best choice for $f(0)$ is 0. It must therefore be true that $\|z_i^k\|_p \leqslant \|de_i^k\|_q^\alpha$, that is, $(1-2^{-2k})^{1/p} \leqslant 2^{1/p'-\alpha/q}$ and this must hold for integers k since we are assuming that $e(L^q, L^p; \alpha)$ always holds. Therefore $2^{1/p'-\alpha/q} \geqslant 1$ or $q/p' \geqslant \alpha$.

For $1 < p \leqslant 2 \leqslant q < \infty$ define f by $f(2^{1/\alpha p'-1/\alpha' }z_i^k) = z_i^k$; for $2 \leqslant p$, $q < \infty$ define f by $f(2^{1/\alpha p-1/q'}z_i^k) = e_i^k$; and for $1 < q \leqslant 2 \leqslant p < \infty$ define $f(2^{1/\alpha p-1/q}e_i^k) = e_i^k$. In each case extension of f to zero implies sharpness of the corresponding part of Theorem 1.

Notice, for example in the case $1 < q \leqslant 2 \leqslant p < \infty$, that we have constructed functions $f \in \operatorname{Lip}(D, L^p; \alpha)$ for which the best choice for extension to the point zero does not lie on the convex hull of $f(D)$.

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Reçu par la Rédaction le 3. 1. 1970

STUDIA MATHEMATICA, T. XXXIX. (1971)

Conjugate kernels and convergence of harmonic singulatr inegrals

by

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Introduction. In this paper we shall be concerned with singular integrals having the form P.V. $\Omega(X')|X|^{-m} * f$, where $\Omega(X')$ is a spherical harmonic.

The purpose of this paper is twofold.

First, to show that if $K_\varepsilon(X) = \Omega(X')|X|^{-m}$ for $|X| \geqslant \varepsilon$ and zero otherwise, there exists a unique radial function $k(|X|)$ belonging to L^1 such that

$$\text{P.V.} \int_{R^m} \Omega(Y')|Y|^{-m}\varepsilon^{-m}k(\varepsilon^{-1}|X-Y|)\,dY = K_\varepsilon(X) \quad \text{a.e.}$$

The function $k(|X|)$ is the same for any spherical harmonic $\Omega(X')$ of a fixed degree.

Second, we shall use this representation to study the pointwise convergence of harmonic singular integrals at individual points by giving conditions on $K(f)$ only. The kernel $k(|X|)$ is also studied.

NOTATION

1. $X = (x_1, \ldots, x_m)$ will denote a point in the m-dimensional Euclidean space and $dX = dx_1 \ldots dx_m$ the element of volume there.

2. Σ will denote the surface of the unit sphere in $\boldsymbol{R}^m$, X' any point there and $d\sigma$ the "area" element on Σ.

3. $f*g$ will denote the convolution of f and g, namely

$$\int_{R^m} f(X-Y)g(Y)\,dY;$$

$f(\lambda X) = f(\lambda x_1, \ldots, \lambda x_m)$ for any real λ and any function f defined on $\boldsymbol{R}^m$.

4. $$\int_{R^m} \exp(-i2\pi\langle X, Y\rangle)f(Y)\,dY = \hat{f}(X)$$

will be the Fourier transform used here; $\langle X, Y\rangle = \sum_{j=1}^{m} x_j y_j$.

5. L^p, $p \geqslant 1$ will denote the usual L^p spaces defined on $\boldsymbol{R}^m$; C_0^∞ the space of infinitely differentiable and compact supported functions; C_0 the space of continuous and compact supported functions.

6. If $K(X) = \Omega(X')|X|^{-m}$ is a singular kernel, then we write

$$K(f) = \mathrm{P.V.} \int_{\boldsymbol{R}^m} K(X-Y)f(Y)\,dY;$$

also $K_\varepsilon(X) = K(X)$ if $|X| \geqslant \varepsilon$ and zero otherwise; similarly $K_\varepsilon(f) = K_\varepsilon * f$.

7. The symbol of K, as usual, will be $\widehat{\mathrm{P.V.}K} = \lim \hat{K}_\varepsilon$, and the characteristic, the function $\Omega(X')$.

