by functions $f_n\in H^\infty$ such that $\|f_n\|_\infty<M$ for all n. Then for each n, $f_n=B_ng_n$, say, where $g_n\in H^\infty$ and B_n is the usual Blaschke product. Thus,

$$1=|f_n(z_n)|=|B_n(z_n)|\,|g_n(z_n)|\leqslant|B_n(z_n)|\,\|g_n\|_\infty<M|B_n(z_n)|.$$

It follows, as before, that $\{z_n\}$ is interpolating. Let $x\in m$ be arbitrary. Choose $f\in H^\infty$ such that f interpolates x. For any $g\in H^p$, it follows that $fg\in H^p$ and $(fg)(z_n)=f(z_n)g(z_n)$. Therefore, $\{x_ny_n\}\in H^p(w)$ for every $y=\{y_n\}\in H^p(w)$. It is known that the latter implies that $\{\delta^n\}$ is an unconditional basis for $H^p(w)$. (See [11] or [3].)

5.2. Corollary. (i) *If $\{\delta^n\}$ is a basis for $H^p(w)$, then $H^p(w)$ is coercive.*

(ii) *If $H^p(w)$ is coercive, then it is conull (for $p<\infty$).*

Proof. See 5.1 and 4.6.

In § 6 it will be shown that the converse of 5.2 (i) is false. The converse of 5.2 (ii) can also be shown to be false.

5.3. Remark. Note that $\{\delta^n\}$ need not be an orthogonal sequence in the Hilbert space $H^2(w)$, even if $\{\delta^n\}$ is a basis. For instance, suppose $0<z_n<1$ for all n. For $n\neq m$, as in the proof of 3.2,

$$B_n(z_n)B_m(z_m)(\delta^n,\delta^m)$$

$$=(B_n-z_nB,\ B_m-z_mB)=(B_n,B_m)-z_n(B,B_m)$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{z_n-z}{1-z_nz}\,\frac{z_m-\bar z}{1-z_m\bar z}\,dt-z_n\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{z_m-z}{1-z_mz}\,dt$$

(where $z=e^{it}$)

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{z_nz_m-z_n\bar z-z_mz+1}{1-z_m\bar z-z_nz+z_nz_m}\,dt-z_nz_m$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{z_mz^2-(1+z_nz_m)z+z_n}{z_nz^2-(1+z_nz_m)z+z_m}\,dt-z_nz_m$$

$$=\frac{1}{2\pi i}\int_C\frac{z_mz^2-(1+z_nz_m)z+z_n}{z_nz^3-(1+z_nz_m)z^2+z_mz}\,dz-z_nz_m$$

(where C is the unit circle)

$$=\frac{z_n}{z_m}+\frac{z_m^3-(1+z_nz_m)z_m+z_n}{z_m(z_nz_m-1)}-z_nz_m$$

(using the residue theorem)

$$=\frac{(1-z_m^2)(1-z_n^2)}{1-z_nz_m}\neq 0.$$

§ 6. AN INTERPOLATION PROBLEM

It is shown that there exists a sequence $w=\{z_n\}$ such that $H^2(w)$ is coercive but $H^\infty(w)$ is not. The method consists of choosing a sequence w which is not interpolating and yet $\sum\|\delta^n\|<\infty$ in $H^2(w)$.

In this section let B_n denote the product

$$B_n(z)=\prod_{k\neq n}\frac{\bar z_k}{z_k}\,\frac{z_k-z}{1-\bar z_kz}$$

for each sequence $\{z_n\}$ under consideration.

6.1. LEMMA. *For* $0 < x < y < 1$, $\frac{y-x}{1-xy}$ *defines a decreasing function of* x *and an increasing function of* y.

Choose a sequence $\{\gamma_j\}$ satisfying the following conditions:

(i) $0 < \gamma_j < \gamma_{j+1} < 1$ for all j;

(ii) $\{\gamma_j\}$ is an interpolating sequence.

Since then $\gamma_j \to 1$ exponentially, it follows from the ratio test that

(iii) $\sum_{j=1}^{\infty} (1-\gamma_j^2)^{1/2} < \infty$.

Let $\alpha_j = \gamma_{2j-1}$ for all j.

6.2. LEMMA. *There exists a sequence* $\{\beta_j\}$ *such that*

(i) $\alpha_j < \beta_j < \alpha_{j+1}$ *for all* j;

(ii) $P(\beta_j) \to 0$ *monotonely*;

(iii) $\sum_{j=1}^{\infty} \frac{(1-\alpha_j^2)^{1/2}}{P(\beta_j)^2} < \infty$

where P *is the Blaschke product* $P(z) = \prod_{j=1}^{\infty} \left(\frac{\alpha_j - z}{1-\alpha_j z}\right)^2$.

