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Polynomials and multilinear mappings in topological vector spaces

by

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Introduction. In our forthcoming papers we want to give a unified presentation of a theory of analytic functions defined in open subsets of a complex or real topological vector space E with values in a locally convex topological vector space F. As a special case we will get the theory developed in [1] and [8] for Banach spaces.

In this paper we gather the most important facts about polynomials and multilinear mappings which are basic for the further developement of the theory. Polynomials are the simplest analytic functions which are used to bild up locally any other analytic mapping (by expanding it into a series of homogeneous polynomials). Therefore their properties should be examined first.

We hope that some of the results of this paper may be interesting for functional analysis apart from their application in the theory of analytic functions.

In Section 1 we give several necessary and sufficient conditions for a polynomial or multilinear mapping to be continuous (Theorem 1). Also here the Banach-Steinhaus theorem is extended to homogeneous polynomials of fixed degree $k \geqslant 1$ (Theorem 2).

The natural domains of existence of analytic mappings are domains in complex spaces. It appears that in order to prove some facts concerning real analytic functions it is convenient to complexify the given function at first. The problem of complexification of real vector spaces, multilinear mappings and polynomials is treated in Section 2.

Section 3 is devoted to existence of Gateaux differentials and their continuity. This is of first importance when we ask whether a given function may be locally developed into a series of homogeneous polynomials.

While preparing this paper we have been much inspired by [1], [8] and [9].

1. Polynomials and multilinear mappings. In the sequel K denotes either the field of complex numbers $\boldsymbol{C}$ or the field of real numbers $\boldsymbol{R}$. Letters E, F, G will denote vector spaces over K. If the field is not strictly indicated the results are valid in both cases, $K = \mathbf{R}$ and $K = \mathbf{C}$. We assume that all the topological spaces considered in this paper are Hausdorff spaces.

DEFINITION 1. A mapping $f\colon E \to F$ is called a *homogeneous polynomial of degree* k if and only if there exists a k-linear symmetrical mapping $\bar{f}\colon E^k \to F$ such that $f(x) = \bar{f}(x, \ldots, x)$, $x \in E$. We say that $\bar{f}$ is *associated with* f *or that* $\bar{f}$ *generates* f.

Let us denote by $\mathrm{Hom}(E_1, \ldots, E_k; F)$, $\mathrm{Hom}^k(E, F)$, $\mathrm{Hom}^k_s(E, F)$ and $Q^k(E, F)$ the vector spaces of k-linear mappings of $E_1 \times \ldots \times E_k$ into F, k-linear mappings of E^k into F, k-linear symmetric mappings of E^k into F and homogeneous polynomials of E into F of degree k, respectively. The corresponding spaces of continuous mappings will be denoted by $L(E_1, \ldots, E_k; F)$, $L^k(E, F)$, $L^k_s(E, F)$ and $P^k(E, F)$, respectively.

If k is a fixed positive integer, then $\pi_k\colon E \to E^k$ will denote the mapping given by $\pi_k(x) = (x, \ldots, x)$.

A mapping $f\colon E \to F$ is a homogeneous polynomial of degree k if and only if there exists a k-linear (not necessarily symmetric) mapping $g\colon E^k \to F$ such that $f = g \circ \pi_k$. Indeed, the function

$$g_*(x_1, \ldots, x_k) = \frac{1}{k!} \sum g(x_{j_1}, \ldots, x_{j_k}), \qquad (x_1, \ldots, x_k) \in E^k,$$

where the sum is spread over all the permutations $(j_1, \ldots, j_k)$ of the sequence $(1, \ldots, k)$, is k-linear, symmetric and, moreover, $f = g \circ \pi_k = g_* \circ \pi_k$, whence it follows that $f \in Q^k(E, F)$. The opposite implication is obvious.

The following known theorem (Mazur-Orlicz [9]) plays a fundamental role in our study.

THEOREM A. *If* $f \in Q^k(E, F)$ *and* $f = \bar{f} \circ \pi_k$, $\bar{f} \in \mathrm{Hom}^k_s(E, F)$, *then*

$$(1) \quad \bar{f}(x_1, \ldots, x_k) = \frac{1}{k!} \sum_{\varepsilon_1, \ldots, \varepsilon_k = 0}^{1} (-1)^{k-(\varepsilon_1 + \ldots + \varepsilon_k)} f(x_0 + \varepsilon_1 x_1 + \ldots + \varepsilon_k x_k),$$

where x_0 *is an arbitrary fixed point of* E.

Proof. Let us denote the right-hand side of (1) by $g(x_1, \ldots, x_k)$. We have to show that $g = \bar{f}$. To this end observe that by k-linearity and symmetry of $\bar{f}$ we have

$$\begin{aligned} f(x_0 + \varepsilon_1 x_1 + \ldots + \varepsilon_k x_k) &= \bar{f}(x_0 + \varepsilon_1 x_1 + \ldots + \varepsilon_k x_k, \ldots, x_0 + \varepsilon_1 x_1 + \ldots + \varepsilon_k x_k) \\ &= \sum_{j_0 + \ldots + j_k = k} \frac{k!}{j_0! \ldots j_k!} \varepsilon_0^{j_0} \ldots \varepsilon_k^{j_k} \bar{f}(\underbrace{x_0, \ldots, x_0}_{j_0}, \ldots, \underbrace{x_k, \ldots, x_k}_{j_k}), \end{aligned}$$

where $\varepsilon_0 = 1$ and $\varepsilon_s^{j_s} = 1$ if $j_s = 0$. Therefore

$$g(x_1, \ldots, x_k) = \sum_{j_0 + \ldots + j_k = k} \frac{A_{j_0 \ldots j_k}}{j_0! \ldots j_k!} \bar{f}(\underbrace{x_0, \ldots, x_0}_{j_0}, \ldots, \underbrace{x_k, \ldots, x_k}_{j_k}),$$

where

$$A_{j_0 \ldots j_k} = \sum_{\varepsilon_1, \ldots, \varepsilon_k = 0}^{1} (-1)^{k-(\varepsilon_1 + \ldots + \varepsilon_k)} \varepsilon_0^{j_0} \ldots \varepsilon_k^{j_k}.$$

Suppose $j_s = 0$ for some $s \geqslant 1$. Without loss of generality we may take $s = 1$. Then

$$A_{j_0 1 j_2 \ldots j_k} = \sum_{\varepsilon_2, \ldots, \varepsilon_k = 0}^{1} \left((-1)^{k-(\varepsilon_2 + \ldots + \varepsilon_k)} + (-1)^{k-(\varepsilon_2 + \ldots + \varepsilon_k) - 1}\right) \varepsilon_2^{j_2} \ldots \varepsilon_k^{j_k} = 0.$$

Therefore $A_{j_0 \ldots j_k}$ may be different from zero only if $j_1 \ldots j_k \neq 0$. But then $j_1 = \ldots = j_k = 1$, $j_0 = 0$, because $j_0 + \ldots j_k = k$. Thus $g = \bar{f}$. Q.E.D.