8. By $\Omega^{(n,j)}(X')$, $j = 1, \ldots, L$, $L = \binom{n+m-1}{m-1} - \binom{n+m-3}{m-1}$ we write any complete system of spherical harmonics of degree n on the unit sphere Σ of $\boldsymbol{R}^m$ normalized by the conditions

$$\int_\Sigma \Omega^{(n,i)}(X')\,\Omega^{(n,j)}(X')\,d\sigma = \delta_{ij}.$$

STATEMENT OF RESULTS

THEOREM 1. *If $K_\varepsilon(X)$ denotes the truncated singular kernel whose characteristic is $\Omega(X')$ a spherical harmonic of degree n, then there exists a unique radial function $k(|X|)$ belonging to L^1 such that*

(i) $K_\varepsilon(X) = \mathrm{P.V.} \int_{\boldsymbol{R}^m} K(X-Y)\,\varepsilon^{-m} k(\varepsilon^{-1}|Y|)\,dY.$

(ii) *$k(|X|)$ is the same for any harmonic $\Omega(X')$ of a fixed degree n.*

THEOREM 2. *If $f \in L^p$, $p > 1$, $p < \infty$, then*

(i) *$K_\varepsilon(f)$ converges to $K(f)$ at the point X provided that*

$$\lim_{\varepsilon \to 0} \varepsilon^{-m} \int_{|Y|<\varepsilon} |K(f)(X-Y) - K(f)(X)|^p\,dY = 0$$

in particular, if $K(f)(Y)$ is continuous at X.

(ii) $$\sup_\varepsilon |K_\varepsilon(f)(X)|^p \leqslant C_p \sup_\varepsilon \varepsilon^{-m} \int_{|Y|<\varepsilon} |K(f)(X-Y)|^p\,dY$$

C_p depends on p only.

THEOREM 3. (*Behavior of $\varepsilon^{-m}k(\varepsilon^{-1}|X|) = k_\varepsilon(|X|)$*). *If f and $|f|\log^+|f|$ belong to $L^1(\boldsymbol{R}^m)$, then*

(i) *$k_\varepsilon * f \to f$ a.e. as $\varepsilon \to 0$,*

furthermore if $f^* = \sup_\varepsilon |k_\varepsilon * f|$ we have

(ii) $|E(f^* > \lambda)| \leqslant (C_1/\lambda^2) \int_{\boldsymbol{R}^m} |f|\,dX + (C_2/\lambda) \int_{\boldsymbol{R}^m} |f|\,dX + (C_3/\lambda) \int_{\boldsymbol{R}^m} |f| \log^+ |f|\,dX,$

where C_1, C_2 and C_3 do not depend on f.

If $f \in L^p$ and $p > 1$, we have

(iii) *$k_\varepsilon * f \to f$ a.e. as $\varepsilon \to 0$ and $\|f^*\|_p < C_p \|f\|_p$ the constant C_p depends on p only.*

1. Proof of Theorem 1. We start proving some auxiliary lemmae:

1.1. LEMMA. *If $f(|X|)$ is a radial function defined on $\boldsymbol{R}^m$ and $\Omega(X')$ is a spherical harmonic of degree n defined over the unit sphere of $\boldsymbol{R}^m$, then we have for $f \in L^p$, $1 \leqslant p \leqslant 2$:*

(i) $$\{f(|X|) \cdot \Omega(X')\}^\wedge = \Psi(|X|) \cdot \Omega(X'),$$

where $\Omega(X')$ is the same harmonic and $\Psi(|X|)$ is also a radial function.

(ii) $$\Big(\int_{\boldsymbol{R}^m} |\Psi(|X|)|^q\,dX\Big)^{1/q} \leqslant \bar{C}_p \Big(\int_{\boldsymbol{R}^m} |f(|X|)|^p\,dX\Big)^{1/p}, \qquad 1/p + 1/q = 1,$$

furthermore

$$\Psi(t) = \lim_{N\to\infty} \Psi_N(t) = \lim_{N\to\infty} 2\pi i^n t^{1-m/2} \int_0^N J_{n+m/2-1}(2\pi ts) s^{m/2} f(s)\,ds,$$

where the limit must be understood pointwise if $p = 1$ and in mean of order q if $f \in L^p$ $(1/p + 1/q = 1)$. The L^q means are taken in $\boldsymbol{R}^m$ and $t = |X| \cdot J_{n+m/2-1}$ denotes the Bessel function of order $n + m/2 - 1$.