Proof. Since $\{\gamma_j\}$ is interpolating, it follows easily that $P(\gamma_{2j})$ is bounded away from zero. Now the function P is continuous on the interval, $[0, 1)$ and vanishes on $\alpha_j = \gamma_{2j-1}$. By the intermediate value theorem for continuous functions it can be arranged that $\alpha_j < \beta_j < \alpha_{j+1}$ for all j and $P(\beta_j) \to 0$ arbitrarily slowly. In particular, since

$$\sum_{j=1}^{\infty} (1-\alpha_j^2)^{1/2} < \infty$$

it can be arranged that (iii) holds.

Let $\{z_j\}$ be the sequence $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots\}$, and let $\{B_k\}$ be the usual Blaschke products associated with $\{z_j\}$.

6.3. LEMMA. $P(\beta_n)^2 \leqslant |B_{2n}(z_{2n})|$ *and* $P(\beta_n)^2 \leqslant |B_{2n-1}(z_{2n-1})|$ *for all* n.

Proof.

$$|B_{2n}(z_{2n})| = |B_{2n}(\beta_n)| = \prod_{j=1}^{2n-1} \frac{\beta_n - z_j}{1-\beta_n z_j} \cdot \prod_{j=2n+1}^{\infty} \frac{z_j - \beta_n}{1-z_j\beta_n}$$

$$\geqslant \prod_{j=1}^{n} \left(\frac{\beta_n - \alpha_j}{1-\beta_n\alpha_j}\right)^2 \cdot \prod_{j=n+1}^{\infty} \left(\frac{\alpha_j - \beta_n}{1-\alpha_j\beta_n}\right)^2 = P(\beta_n) \geqslant P(\beta_n)^2,$$

using 6.1.

Similarly,

$$|B_{2n-1}(z_{2n-1})| = |B_{2n-1}(\alpha_n)| = \prod_{j=1}^{2n-2} \frac{\alpha_n - z_j}{1-\alpha_n z_j} \prod_{j=2n}^{\infty} \frac{z_j - \alpha_n}{1-z_j\alpha_n}$$

$$\geqslant \frac{\alpha_n - \beta_{n-1}}{1-\alpha_n\beta_{n-1}} \cdot \prod_{j=1}^{2n-3} \frac{\beta_{n-1} - z_j}{1-\beta_{n-1}z_j} \cdot \frac{\beta_n - \alpha_n}{1-\beta_n\alpha_n} \cdot \prod_{j=n+1}^{\infty} \left(\frac{\alpha_j - \beta_n}{1-\alpha_j\beta_n}\right)^2$$

$$\geqslant \frac{\alpha_n - \beta_{n-1}}{1-\alpha_n\beta_{n-1}} \cdot \frac{\beta_n - \alpha_n}{1-\beta_n\alpha_n} \cdot \prod_{j=1}^{n-1} \left(\frac{\beta_{n-1} - \alpha_j}{1-\beta_{n-1}\alpha_j}\right)^2 \cdot \prod_{j=n+1}^{\infty} \left(\frac{\alpha_j - \beta_n}{1-\alpha_j\beta_n}\right)^2$$

$$\geqslant \prod_{j=1}^{n} \left(\frac{\alpha_j - \beta_{n-1}}{1-\alpha_j\beta_{n-1}}\right)^2 \prod_{j=n}^{\infty} \left(\frac{\alpha_j - \beta_n}{1-\alpha_j\beta_n}\right)^2 \geqslant P(\beta_{n-1})P(\beta_n) \geqslant P(\beta_n)^2,$$

using 6.1 and 6.2 (ii).

6.4. THEOREM. *Let* $w = \{z_k\}$ *be the sequence defined above. Then* $H^2(w)$ *is coercive and* w *is not an interpolating sequence.*

Proof.

$$|B_{2n}(z_{2n})| = \prod_{j=1}^{\infty} \left|\frac{\alpha_j - \beta_n}{1-\alpha_j\beta_n}\right| \prod_{j \neq n} \left|\frac{\beta_j - \beta_n}{1-\beta_j\beta_n}\right|$$

$$\leqslant \prod_{j=1}^{\infty} \left|\frac{\alpha_j - \beta_n}{1-\alpha_j\beta_n}\right| = (P(\beta_n))^{1/2} \to 0 \text{ as } n \to \infty,$$

so $\{z_k\}$ is not interpolating.