It is obvious that the mapping given by (1) is an isomorphism between $Q^k(E, F)$ and $\mathrm{Hom}^k_s(E, F)$. Moreover, $\bar{f}$ is continuous if and only if f is continuous.

If E and F are Banach spaces, then $P^k(E, F)$ and $L^k(E, F)$ are also Banach spaces, if the corresponding norms are defined by

$$\|f\| = \sup\{\|f(x)\|\colon \|x\| \leqslant 1\}, \quad f \in P^k(E, F),$$

$$\|f\| = \sup\{\|f(x_1, \ldots, x_k)\|\colon \|x_j\| \leqslant 1 \ (j = 1, \ldots, k)\}, \quad f \in L^k(E, F).$$

PROPOSITION 1. 1° *If E and F are Banach spaces over K, then mapping* (1) *is a topological isomorphism between the Banach spaces* $P^k(E, F)$ *and* $L^k_s(E, F)$.

2° *If E and F are Banach spaces over* $\mathbf{R}$ *and the norm in E is defined by a scalar product, then* (1) *is an isometry between* P^k *and* L^k_s.

Proof. It is obvious that $\|f\| \leqslant \|\bar{f}\|$.

1° It follows from (1) that

$$\begin{aligned} \|\bar{f}(x_1, \ldots, x_k)\| &\leqslant \frac{\|f\|}{k!} \sum_{\varepsilon_1, \ldots, \varepsilon_k = 0}^{1} \|\varepsilon_1 x_1 + \ldots + \varepsilon_k x_k\|^k \\ &\leqslant \frac{1}{k!} \sum_{s=0}^{k} \binom{k}{s} s^k \|f\| \leqslant \frac{(2k)^k}{k!} \|f\|, \end{aligned}$$

where $\|x_j\| \leqslant 1$ $(j = 1, \ldots, k)$. So $\|\bar{f}\| \leqslant \frac{(2k)^k}{k!} \|f\|$ and this implies the first part of the proposition.

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2° First assume that $\dim E < \infty$. Let $S = \{x \in E: \|x\| \leqslant 1\}$. If $k = 2$ and x, y are points of S such that $\|\bar{f}\| = \|f(x, y)\|$, then $\|f(x \pm y)\| = \|\bar{f}\| \, \|x \pm y\|^2$. Indeed, suppose e.g. $\|f(x-y)\| < \|\bar{f}\| \, \|x-y\|^2$. Since $\bar{f}(x, y) = \frac{1}{4}f(x+y) + \frac{1}{4}f(x-y)$, we have

$$\|\bar{f}\| = \|\bar{f}(x, y)\| < \|\bar{f}\| \tfrac{1}{4}(\|x+y\|^2 + \|x-y\|^2) \leqslant \|\bar{f}\| .$$

This contradiction gives the result.

Given arbitrary $k \geqslant 2$, we can find $\tilde{x}_1, \ldots, \tilde{x}_k \in S$ such that $\|\bar{f}\| = \|\bar{f}(\tilde{x}_1, \ldots, \tilde{x}_k)\|$. Take $a \in S$ such that the scalar product $\langle a, x_j \rangle \neq 0$ $(j = 1, \ldots, k)$. Then there exists $\varepsilon > 0$ such that the set

$$A = \{(x_1, \ldots, x_k) \in S^k: \langle a, x_j \rangle \geqslant \varepsilon \ (j = 1, \ldots, k), \|\bar{f}(x_1, \ldots, x_k)\| = \|\bar{f}\|\}$$

is not empty. There exists a point $(x_1^*, \ldots, x_k^*) \in A$ at which $\sum_{j=1}^{k} \langle a, x_j \rangle$ attains its maximum on A. It is enough to show that $\varepsilon_1 x_1^* = \ldots = \varepsilon_k x_k^*$, where $\varepsilon_j = \pm 1$. Suppose that $x_p^* + x_q^* \neq 0$ and $x_p^* - x_q^* \neq 0$ for a pair of different indices. Then at the point

$$(x_1', \ldots, x_k') \in A, \quad x_j' = x_j^* \quad (j \neq p, q), \quad x_p' = x_q' = (x_p^* + x_q^*)/\|x_p^* + x_q^*\|$$

we would have

$$\sum_{j=1}^{k} \langle a, x_j' \rangle > \sum_{j=1}^{k} \langle a, x_j^* \rangle$$

(because $\|x_p^* + x_q^*\| < 2$), what is impossible.

Let now the dimension of E be infinite. Let $a_1, \ldots, a_k$ be arbitrary fixed points of the unit ball $S \subset E$. Let V be a k-dimensional subspace of E such that $a_1, \ldots, a_k \in V$. We take the scalar product in V obtained by the restriction to V the scalar product in E. Then

$$\|f|V\| = \sup\{\|f(x)\|: x \in S \cap V\} .$$

Since the theorem holds for finitely-dimensional case, we have

$$\|\bar{f}(a_1, \ldots, a_k)\| \leqslant \|f|V\| \leqslant \|f\|.$$

Therefore $\|\bar{f}\| \leqslant \|f\|$. The proof is concluded.

Remark. Proposition 1 (2°) has been proved by Banach [2] for seperable unitary spaces. The method of proof of the finite-dimensional case given here is due to S. Łojasiewicz.

Definition 2. If $f: E \to F$ is a finite sum $f = \sum_{k=0}^{m} f_k$ of homogeneous polynomials $f_k \in Q^k(E, F)$, then f is called a *polynomial* (of degree at most m).

Proposition 2. *Superposition of two polynomials is a polynomial.*

Proof. Let $f: E \to F$, $g: F \to G$ be polynomials. We want to prove that $g \circ f$ is a polynomial. Without loss of generality we may assume that $g \in Q^k(F, G)$ and $f = f_0 + \ldots + f_s$, $f_j \in Q^j(E, F)$. Since

$$g \circ f(x) = \bar{g}(f(x), \ldots, f(x)) = \sum_{j_1, \ldots, j_s = 0}^{s} \bar{g}(\bar{f}_{j_1} \circ \pi_{j_1}(x), \ldots, \bar{f}_{j_k} \circ \pi_{j_k}(x)), \quad x \in E$$

it is enough to show that $\bar{g} \circ (\bar{f}_{j_1} \circ \pi_{j_1}, \ldots, \bar{f}_{j_k} \circ \pi_{j_k})$ is a homogeneous polynomial. This, however, follows immediately from the fact that the mapping

$$E^{j_1} \times \ldots \times E^{j_k} \ni (x_1, \ldots, x_k) \to \bar{g}(\bar{f}_{j_1}(x), \ldots, \bar{f}_{j_k}(x)) \in G$$

is $(j_1 + \ldots + j_k)$-linear.