(iii) *The relationship between f and Ψ is independent of the spherical harmonic provided that the degree n has been fixed.*

Proof. The case $p = 1$ is Theorem (2.6.1) in [1], p. 38. If $p \neq 1$ and according to Lemma (2.6.5) in [1], p. 37, we have

(1.1.1) $$\int_{|X|\leqslant N} \exp(-2\pi i |X|\,|Y| \langle X', Y' \rangle)\,\Omega(X') f(|X|)\,dX$$
$$= \Omega(Y') 2\pi i^n |Y|^{-m/2+1} \int_0^N J_{n+m/2-1}(2\pi |X|\,|Y|)\,|X|^{m/2} f(|X|)\,d|X|.$$

The Hausdorff-Young inequality gives

(1.1.2) $$\Big(\int_{\boldsymbol{R}^m} |\Omega(Y')|^q \cdot |\Psi_N(|Y|)|^q\,dY\Big)^{1/q} \leqslant C_p \Big(\int_{0<|X|<N} |\Omega(X')|^p |f(|X|)|^p\,dX\Big)^{1/p}$$

and also

$$\Big(\int_{\boldsymbol{R}^m} |\Omega(Y')|^q |\Psi_M(|Y|) - \Psi_N(|Y|)|^q\,dY\Big)^{1/q} \leqslant C_p \Big(\int_{N<|X|<M} |\Omega(X')|^p |f(|X|)|^p\,dX\Big)^{1/p}.$$

According to the fact that $\Psi_M(|Y|)$ is radial and taking into account that $\Omega(Y')$ is homogeneous of degree zero we get

(1.1.3) $$\Big(\int_{R^m} |\Psi_M(|Y|)-\Psi_N(|Y|)|^q dY\Big)^{1/q} \leqslant \frac{C^p|\Sigma|^{1/q}}{\|\Omega\|_{q,\Sigma}}\Big(\int_{N<|X|<M} |\Omega(X')|^p|f(|X|)|^p dX\Big)^{1/p} \leqslant C_p|\Sigma|^{1/q-1/p}\frac{\|\Omega\|_{p,\Sigma}}{\|\Omega\|_{q,\Sigma}}\Big(\int_{N<|X|<M}|f(|X|)|^p dX\Big)^{1/p}$$

which shows that $\Psi_M(|Y|)$ is a Cauchy sequence in L^q, which gives (i) and (ii). Now taking $N = 0$ we get the estimates for the L^q norm of Ψ letting M tend to infinity.

Notice that the formula for Ψ involves the degree of the harmonic and not the harmonic itself which gives (iii). In order to get uniform constants for inequality (1.1.3) we may take $\Omega_0(X')$ a suitable spherical harmonic of degree n and

(1.1.4) $$\bar{C}_p = |\Sigma|^{1/p-1/q}\frac{\|\Omega_0\|_{p,\Sigma}}{\|\Omega_0\|_{q,\Sigma}}\cdot C_p.$$

The bound $\bar{C}_p$ will depend on the degree n only.

1.2. Lemma. *Let us consider now a complete system of spherical harmonics of degree n and also normalized by the conditions*

(1.2.1) $$\int_\Sigma \Omega^{n,j}(X')\Omega^{n,i}(X')\,d\sigma = \delta_{ij}, \quad i = 1, \ldots, L.$$

Then the following relation holds

(1.2.2) $$\sum_{j=1}^{L}\{\Omega^{n,j}(X')\}^2 = L/|\Sigma|$$

(see for example [3], p. 33).

On the other hand, considering $K^{n,j}(X) = |X|^{-m}\Omega^{n,j}(X')$, the symbol of this singular kernel is

(1.2.3) $$(\text{P.V. } K^{n,j})^\wedge = i^n\pi^{m/2}\Gamma(n/2)\Gamma^{-1}\big((n+m)/2\big)\,\Omega^{n,j}(X') = \gamma_n\Omega^{n,j}(X')$$

(see [3], p. 36).

(1.2.2) and (1.2.3) give the following lemma:

(1.2.4) Lemma. $\gamma_n^{-2}(|\Sigma|/L)\sum_{j=1}^{L}K^{n,j}(K^{n,j}f) = f$ *a.e. for f in L^p, $p>1$.*

Proof. According to (1.2.2) and (1.2.3) for all $f_k \in L^2$ we have

$$\Big\{\gamma_n^{-2}(|\Sigma|/L)\sum_{j=1}^{L}K^{n,j}(K^{n,j}f_k)\Big\}^\wedge = (|\Sigma|/L)\sum_{j=1}^{L}\{\Omega^{n,j}(X')\}^2\hat{f}_k(X) = \hat{f}_k(X).$$

Now taking into account that $K^{n,j}$ maps continuously L^p into L^p, $p>1$, $j = 1, \ldots, L$, and taking a sequence $f_k \to f$ in L^p and $f_k \in L^2 \cap L^p$ for all k we get