Next, in $H^2(w)$

$$\|\delta^{2n}\| = \frac{(1-z_{2n}^2)^{1/2}}{|B_{2n}(z_{2n})|} = \frac{(1-\beta_n^2)^{1/2}}{|B_{2n}(z_{2n})|} \leqslant \frac{(1-\alpha_n^2)^{1/2}}{|B_{2n}(z_{2n})|} \leqslant \frac{(1-\alpha_n^2)^{1/2}}{P(\beta_n)^2},$$

according to 6.3. Similarly,

$$\|\delta^{2n-1}\| \leqslant \frac{(1-\alpha_n^2)^{1/2}}{P(\beta_n)^2}.$$

By 6.2 (iii),

$$\sum_{n=1}^{\infty} \|\delta^n\| < \infty.$$

It follows easily that $H^2(w)$ is coercive.

Using similar techniques it can be shown that there exists $w = \{z_n\}$ such that $H^2(w)$ is conull but not coercive. The method consists of choosing $\{z_n\}$ so that $\|\delta^n\| \nrightarrow 0$ and yet $\|\psi^n\|$ is bounded. The result follows from 4.6 and 4.11.

§ 7. CONCLUDING REMARKS AND QUESTIONS

7.1. The sequence space $H^p(w)$ may be approached from a slightly different point of view. Consider L^1 as a Banach algebra under convolution

$$(f*g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(x-t)\,dt.$$

Let $\{P_r\}$ be the Poisson kernel

$$P_r(t) = \mathrm{Re}\,\frac{1+re^{it}}{1-re^{it}}.$$

Then for $0 \leqslant r_n < 1$, $r_n \to 1$, the sequence of functions $\{P_{r_n}\}$ is an approximate identity for L^1, i. e. $\|P_{r_n}\|_1 \leqslant 1$ for all n and $\|P_{r_n}*f-f\|_1 \to 0$ for all $f \in L^1$.

Now let $w = \{z_n\}$ be a sequence of distinct points in D with $|z_n| \to 1$. Let $z_n = r_n e^{i\theta_n}$ for each n. Then

$$H^1(w) = \{\{(f*P_{r_n})(\theta_n)\}:\ f \in H^1\},$$

where here H^1 is being considered as a family of functions on $[-\pi, \pi]$ or on the circle $\{z: |z| = 1\}$. This suggests the following general setting: Let G be a compact abelian group, let $g = \{g_n\}$ be a sequence of points in G, and let $Q = \{Q_n\} \subset L^\infty(G)$ be an approximate identity for the Banach algebra $L^1(G)$. Let S be some closed subspace of $L^p(G)$, for instance, the abstract Hardy class H^p generated by an appropriate Dirichlet algebra on G. (See [2], p. 54.) Finally let

$$S(g, Q) = \{\{(f*Q_n)(g_n)\}:\ f \in S\}.$$

What properties of $H^p(w)$ extend to the setting of $S(g, Q)$?

One result in this direction is the following generalization of Theorem 3.2 of [7]: *If $L^\infty(g, Q)$ is contained in the convergence domain of a regular matrix, then the set of limit points of the sequence $g = \{g_n\}$ has positive Haar measure in G.* The proof will be given elsewhere.

7.2. The space $H^p(w)$ may be generalized in a different direction by replacing the unit disk D by some other subset of the plane. Let X be a compact subset of the plane with connected complement, and let S be an appropriate family of analytic functions on the interior of X. Finally, for a sequence $w = \{z_n\}$ in the interior of X, let

$$S(w) = \{\{f(z_n)\}:\ f \in S\}.$$

7.3. According to 4.3 a space with property A must have property G and be a C-space. However, a space with the AD property may fail to have property G or be a C-space.

It is not hard to find a C-space with the AD property but without property G. Define a matrix $A = (a_{nk})$ by

$$a_{n,2k-1} = \begin{cases} 2^{-k} & \text{for} \quad 2k-1 \leqslant n, \\ 0 & \text{for} \quad 2k-1 > n, \end{cases}$$

$$a_{2n,2n} = a_{2n+2,2n} = (-1)^n \quad \text{for all } n,$$

$$a_{nk} = 0 \quad \text{otherwise}.$$

The BK space c_A has the required properties.

Is there a space with property G which is coregular (or which is not a C-space)? For which matrices A does c_A have property G?

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