Corollary 1. *Superposition of continuous polynomials is a continuous polynomial. If $f \in Q^r(E, F)$, $g \in Q^k(F, G)$, then $g \circ f \in Q^{k \cdot r}(E, G)$. If $a \in E$ is a fixed point and f is a polynomial, then the mapping $E \ni x \to f(a+x) \in F$ is also a polynomial.*

Lemma 1. *Let F be an arbitrary vector space over K. If $f: K^m \to F$ is a polynomial with respect to each variable seperately, then f is a polynomial.*

Proof. Suppose the lemma holds for functions of m variables. Given $f: K^{m+1} \ni (u, t) \to f(u, t) \in F$ satisfying assumptions of the lemma, then for any fixed $t \in K$ the function $K^m \ni u \to f(u, t) \in F$ is a polynomial of m variables. Put $X_n = \{t \in K: f(u, t)$ is a polynomial of degree $\leqslant n$ with respect to each variable $u_1, \ldots, u_m\}$, $n = 1, 2, \ldots$ It is obvious that $X_n \subset X_{n+1}$ and $K = \bigcup_{n=1}^{\infty} X_n$. Therefore there exists n such that X_n is infinite. Let

$$U_s = \{u_{s0}, u_{s1}, \ldots, u_{sn}\} \quad (s = 1, \ldots, m)$$

be a system of $n+1$ different points of K. Let $L^{(j_s)}(u_s, U_s)$ $(j_s = 0, \ldots, n)$ denote the fundamental interpolation polynomials of Lagrange corresponding to the system U_s of nodes $u_{s0}, \ldots, u_{sn}$. Then the mapping $g: K^{m+1} \to F$ given by

$$g(u, t) = \sum_{j_1, \ldots, j_m = 0}^{n} L^{(j_1)}(u_1, U_1) \ldots L^{(j_m)}(u_m, U_m) f(u_{1j_1}, \ldots, u_{mj_m}, t)$$

is a polynomial from K^{m+1} to F. Moreover, by the Lagrange interpolation formula, $f(u, t) = g(u, t)$, $u \in K^m$, $t \in X_n$. Since, for every fixed $u \in K^m$, both f and g are polynomials in $t \in K$ which have the same values at every point of the infinite set X_n, so $f(u, t) = g(u, t)$ for every $t \in K$. This concludes the proof.

Corollary 2. *If $f: E \to F$ is a polynomial on every affine line in E, then for every subspace $V \subset E$, $\dim E < \infty$, the function $f|V$ is a polynomial.*

PROPOSITION 3. *Let E, F be arbitrary vector spaces over K and let $f\colon E\to F$ be a polynomial on every affine line contained in E. Then there exists a sequence of homogeneous polynomials $f_k \in Q^k(E,F)$ $(k=0,1,\ldots)$ such that*

$$f(x)=\sum_{k=0}^{\infty} f_k(x),\qquad x\in E,$$

where for every fixed $x\in E$ only finitely many components of the series do not vanish.

Proof. For a fixed $x\in E$ the mapping $K\ni t\to f(tx)\in F$ is a polynomial. Then

$$f(tx)=\sum_{k=0}^{\infty} t^k f_k(x),\qquad t\in K,$$

where only finitely many components of the series do not vanish. Since

$$f(t\lambda x)=\sum f_k(tx)\lambda^k=\sum\big(f_k(x)t^k\big)\lambda^k,\qquad x\in E,\ t,\lambda\in K$$

so $f_k(tx)=t^kf_k(x)$, $t\in K$, $x\in E$, $k\geqslant 0$. We want to show that $f_k\in Q^k(E,F)$. Given f_k, let $\bar f_k$ be defined by formula (1). The mapping $\bar f_k$ is obviously symmetric. It is enough to show that it is k-linear. To this end, given $x_1, x_1', x_2, \ldots, x_k\in E$, take a subspace $V\subset E$ such that $\dim V\leqslant k+1$ and $x_1, x_1', x_2, \ldots, x_k\in V$. Since according to the Corollary after Lemma 1, the mapping $f_k|V$ is a polynomial, so $\bar f_k|V^k$ is k-linear. Therefore $\bar f_k(ux_1+vx_1', x_2,\ldots,x_k)=u\bar f_k(x_1,x_2,\ldots,x_k)+v\bar f_k(x_1',x_2,\ldots,x_k)$, $u,v\in K$. Hence $\bar f_k\in \operatorname{Hom}^k(E,F)$ and the proof is completed.

COROLLARY 3. [9]. *Given a function $f\colon E\to F$ and a positive integer m consider the two following conditions:*

(P) *For every fixed x, $h\in E$ there exists $a_j(x,h)\in F$ $(j=0,\ldots,m)$ such that*

$$f(x+th)=\sum_{j=0}^{m} t^j a_j(x,h),\qquad t\in K,$$

i.e. f is a polynomial of degree at most m on every affine line contained in E.

(H) $f(tx)=t^m f(x)$, $x\in E$, $t\in K$.

Condition (P) *is necessary and sufficient that f be a polynomial of degree at most m.*

Conditions (P) *and* (H) *are necessary and sufficient that f be a homogeneous polynomial of degree m.*

COROLLARY 4. *Let F^* be a set of linear forms $u\colon F\to K$ such that $u(x_1)=u(x_2)$ for every $u\in F^*$ if and only if $x_1=x_2$.*

If $f\colon E\to F$ is a mapping such that $u\circ f$ is a polynomial of degree $\leqslant m$ for every $u\in F^*$, then f is a polynomial of degree $\leqslant m$.

Proof. Let $T_m=\{t_0,\ldots,t_m\}$ be a system of $m+1$ different points of K. By the Lagrange interpolation formula

$$u\circ f(x+th)=\sum_{j=0}^{m}L^{(j)}(t,T_m)u\circ f(x+t_jh)=u\Big(\sum_{j=0}^{m}L^{(j)}(t,T_m)f(x+t_jh)\Big)$$

for $x,h\in E$, $t\in K$, $u\in F^*$. Therefore $K\ni t\to f(x+th)=\sum_{j=0}^{m}L^{(j)}(t,T_m)f(x+t_jh)\in F$ is a polynomial of degree $\leqslant m$ for any fixed x, $h\in F$. By the first Corollary f is a polynomial of degree $\leqslant m$.

Remark. If E, F and f satisfy the assumptions of Proposition 3 and moreover $f|U=0$ for an open non-empty set $U\subset E$, then $f=0$.

Indeed, let $x\in E$ and $a\in U$ be fixed. The function $g\colon K\ni t\to f\big(x+t(a-x)\big)\in F$ is a polynomial which vanishes in a neighborhood of $t=0$. Therefore $f\big(x+t(a-x)\big)=0$ for $t\in K$. In particular $f(x)=0$.

LEMMA 2. *Let E, F, E_j $(j=1,\ldots,k)$ be vector spaces over K and let V be a convex symmetric subset of F.*

(i) *If $f\in\operatorname{Hom}(E_1,\ldots,E_k;F)$, $U=U_1\times\ldots\times U_k$, $0\in U_j\subset E_j$ and $f(a+U)\subset V$, then $f(\frac{1}{2}U)\subset V$.*

(ii) *If $f\in Q^k(E,F)$, $U\subset E$ is balanced and $f(a+U)\subset V$, then $f\left(\frac{1}{2e}U\right)\subset V$.*

Proof. (i) Let $a=(a_1,\ldots,a_k)$. For any fixed $x=(x_1,\ldots,x_k)\in U$ we have $-f(a_1,a_2+x_2,\ldots,a_k+x_k)\in V$. The convexity of V implies that $f(a_1+x_1,\ldots,a_k+x_k)-f(a_1,a_2+x_2,\ldots,a_k+x_k)=f(x_1,a_2+x_2,\ldots,a_k+x_k)\in 2V$. By the induction with respect to k we get $f(x)\in 2^kV$, $x\in U$. Therefore $f(\frac{1}{2}U)\subset V$.