(1) $$\|K^{n,j}K^{n,j}f_k - K^{n,j}K^{n,j}f\|_p \to 0, \quad j = 1, \ldots, L,$$

therefore

(2) $$\gamma_n^{-2}(|\Sigma|/L)\sum_{j=1}^{L}K^{n,j}K^{n,j}f_k \xrightarrow{L^p} \gamma_n^{-2}(|\Sigma|/L)\sum_{j=1}^{L}K^{n,j}K^{n,j}f.$$

On the other hand, since $f_k \in L^2$ for all k, we have

$$\gamma_n^{-2}(|\Sigma|/L)\sum_{j=1}^{L}K^{n,j}K^{n,j}f_k = f_k \quad \text{a.e.}$$

Taking L^p limit in both sides we get the desired result.

1.3. Lemma. *If $\Omega(Y')$ is a spherical harmonic of degree n there exists a radial function $k(|Y|)$ belonging to L^p for all $p \geqslant 1$, $p \neq \infty$ such that*

(1.3.1) $$\text{P.V.}\int_{R^m}\Omega(Y')|Y|^{-m}k(|X-Y|)\,dY = \Omega(X')|X|^{-m} \quad \text{if } |X|\geqslant 1$$

and zero otherwise. Furthermore $k(|X|)$ is the same function for any spherical harmonic of degree n.

Proof. Without loss of generality we may assume that

(1.3.2) $$\int_\Sigma |\Omega(Y')|^2 d\sigma = 1.$$

Let's construct now a complete and normalized system of spherical harmonics of degree n, $\Omega^{n,j}(X')$, $j = 1, \ldots, L$, such that $\Omega(Y') = \Omega^{n,1}(Y')$.

Call $\theta(|X|)$ the function which is equal to $|X|^{-m}$ if $|X|\geqslant 1$ and zero otherwise;

(1.3.3) $$K_1^{n,j}(X) = \Omega^{n,j}(X')\cdot\theta(|X|), \quad j = 1, \ldots, L.$$

Now, according to Lemma 1.1

(1.3.4) $$\{K^{n,j}(X)\}^\wedge = \Omega^{n,j}(X')\Psi(|X|), \quad j = 1, \ldots, L;$$

where

$$\Psi(|X|) = \lim_{N\to\infty} 2\pi i^n|X|^{1-m/2}\int_0^N j_{n+m/2-1}(2\pi|X|s)\cdot s^{m/2}\theta(s)\,ds$$

and the limit must be understood in L^2 norm in $\boldsymbol{R}^m$, since $\theta(|X|)\in L^2$.

Again, according to Lemma (1.1)

(1.3.5) $$\|\Psi(|X|)\|_2 < \infty$$

because $\theta(|X|)\in L^2(\boldsymbol{R}^m)$.

Now, $\Omega^{n,j}(X')\Psi(|X|)$ may be interpreted in two different forms:

(1) $$\Omega^{n,j}(X')\Psi(|X|) = \big(\text{P.V. } K^{n,j} * \eta(|X|)\big)^{\wedge},$$

where $\eta(|X|)$ is radial and $\{\eta(|X|)\}^{\wedge} = \gamma_n^{-1}\Psi(|X|)$,

(2) or simply as $\big(K_1^{n,j}(X)\big)^{\wedge}$.

On the other hand, $\eta(|X|)\in L^2$ since $\Psi(|X|)$ also does. Consider

$$\sum_{j=1}^{L} \text{P.V. } K^{n,j} * K^{n,j} \in L^p \quad \text{for all } p>1,$$

since $K_1^{n,j}\in L^p$, for all $p>1$, $j=1,\ldots,L$.

Now according to (1)

$$\sum_{j=1}^{L} \text{P.V. } K_1^{n,j} * K_1^{n,j} = \sum_{j=1}^{L} K^{n,j}K^{n,j}\big(\eta(|X|)\big) = \gamma_n^2(L/|\Sigma|)\,\eta(|X|)$$

a.e. which shows that $\eta(|X|)\in L^p$, for all $p>1$.