(ii) From formula (1) we get

$$f(x)=\frac{1}{k!}\sum_{s=0}^{k}(-1)^{k-s}\binom{k}{s}f(a+sx).\tag{2}$$

But $(-1)^{k-s}f(a+sx)\in V$, when $x\in\frac{1}{k}U$, $s=0,\ldots,k$. Therefore $f(x)\in cV$, $x\in\frac{1}{k}U$, where $c=\frac{1}{k!}\sum_{s=0}^{k}\binom{k}{s}=2^k/k!$. Finally, $f(x)\in\frac{(2k)^k}{k!}V\subset(2e)^kV$, as $x\in U$, whence $f\left(\frac{1}{2e}U\right)\subset V$.

Given a locally convex topological vector space E (shortly l.c.s. E), denote by $\Gamma(E)$ any filtrant set of seminorms determining the topology of E.

THEOREM 1. *Assume that F, E, E_j $(j = 1, \ldots, k)$ are topological vector spaces over K, F is locally convex, $f \in \operatorname{Hom}(E_1, \ldots, E_k; F)$ $(f \in Q^k(E, F))$. Then the following conditions are equivalent:*

(i) *For every $q \in \Gamma(F)$ there exists a non-empty open set U in $E_1 \times \ldots \times E_k$ (in E) such that $q \circ f$ is bounded on U.*

(ii) *For every $q \in \Gamma(F)$ there exists a neighborhood U of zero in $E_1 \times \ldots \times E_k$ (in E) such that $q \circ f$ is bounded on U.*

(iii) *f is continuous at zero of $E_1 \times \ldots \times E_k$ (of E).*

(iv) *f is continuous at a point a of $E_1 \times \ldots \times E_k$ (of E).*

(v) *f is continuous.*

Proof. Implications (v) $\Rightarrow$ (i), (v) $\Rightarrow$ (iv), (iv) $\Rightarrow$ (i) are trivial, so it remains to prove only that (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (v).

Case I: $f \in \operatorname{Hom}(E_1, \ldots, E_k; F)$. Let $U = U_1 \times \ldots \times U_k$ be a neighborhood of zero in $E_1 \times \ldots \times E_k$ such that $q \circ f(x) \leqslant M$ in $a + U$. Then according to (i) of Lemma 2, $q \circ f(u) \leqslant 2^k M$ for $u \in U$. Thus (i) $\Rightarrow$ (ii).

To prove that (ii) $\Rightarrow$ (iii) it is enough to show that for any seminorm $q \in \Gamma(F)$ the function $q \circ f$ is continuous at 0. Let us fix $\varepsilon > 0$ and suppose that $q \circ f(x) \leqslant M$ on $U = U_1 \times \ldots \times U_k$, U_j being a balanced neighborhood of $0 \in E_j$. Then for every ξ $q \circ f(y) \leqslant \xi^k M$ for $y \in \xi U = \xi U_1 \times \ldots \times \xi U_k$. It is obvious that $q \circ f(y) \leqslant \varepsilon$, $y \in \xi U$, if ξ is sufficiently small. So $q \circ f$ is continuous at $0 \in E_1 \times \ldots \times E_k$.

(iii) $\Rightarrow$ (v). Fix $a = (a_1, \ldots, a_k) \in E_1 \times \ldots \times E_k$, $\varepsilon > 0$, $q \in \Gamma(F)$. Assume that (iii) $\Rightarrow$ (v) holds for $(k-1)$-linear mappings and observe that if f is continuous at $0 \in E_1 \times \ldots \times E_k$, then the mapping $g\colon E_2 \times \ldots \times E_k \ni (x_2, \ldots, x_k) \to f(a_1, x_2, \ldots, x_k) \in F$ is continuous at 0. Indeed, there exist balanced neighborhoods V_j of zero in E_j such that

$$q \circ f(x_1, \ldots, x_k) < \varepsilon, \quad x_j \in V_j \ (j = 1, \ldots, k).$$

Let $\theta \in (0, 1)$ be so small that $\theta a_1 \in V_1$. Then

$$q \circ f(\theta a_1, x_2, \ldots, x_k) = q \circ f(a_1, \theta x_2, x_3, \ldots, x_k) \quad \text{for } x_j \in V_j \ (j = 2, \ldots, k).$$

Hence

$$q \circ f(a_1, x_2, \ldots, x_k) < \varepsilon \quad \text{for } x_2 \in \theta V_2,\ x_j \in V_j \ (j = 3, \ldots, k).$$

By the induction assumption the mapping g is continuous in $E_2 \times \ldots \times E_k$, in particular

$$(+) \qquad q\big(f(a_1, a_2 + x_2, \ldots, a_k + x_k) - f(a_1, \ldots, a_k)\big) < \varepsilon/2$$

for $x_j \in V_j'$, V_j' being a neighborhood of 0 in E_j.

Condition (iii) implies that there exist neighborhoods U_j of 0 in E_j such that

$$(++) \qquad q \circ f(y_1, \ldots, y_k) < \varepsilon/2, \quad y_j \in U_j \ (j = 1, \ldots, k).$$

Let V_j'' be a balanced neighborhood of zero in E_j such that $V_j'' + V_j'' \subset U_j$ and take $\eta \in (0, 1)$ so small that $\eta a_j \in V_j''$ $(j = 1, \ldots, k)$. Then

$$(+++) \qquad q \circ f(x_1, a_2 + x_2, \ldots, a_k + x_k) < \varepsilon/2, \quad x_j \in \eta^k V_j'' \ (j = 1, \ldots, k).$$

Indeed,

$$\eta^{-k} x_1 \in V_1'' \subset U_1, \quad \eta a_j + \eta x_j \in V_j'' + V_j'' \subset U_j \quad (j = 1, \ldots, k),$$

so according to (++),

$$q \circ f(x_1, a_2 + x_2, \ldots, a_k + x_k) = \eta q \circ f\big(\eta^{-k} x_1, \eta(a_2 + x_2), \ldots, \eta(a_k + x_k)\big) < \varepsilon/2.$$

Inequalities (+) and (+++) imply that $q\big(f(a + x) - f(a)\big) < \varepsilon$, $x \in V_1 \times \ldots \times V_k$, where $V_j = V_j' \cap \eta^k V_j''$. Therefore f is continuous at every point $a \in E_1 \times \ldots \times E_k$.

Case II. $f \in Q^k(E, F)$. Implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) may be proved analogously as in case I by using (ii) of Lemma 2.

(iii) $\Rightarrow$ (v). It follows from formula (1) that the k-linear mapping $\bar{f}$ associated with f is continuous at $0 \in E^k$. Therefore by virtue of case I, $\bar{f}$ is continuous. This, however, implies that $f = \bar{f} \circ \pi_k$ is continuous.

PROPOSITION 4. *Assume that E and F are topological vector spaces over K, E is a Baire space, F is locally convex and $f\colon E \to F$ is continuous. If f is a polynomial on every affine line in E, then f is a polynomial on E.*

Proof. By Proposition 3 one can find polynomials $x \in E$, $f_k \in Q^k(E, F)$ $(k = 0, 1, \ldots)$ such that

$$(\S) \qquad f(x) = \sum_{k=0}^{\infty} f_k(x).$$

At first we shall show that the polynomials f_k are continuous. It is obvious that

$$f_k(x) = \lim_{t \to 0} \frac{1}{t^k} \big(f(tx) - f(0) - f_1(tx) - \ldots - f_{k-1}(tx)\big), \quad x \in E,$$

because for any fixed x there are only finitely many components of series (§) which do not vanish.

Assume that $f_1, \ldots, f_{k-1}$ are continuous. Given $t \in K$ $(t \neq 0)$ and $q \in \Gamma(F)$, the scalar function

$$g(t, x) = q \circ \left(\frac{1}{t^k} \big(f(tx) - f(0) - f_1(tx) - \ldots - f_{k-1}(tx)\big)\right)$$

is continuous in E and $\lim_{t \to 0} g(t, x) = q \circ f_k(x)$, $x \in E$. Since E is a Baire space, there exists an open set $U \subset E$ on which $q \circ f_k$ is bounded. By Theorem 1 the polynomial f_k is continuous.

Put

$$E_n = \{x \in E\colon f_k(x) = 0 \text{ for } k > n\}, \quad n = 1, 2, \ldots$$

The set E_n is closed, $E_n \subset E_{n+1}$ and $E = \bigcup_{n=1}^{\infty} E_n$. Therefore $\operatorname{int}(E_n) \neq \emptyset$, if n is sufficiently large. By the Remark after Proposition 3 $f_k = 0$ for $k > n$, where n is sufficiently large. This, however, implies that $f = \sum_{k=0}^{n} f_k$ is a polynomial.

EXAMPLE. Proposition 4 breaks off to be true if E is an arbitrary topological vector space. For instance, let E be the subspace of l^2 composed of those $x = (x_1, x_2, \ldots) \in l^2$ which have only a finite number of coordinates different from zero. It can be easily checked that the series of homogeneous polynomials

$$g(x) = \sum_{k=1}^{\infty} x_k^k, \quad x \in l^2,$$

is convergent locally uniformly in l^2 and its sum g is continuous in l^2. The function $f = g|E$ is continuous in E and it is a polynomial on every subspace $V \subset E$, $\dim V < \infty$. However, f is not a polynomial on E, because its restriction to the vector line $x = te_j$, $t \in K$, where $e_j = (\delta_{j1}, \delta_{j2}, \ldots)$ ($\delta_{jk} = 0$ if $j \neq k$, $\delta_{jj} = 1$) is a polynomial of degree j ($j = 1, 2, \ldots$).

Given any vector space E over K, $\dim E = \infty$, one may find a function $f\colon E \to F$ such that its restriction to every affine line $V_1 \subset E$ is a polynomial but f is not a polynomial. If E is a Baire space, then such a function must be discontinuous.

THEOREM 2. *Assume that E and F are vector spaces over K, $f_n \in Q^k(E, F)$, $n = 1, 2, \ldots$ and $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \in E$. Then*

(i) $f \in Q^k(E, F)$.

(ii) *If E and F are topological vector spaces, E is a Baire space and F is locally convex, $f_n \in P^k(E, F)$, $n = 1, 2, \ldots$, then $f \in P^k(E, F)$.*

Proof. (i) follows directly from formula (1).

(ii) Let $q \in \Gamma(F)$. Then $q \circ f$, as a limit of the sequence of continuous real functions $q \circ f_n$ in the Baire space E, is bounded on an open subset of E. Therefore by Theorem 1 the polynomial f is continuous.

PROPOSITION 5. *Let E be metrizable, let F be locally convex space and let $f \in Q^k(E, F)$. If for every continuous linear form $u \in F'$ the function $u \circ f$ is continuous, then the polynomial f is continuous.*

Proof. Suppose condition (i) of Theorem 1 is not satisfied. Then there exists a seminorm $q \in \Gamma(F)$ such that for every $n \in \boldsymbol{N}$ there exists a point $x_n \in E$ such that $q \circ f(x_n) > n$ and the distance of x_n to 0 is smaller then $1/n$. Since for every $u \in F'$ the sequence $\{u \circ f(x_n)\}$ is bounded (because it is convergent to 0), so the sequence $\{q \circ f(x_n)\}$ is bounded. This gives a contradiction, which ends the proof.

LEMMA 3. *Let A be a topological Baire space and let B be a metric space. If a function $f\colon A \times B \to K$ is continuous with respect to each variable separately, then it is bounded on a non-empty open subset of $A \times B$.*

Proof (following Noverraz [10], p. 27). Given $m \in \boldsymbol{N}$ and $y \in B$, let us put

$$A_y(m) = \{x \in A\colon |f(x, y)| \leqslant m\}.$$

Let B_n ($n = 1, 2, \ldots$) be a ball with center $y_0 \in B$ and radius $1/n$. The set $A_{m,n} = \bigcap_{y \in B_n} A_y(m)$ is closed, because every set $A_y(m)$ is closed. We claim that $A = \bigcup_{m,n=1}^{\infty} A_{m,n}$. Indeed, for a fixed $x \in A$ the function $B \ni y \to f(x, y) \in K$ is continuous and therefore $|f(x, y)| \leqslant m_0$ for all $y \in B_{n_0}$, m_0 and n_0 being integers sufficiently large. Hence $x \in A_{m_0, n_0}$. Since A is a Baire space there exists $m_1, n_1 \in \boldsymbol{N}$ such that an open subset $U \subset A$ is contained in A_{m_1, n_1}. Consequently $|f(x, y)| \leqslant m_1$ in $U \times B_{n_1}$.

COROLLARY 5 (see [5]). *Let E, F, G be topological vector spaces over K, E — Baire, F — metrizable, G — locally convex. Then every bilinear mapping $f\colon E \times F \to G$ seperately continuous is continuous.*

Proof. By Lemma 3 for every $q \in \Gamma(F)$ the function $q \circ f$ is bounded on an open subset of $E \times F$. By Theorem 1 the mapping f is continuous.

COROLLARY 6. *Let E_j ($j = 1, \ldots, k$) be Baire metrizable vector spaces and let G be a locally covex topological vector space. Then $f \in \operatorname{Hom}(E_1, \ldots \ldots, E_k; G)$ is continuous if and only if it is continuous with respect to each variable separately.*

2. Complexification. Let E be a vector space over $\boldsymbol{R}$. It is easy to check that the space $E \times E$ furnished with the addition operation of its elements and with the multiplication by complex scalars according to the following formulae

$$(+) \quad (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \quad (x_1, y_1), (x_2, y_2) \in E \times E,$$

$$(\times) \quad (u + iv)(x, y) = (ux - vy, vx + uy), \quad u, v \in \boldsymbol{R}, (x, y) \in E \times E,$$

is a vector space over $\boldsymbol{C}$. We denote this space by $\tilde{E}$ and call it the *complexification* of E. If E is a topological vector space then operations (+) and (×) are consistent with product topology so then E is a topological vector space over $\boldsymbol{C}$.