Our next step is to show that $\eta(|X|)$ belongs also to L^1. Consider now a C_0^∞ and radial function $\delta(|X|)$ supported on the sphere having radius $\frac{1}{4}$ centered at the origin. According to the smoothness of $K^{n,j}(X) = |X|^{-m}\Omega^{n,j}(X)$ outside the origin we have

(1.3.6) $$|K_1^{n,j}(X) - \text{P.V. } K^{n,j} * \delta| \leqslant C/(1+|X|^{m+1}), \qquad j=1,\ldots,L.$$

According to Lemma 1.1 we have also

(1.3.7) $$\{\text{P.V. } K^{n,j} * \delta\}^{\wedge} = \gamma_n\,\Omega^{n,j}(X')\cdot\widehat{\delta(|X|)}$$

and

$$\{\gamma_n\,\Omega^{n,j}(X')\hat{\delta}(|X|)\}^{\vee} = z(|X|)\,\Omega^{n,j}(-X'),$$

where the symbol $^{\vee}$ means the Fourier antitransform. The sign — in the argument of $\Omega^{n,j}(-X')$ appears due to the fact that the inversion formula is

$$\int_{R^m} \exp\big(2\pi i\langle X, Y\rangle\big)\hat{f}(Y)\,dY$$

and Lemma 1.1 has been proved for

$$\int_{R^m} \exp\big(-2\pi i\langle X, Y\rangle\big)f(Y)\,dY.$$

According to (1.3.7) we have

(1.3.8) $$\int_{N<|X|<M} (K^{n,j} * \delta)\,dX = \int_{N<|X|<M} \Omega^{n,j}(-X')z(|X|)\,dX = 0.$$

This last inequality follows because $z(|X|)$ is radial and $\Omega^{n,j}(-X')$ has mean value 0 over the unit sphere.

From (1.3.8) it follows

(1.3.9) $$\int_{N<|X|<M} \{K_1^{n,j}(X) - \text{P.V. } K^{n,j} * \delta\}\,dX = 0, \qquad j=1,\ldots,L$$

for every pair N, M such that $0\leqslant N\leqslant M<\infty$, but also

(1.3.10) $$K_1^{n,j}(X) - \text{P.V. } K^{n,j} * \delta = K^{n,j}\big(\eta(|X|) - \delta(|X|)\big).$$

Taking into account (1.3.6) we see that the following integral is finite:

(1.3.11)

$$\int_{R^m} \big|K^{n,j}\big(\eta(|X|)-\delta(|X|)\big)\big|\cdot\big(1+\log^+|X|+\log^+\big|K^{n,j}\big(\eta(|X|)-\delta(|X|)\big)\big|\big)\,dX.$$

This together with (1.3.9) shows that

(1.3.12) $$K^{n,j}K^{n,j}\big(\eta(|X|)-\delta(|X|)\big)\in L^1, \qquad j=1,\ldots,L.$$

(See theorems 5 and 7 in [2], p. 103, and 108, respectively).

Finally, since

$$\sum_{j=1}^{L} K^{n,j}K^{n,j}\big(\eta(|X|)-\delta(|X|)\big) = \gamma_n^2(L/|\Sigma|)\cdot\big(\eta(|X|)-\delta(|X|)\big)$$

it follows that $\eta(|X|)\in L^1$.

Now we take $\eta(|X|)$ to be $k(|X|)$ and the lemma is proved.

1.4. End of the proof of Theorem 1. According to Lemma 1.3:

$$K_1(X) = \text{P.V. } \int_{R^m} K(X-Y)\,k(|Y|)\,dY.$$

Also according to the homogeneity of $K(X)$

(1.4.1) $$\varepsilon^{-m}K_1(\varepsilon^{-1}X) = K_\varepsilon(X).$$

Taking the Fourier transform we get

(1.4.2) $$\hat{K}_1(X) = \gamma_n\,\Omega(X')\cdot\hat{k} = \Omega(X')\gamma_n\widehat{k(|X|)}$$

also

$$\hat{K}_1(X) = \Omega(X')\Psi(|X|),$$

where $\Psi(|X|) = \gamma_n\hat{k}(|X|)$.

According to (1.4.2) and (1.4.1) we have

(1.4.3) $$\hat{K}_\varepsilon(X) = \Omega(X')\Psi(\varepsilon|X|) = \Omega(X')\gamma_n\hat{k}(\varepsilon|X|).$$

Antitransforming we have

$$K_\varepsilon(X) = \text{P.V. } \int_{R^m} K(X-Y)\,\varepsilon^{-m}k(\varepsilon^{-1}|Y|)\,dY \quad \text{a.e.}$$

This finishes the proof of Theorem 1.