Every element $(x, y) \in \tilde{E}$ may be written in a unique way in the form $(x, y) = (x, 0) + i(y, 0)$. We may treat E as a subspace of $\tilde{E}$ by identifying

$x \in E$ with $(x, 0) \in \tilde{E}$. Therefore we may treat $\tilde{E}$ as a direct topological product of E and iE and write every element $(x, y) \in \tilde{E}$ in the form $x+iy$.

We may also identify $\tilde{E}$ with the tensor product $E \otimes_{\boldsymbol{R}} \boldsymbol{C}$.

Given any seminorm q on E, we put

$$\tilde{q}(\tilde{x}) = \inf\Big\{\sum |t_j|\, q(x_j) \colon \tilde{x} = \sum t_j x_j,\ x_j \in E,\ t_j \in \boldsymbol{C}\Big\}, \qquad \tilde{x} \in \tilde{E}.$$

One may easily show that $\tilde{q}$ is a seminorm on $\tilde{E}$ and

$$\max\big(q(x), q(y)\big) \leqslant \tilde{q}(x+iy) \leqslant q(x)+q(y), \qquad x+iy \in \tilde{E}.$$

Therefore, if $\Gamma(E) = \{q_l\}_{l \in I}$ is the set of seminorms defining the topology of a locally convex space E, then $\Gamma(\tilde{E}) = \{\tilde{q}_l\}_{l \in I}$ defines the topology of $\tilde{E}$.

If q is a norm in E, then $\tilde{q}$ is a norm in $\tilde{E}$.

THEOREM 3. *Let E, E_j $(j = 1, \ldots, k)$ be topological vector spaces over $\boldsymbol{R}$. Let F be a topological vector space over $\boldsymbol{C}$. Given $f \in \mathrm{Hom}_{\boldsymbol{R}}(E_1, \ldots, E_k; F)$ $(f \in Q^k_{\boldsymbol{R}}(E, F))$, there exists exactly one element $\tilde{f} \in \mathrm{Hom}_{\boldsymbol{C}}(\tilde{E}_1, \ldots, \tilde{E}_k; F)$ $(\tilde{f} \in Q^k_{\boldsymbol{C}}(\tilde{E}, F))$ such that $\tilde{f}|E_1 \times \ldots \times E_k = f$ $(\tilde{f}|E = f)$. The mapping f is continuous if and only if the mapping $\tilde{f}$ is continuous.*

Proof. Case I. $f \in \mathrm{Hom}_{\boldsymbol{R}}(E_1, \ldots, E_k; F)$. Put

$$\text{(C)} \qquad \tilde{f}(x_1^0 + ix_1^1, \ldots, x_k^0 + ix_k^1) = \sum_{\varepsilon_1, \ldots \varepsilon_k = 0}^{1} i^{\Sigma \varepsilon_j} f(x_1^{\varepsilon_1}, \ldots, x_k^{\varepsilon_k}),$$
$$x_j^0 + ix_j^1 \in \tilde{E}_j \qquad (j = 1, \ldots, k).$$

It is obvious that $\tilde{f}$ is a continuation of f and $\tilde{f}$ is k-linear over $\boldsymbol{R}$. Observe that $\tilde{f}(u_1, \ldots, iu_s, \ldots, u_k) = i\tilde{f}(u_1, \ldots, u_k)$, $(u_1, \ldots, u_k) \in \tilde{E}_1 \times \ldots \times \tilde{E}_k$. This implies that $\tilde{f}$ is k-linear over $\boldsymbol{C}$.

Formula (C) implies that f and $\tilde{f}$ are simultaneously continuous.

Case II. $f \in Q^k_{\boldsymbol{R}}(E, F)$. Let $\bar{f}$ be the k-linear mapping associated with f. Then $\tilde{f}(u) = \tilde{\bar{f}}(u, \ldots, u)$, $u \in \tilde{E}$, is the desired homogeneous polynomial.

Remark. If $E_1, \ldots, E_k, F$ are vector spaces over $\boldsymbol{R}$, then the element $f \in \mathrm{Hom}(E_1, \ldots, E_k; F)$ may be interpreted also as an element of $\mathrm{Hom}_{\boldsymbol{R}}(E_1, \ldots, E_k; \tilde{F})$. So the mapping $\tilde{f}$ also in this case may be correctly defined. The same remark is valid for polynomials.

LEMMA 4. *Assume that E, F are vector spaces over $\boldsymbol{R}$, $U \subset E$ is convex, $V \subset F$ is convex and balanced, $f \in Q^k(E, F)$, $f(U) \subset V$. Then $\tilde{f}\left(\frac{1}{4e}(U+iU)\right) \subset V+iV$, where $\tilde{f} \in Q^k(\tilde{E}, \tilde{F})$ denotes the complexification of f.*

Proof. It follows from (1) and (C) that

$$\tilde{f}(x+iy) = \sum_{s=0}^{k} i^s \binom{k}{s} \bar{f}(\underbrace{x, \ldots, x}_{s}, \underbrace{y, \ldots, y}_{k-s})$$
$$= \frac{1}{k!} \sum_{\varepsilon_1, \ldots, \varepsilon_k = 0}^{1} (-1)^{k-\Sigma\varepsilon_j} \sum_{s=0}^{k} i^s \binom{k}{s} f\big((\varepsilon_1 + \ldots + \varepsilon_s)x + (\varepsilon_{s+1} + \ldots + \varepsilon_k)y\big)$$
$$= \frac{1}{k!} \sum_{l=1}^{k} (-1)^{k-l} \sum_{\varepsilon_1 + \ldots + \varepsilon_k = l} \sum_{s=0}^{k} i^s \binom{k}{s} (\varepsilon_1 + \ldots + \varepsilon_k)^k \times$$
$$\times f\left(\frac{\varepsilon_1 + \ldots + \varepsilon_s}{\varepsilon_1 + \ldots + \varepsilon_k} x + \frac{\varepsilon_{s+1} + \ldots + \varepsilon_k}{\varepsilon_1 + \ldots + \varepsilon_k} y\right) = \sum_{\mu} a_\mu w_\mu + i\beta_\mu w_\mu,$$

where $a_\mu \geqslant 0$, $\beta_\mu \geqslant 0$ and w_μ has the form

$$w_\mu = (-1)^\varrho f\left(\frac{\varepsilon_1 + \ldots + \varepsilon_s}{\varepsilon_1 + \ldots + \varepsilon_k} x + \frac{\varepsilon_{s+1} + \ldots + \varepsilon_k}{\varepsilon_1 + \ldots + \varepsilon_k} y\right).$$

By virtue of the inequalities

$$\max\Big(\sum a_\mu, \sum \beta_\mu\Big) \leqslant \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} l^k \sum_{s=0}^{k} \binom{k}{s} \leqslant (4k)^k / k! \leqslant (4e)^k,$$

this implies that

$$\tilde{f}(x+iy) \in (4e)^k V + i(4e)^k V \quad \text{for } x, y \in U,$$

whence

$$\tilde{f}(x+iy) \in V + iV \quad \text{for } x+iy \in \frac{1}{4e}(U+iU).$$

PROPOSITION 6. *Let U be a convex subset of a real vector space E and let q be a seminorm defined on a real vector space F. If $f_k \in Q^k(E, F)$, $k = 0, 1, \ldots$, and $\sum_{k=0}^{\infty} \sup\{q \circ f_k(x) \colon x \in U\} < \infty$, then*

$$\sum_{k=0}^{\infty} \sup\left\{\tilde{q} \circ \tilde{f}_k(x) \colon x \in \frac{1}{4e}(U+iU)\right\} < \infty.$$

Proof. Put $a_k = \sup\{q \circ f_k(x) \colon x \in U\}$, $V_k = q^{-1}([0, a_k])$, $k \geqslant 0$. By Lemma 4

$$\tilde{f}_k(x+iy) \in V_k + iV_k \quad \text{for } x+iy \in \frac{1}{4e}(U+iU).$$

Therefore $\tilde{q}\circ\tilde{f}_k(x+iy) \leqslant 2a_k$, which ends the proof.

LEMMA 4a. *Assume that E, F are vector spaces over R, U is a subset of E, $V \subset F$ is convex and balanced, $f \in Q^k(E, F)$, $f(U) \subset V$ and W is a balanced subset of E such that $x_0+W+W \subset U$, x_0 being a fixed point of U. Then*

$$\tilde{f}\left(\frac{1}{4e}(W+iW)\right) \subset V+iV.$$

Proof. It follows from (1) and (C) that

$$\tilde{f}(x+iy)$$

$$= \frac{1}{k!}\sum_{\varepsilon_1,\ldots,\varepsilon_k=0}^{1}(-1)^{k-\Sigma\varepsilon_j}\sum_{s=0}^{k} i^s\binom{k}{s} f\big(x_0+(\varepsilon_1+\ldots+\varepsilon_s)x+(\varepsilon_{s+1}+\ldots+\varepsilon_k)y\big)$$

$$= \sum \alpha_\mu w_\mu + i\sum \beta_\mu w_\mu,$$

where $\alpha_\mu \geqslant 0$, $\beta_\mu \geqslant 0$ and w_μ has the form $w_\mu = (-1)^\varrho f\big(x_0+(\varepsilon_1+\ldots+\varepsilon_s)x+ +(\varepsilon_{s+1}+\ldots+\varepsilon_k)y\big)$. It is obvious that $x_0+(\varepsilon_1+\ldots+\varepsilon_s)x+(\varepsilon_{s+1}+\ldots+ +\varepsilon_k)y \in U$, $w_\mu \in V$, if $x, y \in \frac{1}{k}W$. By virtue of the inequalities

$$\max\left(\sum \alpha_\mu, \sum \beta_\mu\right) \leqslant \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}\sum_{s=0}^{k}\binom{k}{s} \leqslant 4^k/k!,$$

this implies that $\tilde{f}(x+iy) \in \frac{4^k}{k!}(V+iV)$ for $x, y \in \frac{1}{k}W$, whence $\tilde{f}(x+iy) \in V+iV$ for $x+iy \in \frac{1}{4e}(W+iW)$. The proof is concluded.

PROPOSITION 6a. *Let U be an open subset of a real vector space E and let q be a seminorm defined on a real vector space F. If $f_k \in Q^k(E, F)$, $k = 0, 1, \ldots$, and $\sum_{k=0}^{\infty}\sup\{q\circ f_k(x)\colon x \in U\} < \infty$, then there exists a neighborhood W of zero in E such that*

$$\sum_{k=0}^{\infty}\sup\left\{\tilde{q}\circ\tilde{f}(x)\colon x \in \frac{1}{4e}(W+iW)\right\} < \infty.$$

Proof. By Lemma 4a it is enough to take an arbitrary balanced neighborhood W of zero in E such that $x_0+W+W \subset U$ for a fixed point $x_0 \in U$.

3. Gateaux differentials. In this section E and F are topological vector spaces over K, F is locally convex (shortly l.c.s.), and U is an open subset of E.

DEFINITION 3. We say that a function $f\colon U \to F$ has the *p-th Gateaux differential* (*G-differential*) at a point $x_0 \in U$ (or f is of class G^p at x_0), if

1° For every $h \in E$ the mapping

$$f_h\colon K_h \ni t \to f(x_0+th) \in F,$$

defined in a neighborhood K_h of zero in K, has the p-th derivative at 0;

2° The mapping

$$\delta^k_{x_0}f\colon E \ni h \to \frac{d^k}{dt^k} f(x_0+th)|_{t=0} \in F$$

is a homogeneous polynomial of degree k, $k = 1, \ldots, p$, i.e. $\delta^k_{x_0}f \in Q^k(E, F)$.

We write $f \in G^p(U)$, if f is of class G^p at every point $x_0 \in U$. The polynomial $\delta^k_{x_0}f$ is called the *k-th G-differential* at x_0.

The condition 2° is not implied by 1°.

EXAMPLE. *If $f \in Q^n(E, F)$, then $\delta^k_x f$ exists for every $k \geqslant 1$, $\delta^k_x f = 0$ for $k \geqslant n+1$ and*

$$\delta^k_x f(h) = k!\binom{n}{k}\bar{f}(\underbrace{x, \ldots, x}_{n-k}, \underbrace{h, \ldots, h}_{k}), \qquad k = 1, \ldots, n.$$

In particular, $\delta^n_x f(h) = n!f(h)$.

Proof.

$$f(x+th) = \sum_{s=0}^{n}\binom{n}{s} t^s\bar{f}(\underbrace{x, \ldots, x}_{n-s}, \underbrace{h, \ldots, h}_{s}).$$

DEFINITION 4. We say that $f\colon U \to F$ is of class $\gamma^k(U)$ and write $f \in \gamma^k(U)$ if and only if for every affine subspace $V \subset E$, $\dim V \leqslant k+1$, the function $f|U \cap V$ is of class C^k (For the definition of C^k-functions with values in l.c.s., see [6]).

THEOREM 4. *If $f \in \gamma^k(U)$, then $f \in G^k(U)$.*

Proof. (a) $k = 1$. The derivative

$$\delta^1_x f(h) = \frac{d}{dt} f(x+th)|_{t=0}$$

exists for every $h \in E$, and the restriction of $\delta^1_x f$ to an arbitrary two dimensional subspace of E is a linear function. Therefore $\delta^1_x f$ is linear in E.