1.5. Corollary. *If $g(|X|)$ is a radial function defined on $\boldsymbol{R}^m$ such that*

(i) $$\int_{|X|\leqslant 1} |g| \log^+ |g| \, dX < \infty,$$

(ii) $$\int_{|X|>1} \big|g(|X|)-|X|^{-m}\big| \{1+\log^+|X|+\log^+ \big|g(|X|)-|X|^{-m}\big|\} \, dX < \infty.$$

Then, for any spherical harmonic $\Omega(X')$ we have

$$g(Y)\Omega(Y') = \text{P.V.} \int_{\boldsymbol{R}^m} \Omega(X')|X|^{-m}\theta(|X-Y|)\, dX,$$

where $\theta(|X|)$ is a radial function belonging to L^1. $\theta(|X|)$ is the same for any spherical harmonic of the same degree.

Proof. Without loss of generality we may assume $\|\Omega\|_{2,\Sigma} = 1$. Take $K_1(X) = \Omega(X')|X|^{-m}$ if $|X|>1$ and zero otherwise. Now $\Omega(X')g(X)-K_1(X)$ verifies

(1.5.1) $$\int_{N<|X|<M} \{\Omega(X')g(|X|)-K_1(X)\}\, dX = 0, \quad 0 \leqslant N \leqslant M < \infty.$$

According to (ii) we also have

(1.5.2) $$\int_{\boldsymbol{R}^m} \big|\Omega(X')g(|X|)- - K_1(X)\big| \big(1+\log^+|X|+\log^+\big|\Omega(X')g(|X|)-K_1(X)\big|\big)\, dX < \infty.$$

And if $\Omega^{n,j}(X')$, $j = 1, \ldots, L$, is a complete system of spherical harmonics of degree n normalized by the conditions

(1.5.3) $$\int_{\Sigma} \Omega^{n,i}(X')\,\Omega^{n,j}(X')\, d\sigma = \delta_{ij},$$

where $\Omega^{n,1} = \Omega$, then we have for $\Omega^{n,j}(X')g(|X|)-K_1^{n,j}(X)$, $j = 1, \ldots, L$, relations (1.5.1) and (1.5.2). According to Lemma 1.1 and to Theorem 1

(1.5.4) $$\{\Omega^{n,j}(X')g(|X|)-K_1^{n,j}(X)\}^{\wedge} = \Omega^{n,j}(X')\{\Psi(|X|)-\gamma_n k(|X|)\},$$

where $k(|X|)$ is the "conjugate" of $K_1^{n,j}$, $j = 1, \ldots, L$. On the other hand, according to (1.5.1) and (1.5.2)

$$\text{P.V. } K^{n,j} * \big(\Omega^{n,j}(X')g(|X|)-K_1^{n,j}(X)\big) \in L^1, \quad j = 1, \ldots, L,$$

which says that

$$\gamma_n \sum_{j=1}^{L} \{\Omega^{n,j}(X')\,\Omega^{n,j}(X')\}\big(\Psi(|X|)-\gamma_n k(|X|)\big) = \gamma_n^3 (L/|\Sigma|)\big(\Psi(|X|)-\gamma_n k(|X|)\big)$$

is the Fourier transform of a function in L^1, and since $k(|X|)$ is the Fourier transform of a function in L^1, $\Psi(|X|)$ is also the Fourier transform of a function in L^1. Taking the Fourier antitransform of $\gamma_n^{-1}\Psi(|X|)$ we get the desired result.

2. Proof of Theorem 3. We start with the case L^p, with p **M** 1.

2.1. Let us consider $\varphi \in C_0^\infty$ and

(2.1.1) $$\int_{\boldsymbol{R}^m} \varepsilon^{-m} k\big(\varepsilon^{-1}(X-Y)\big)\varphi(Y)\, dY = \int_{\boldsymbol{R}^m} k_\varepsilon(|X-Y|)\varphi(Y)\, dY.$$

Since φ may be represented a.e. as

(2.1.2) $$\gamma_n^{-2}(|\Sigma|/L)\sum_{j=1}^{L} K^{n,j}(K^{n,j}\varphi) = \varphi \quad \text{a.e.}$$

taking Fourier transforms and recalling that $k_\varepsilon \in L^2$ for each $\varepsilon > 0$, we get that

(2.1.3) $$\hat{k}_\varepsilon \cdot \hat{\varphi} = \sum_{j=1}^{L} \{\gamma_n^{-2}(|\Sigma|/L)K_\varepsilon^{n,j} * K^{n,j}(\varphi)\}^{\wedge}$$

so that

$$k_\varepsilon * \varphi = (|\Sigma|/\gamma_n^2 \cdot L)\sum_{j=1}^{L} K_\varepsilon^{n,j} * K^{n,j}(\varphi) \quad \text{a.e.}$$

Recalling that $k_\varepsilon \in L^p$ for all p such that $1 \leqslant p < \infty$ and also that $K_\varepsilon^{n,j} \in L^p$ for all p such that $1 < p < \infty$, $j = 1, \ldots, L$, and the fact that $K^{n,j}(\varphi)$ is a continuous operator in L^p, $p > 1$, by density we prove (2.1.3) for all $f \in L^p$, $p > 1$. Hölder's inequality shows that for fixed $\varepsilon > 0$, $k_\varepsilon * f$ and $K_\varepsilon^{n,j} * K^{n,j}(f)$ are both continuous functions of X, provided that $f \in L^p$, $p > 1$, in fact

(2.1.4)
$$|(k_\varepsilon * f)(X+h)-(k_\varepsilon * f)(X)| \leqslant \|f\|_p \cdot \Big(\int_{\boldsymbol{R}^m} |k_\varepsilon(X+h-Y)-k_\varepsilon(X-Y)|^q\, dY\Big)^{1/q},$$
$$\big|\big(K_\varepsilon^{n,j} * K^{n,j}(f)\big)(X+h)-\big(K_\varepsilon^{n,j} * K^{n,j}(f)\big)(X)\big| < C_p\|f\|_p \cdot \Big(\int_{\boldsymbol{R}^m} |K_\varepsilon^{n,j}(X+h-Y)-K_\varepsilon^{n,j}(X-Y)|^q\, dY\Big)^{1/q}.$$

Therefore (2.1.3) holds everywhere, provided that $f \in L^p$, $p > 1$. Now, since $K_\varepsilon^{n,j}$ $(j = 1, \ldots, L)$ are truncated singular kernels with C^∞ characteristic we have

(2.1.5) $$\|\sup |K_\varepsilon^{n,j} * K^{n,j} f|\,\|_p \leqslant C_p'\|K^{n,j} f\|_p < C_p' \cdot C_p \cdot \|f\|_p, \quad \text{for } p > 1;$$

according to the representation (2.1.3) we have

(2.1.6)
$$\|f^*\|_p < C_p'' \|f\|_p, \quad p > 1,$$

where $f^* = \sup_{\varepsilon} |k_\varepsilon * f|$.

2.2. Case when f and $|f|\log^+|f|$ belong to L^1.

Without loss of generality we may assume that $f \geqslant 0$. For a given $\lambda > 0$ we are going to make the Calderón-Zygmund decomposition to f (see [2], chapter I).

(2.2.1) There exists a sequence of non-overlapping cubes $\{Q_d\}$ having edges parallel to the coordinate axes, such that

(2.2.2)
$$\lambda < (1/|Q_d|) \int_{Q_d} f\, dX < b_m \cdot \lambda,$$

where b_m depends on the dimension m only.

If we call $D_\lambda = \bigcup_1^\infty Q_d$, then $|f| \leqslant \lambda$ a.e. in $\boldsymbol{R}^m - D_\lambda$. Now we decompose $f = f_1 + f_2$, where

(2.2.3)
$$f_1 = \begin{cases} f & \text{in } \boldsymbol{R}^m - D_\lambda, \\ \sum_{d=1}^\infty \Phi_d(X)\,(1/|Q_d|) \int_{Q_d} f\, dY & \text{in } D_\lambda, \end{cases}$$

$\Phi_d(X)$ denotes the characteristic function of Q_d;

(2.2.4)
$$f_2 = \begin{cases} 0 & \text{in } \boldsymbol{R}^m - D_\lambda, \\ \sum_{d=1}^\infty \Phi_d(X)\{f(X) - (1/|Q_d|) \int_{Q_d} f\, dY\} = \sum_{d=1}^\infty \Psi_d(X) & \text{in } D_\lambda. \end{cases}$$

Given a number $N > 6m^{1/2}$, m is the dimension, we will denote by $ND_\lambda = \bigcup_1^\infty NQ_d$, where NQ_d denotes the cube centered at the same point Y_d, where Q_d is centered and having edges parallel to the coordinated axes whose sizes are N times those of Q_d.

(2.2.5) We are going to show that $K^{n,j}(f_2)$, $j = 1, \ldots, L$, belong to L^1.

In fact, consider $K_\varepsilon^{n,j}(f_2)(X)$ in $\boldsymbol{R}^m - ND_\lambda$

(2.2.6)
$$K_\varepsilon^{n,j}(f_2)(X) = \sum_{S_\varepsilon(X) \cap Q_d = 0} K_\varepsilon^{n,j}(\Psi_d)(X) + \sum_{\tilde S_\varepsilon(X) \cap Q_d \neq 0} K_\varepsilon^{n,j}(\Psi_d)(X),$$

$S_\varepsilon(X)$ denotes the sphere of radius $\varepsilon > 0$ centered at X, and $\tilde S_\varepsilon(X)$ denotes its frontier.