(b) $k \geqslant 2$. We have only to show that

$$\delta^k_x f(h) = \frac{d^k}{dt^k} f(x+th)|_{t=0}$$

is a homogeneous polynomial of degree k, i.e. we have to find $\bar{\delta}_x^k f \in \operatorname{Hom}^k(E, F)$ such that $\bar{\delta}_x^k f(h, \ldots, h) = \delta_x^k f(h)$. To this end we put

$$\bar{\delta}_x^k f(h_1, \ldots, h_k) = \frac{\partial^k}{\partial t_1 \ldots \partial t_k} f\Big(x + \sum_{j=1}^k t_j h_j\Big)\Big|_{t_1 = \ldots = t_k = 0}.$$

We shall prove the theorem, if we show that $\bar{\delta}_x^k f$ so defined is k-linear. Fix $h_2, \ldots, h_k \in E$ and observe that the mapping

$$g\colon U \ni x \to \bar{\delta}_x^{k-1} f(h_2, \ldots, h_k) \in F$$

belongs to $r^1(U)$. Therefore $\delta_x^1 g(h_1)$ is linear for $h_1 \in E$. Since

$$\begin{aligned}\delta_x^1 g(h_1) &= \frac{d}{dt}\bar{\delta}_{x+th_1}^{k-1} f(h_2, \ldots, h_k)|_{t=0} \\ &= \frac{d}{dt_1}\Big(\frac{\partial^{k-1}}{\partial t_2 \ldots \partial t_k} f(x + t_1 h_1 + \ldots + t_k h_k)\Big)\Big|_{t_i = 0} \\ &= \frac{\partial^k}{\partial t_1 \ldots \partial t_k} f\Big(x + \sum_{i=1}^k t_i h_i\Big)\Big|_{t_1 = \ldots = t_k = 0} \\ &= \bar{\delta}_x^k f(h_1, \ldots, h_k),\end{aligned}$$

so $\bar{\delta}_x^k f$ is linear with respect to h_1. But $\bar{\delta}_x^k f$ is obviously symmetric, then it is k-linear. The proof is concluded.

EXAMPLE. We shall show that $f \in G^k(U)$ does not imply that $f \in r^k(U)$. Let $U = \boldsymbol{R}^2$. Put

$$\varphi(x, r) = \begin{cases} \exp\Big(-\dfrac{r^2}{r^2 - x^2}\Big), & \text{if } |x| < r, \\ 0, & \text{if } |x| \geqslant r. \end{cases}$$

Then the function $f(x, y) = \varphi(x - \frac{3}{4}\sqrt{|y|}, \frac{1}{4}\sqrt{|y|})$, $(x, y) \in \boldsymbol{R}^2$, is of G^∞ in $\boldsymbol{R}^2$, but f is not continuous at $(0, 0)$.

LEMMA 5. *If $g\colon K \to F$ has the k-th derivative at 0, then the following Taylor formula holds*

$$\text{(T)} \qquad g(t) = g(0) + \frac{t}{1!} g'(0) + \ldots + \frac{t^k}{k!} g^{(k)}(0) + t^k u(t), \qquad t \in K,$$

where $\lim_{t \to 0} u(t) = 0$.

Proof. One may repeat the proof of Theorem 1 in ([4], Ch. I, § 3), replacing everywhere the norm by an arbitrary seminorm $q \in \Gamma(F)$.

If $K = \boldsymbol{C}$ and $k \geqslant 2$ formula (T) is trivial, because then f is analytic in a neighborhood of 0 (see [3], p. 83).

DEFINITION 5. Let E be a locally convex space over K. We say that E is *sequentially-complete* (or shortly s-complete) if for every Cauchy sequence $\{x_n\}_{n \in \boldsymbol{N}}$, $x_n \in E$, there exists an element $x \in E$ such that $x_n \to x$. A sequence $\{x_n\}_{n \in \boldsymbol{N}}$ is called *Cauchy sequence* if for every $q \in \Gamma(F)$, $q(x_m - x_n) \to 0$ as $m, n \to \infty$.

It is obvious that every l.c.s. which is quasi-complete in sense of [5], is also s-complete. If t.v.s. E over $\boldsymbol{R}$ is a l.c.s. and s-complete, then $\tilde{E}$ is also a l.c.s. and s-complete.

THEOREM 5. 1° *If E is a Baire space, $f\colon U \to F$ is continuous and $f \in G^k(U)$, then for every $x \in U$ the G-differential $\delta_x^k f$ is continuous, i.e. $\delta_x^k f \in P^k(E, F)$.*

2° *If $K = \boldsymbol{C}$, E is an arbitrary topological vector space, F is s-complete, $f \in G^k(U)$ and $q \circ f$ is locally bounded in U for every $q \in \Gamma(F)$, then $\delta_x^k f \in P^k(E, F)$.*

Proof. 1° Fix $x \in U$ and let A be a balanced neighborhood of $0 \in E$ such that $x + A \subset U$. Using the induction with respect to k we may assume that the function

$$F(t, h) = \frac{k!}{t^k}\Big(f(x + th) - f(x) - \frac{t}{1!}\delta_x^1 f(h) - \ldots - \frac{t^{k-1}}{(k-1)!}\delta_x^{k-1} f(h)\Big)$$

is continuous with respect to $h \in A$ for every fixed $t \in K$, $0 < |t| < 1$. Moreover, by formula (T), $\lim_{t \to 0} F(t, h) = \delta_x^k f(h)$, $h \in A$. In particular, for every $q \in \Gamma(F)$ and $t \in K$, $0 < |t| < 1$, we have $\lim q \circ F(t, h) = q \circ \delta_x^k f(h)$. Since E is a Baire space, there is an open subset $V_q \subset A$ on which $q \circ \delta_x^k f$ is bounded. By Theorem 1 the polynomial $\delta_x^k f$ is continuous.

2° If $K = \boldsymbol{C}$ and $f \in G^k(U)$, $k \geqslant 1$, then $f(x + th)$ is analytic in the unit disc $|t| \leqslant 1$ for every fixed $h \in A$, A being any balanced neighborhood of $0 \in E$ such that $x + A \subset U$ ([7], [3]). Therefore

$$\delta_x^k f(h) = k!(2\pi i)^{-1} \int_{|t|=1} f(x + th) t^{-k-1} dt.$$

Given $q \in \Gamma(F)$, $q \circ f$ is locally bounded. So we may choose A in such a way that $q \circ f(x + h) \leqslant M < \infty$ for $h \in A$. Therefore

$$q \circ \delta_x^k f(h) \leqslant k! M \quad \text{for } h \in A.$$

Application of Theorem 1 concludes the proof.

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Analytic functions in topological vector spaces

by

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Contents

Introduction. The aim of this paper is to give a unified and selfcontained exposition of basic concepts and facts concerning analytic functions defined in open subsets of a complex or real topological vector space E with values in a locally convex topological vector space F. As a special case we get the theory developed in [2] and [11] for Banach spaces.

In the present article results of our previous paper [6] on polynomials and multilinear functions are essentially used.

We hope that our exposition may be useful for further study of the theory of analytic functions in topological vector spaces. This theory may be treated in natural way as a branch of non-linear functional analysis and deserves further developement.

In this paper we always assume that E is a topological Hausdorff vector space, F is locally convex and sequentially complete. However, in many places additional assumptions are necessary. Generally speaking, all the results are valid if E is a Baire space (sometimes Baire and metrizable) — in the complex case, or if E and F' (the topological dual of F) are Baire spaces — in the real case.