For those cubes Q_d such that $\tilde S_\varepsilon(X) \cap Q_d \neq 0$ we have (denoting by t_d half of the length of the edge of Q_d):

(2.2.6')
$$\operatorname{dist}(X, Y_d) > N t_d$$

and also $\varepsilon > \operatorname{dist}(X, Q_d) > \varepsilon - 2m^{1/2} t_d$ which gives that $\varepsilon > 4m^{1/2} t_d$ since $N t_d < \operatorname{dist}(X, Y_d) < \varepsilon + 2m^{1/2} t_d$.

Therefore if $\tilde S_\varepsilon(X) \cap Q_d \neq 0$,

(2.2.7)
$$Q_d \subset S_{4\varepsilon}(X) - S_{\varepsilon/2}(X).$$

Now according to the homogeneity of $K^{n,j}(Y)$, we have

(2.2.8)
$$\Big|\sum_{\tilde S_\varepsilon(X) \cap Q_d \neq 0} K^{n,j}(\Psi_d)(X)\Big| \leqslant C\,\varepsilon^{-m} \int_{\varepsilon/2 < |X - Y| < 4\varepsilon} |f_2(Y)|\, dY.$$

Since $f_2 = 0$ in $\boldsymbol{R}^m - ND_\lambda$; then

(2.2.9)
$$\lim_{\varepsilon \to 0} \Big|\sum_{\tilde S_\varepsilon(X) \cap Q_d \neq 0} K_\varepsilon^{n,j}(\Psi_d)(X)\Big| = 0 \quad \text{a.e. in } \boldsymbol{R}^m - ND_\lambda.$$

According to the smoothness of the kernels $K^{n,j}(Y)$, $j = 1, \ldots, L$, and the fact that $\Psi_d(Y)$ has mean value zero, $d = 1, 2, \ldots$

(2.2.10)
$$|K^{n,j}(\Psi_d)(X)| = \Big|\int_{\boldsymbol{R}^m} \{K^{n,j}(X - Y) - K^{n,j}(X - Y_d)\}\Psi_d(Y)\, dY\Big| \leqslant \int_{\boldsymbol{R}^m} |K^{n,j}(X - Y) - K^{n,j}(X - Y_d)|\,|\Psi_d(Y)|\, dY.$$

Also
$$\int_{\boldsymbol{R}^m - NQ_d} |K^{n,j}(\Psi_d)(X)| \leqslant C\|\Psi_d\|_1.$$

Calling
$$\overline{\Psi}(X) = \sum_{d=1}^\infty \int_{\boldsymbol{R}^m} |K^{n,j}(X - Y) - K^{n,j}(X - Y_d)|\,|\Psi_d(Y)|\, dY,$$

we have

(2.2.11)
$$\int_{\boldsymbol{R}^m - ND_\lambda} \overline{\Psi}(X)\, dX < 2C\|f\|_1.$$

Now since $K^{n,j}(Y)$ is a singular kernel with C^∞ characteristic, $\lim_{\varepsilon \to 0} K_\varepsilon^{n,j}(f_2) = K^{n,j}(f)$ a.e. and according to (2.2.9)
$$|K^{n,j}(f_2)(X)| \leqslant \overline{\Psi}(X) \quad \text{a.e. in } \boldsymbol{R}^m - ND_\lambda$$

so that

(2.2.12)
$$\int_{\boldsymbol{R}^m - ND_\lambda} \overline{\Psi}(X)\, dX \geqslant \int_{\boldsymbol{R}^m - ND_\lambda} |K^{n,j}(f_2)|\, dX.$$

(2.2.13) The integrability of $K^{n,j}(f_2)$ in ND_λ will be given by the following formula

(2.2.14)
$$\int_S |K^{n,j}(g)|\, dX \leqslant C_1|S| + C_2 \int_{\boldsymbol{R}^m} |g|\, dX + C_3 \int_{\boldsymbol{R}^m} |g|\log^+|g|\, dX,$$

where the constants C_1, C_2, C_3 do not depend on the function g or on the set S.

This is a consequence of the fact that $K^{n,j}(g)$ is weak type (1, 1) and strong type (2, 2).

According to the following inequalities

$$\text{(2.2.15)}\quad \begin{aligned} &\text{(a)}\quad |g_1+g_2|\log^+|g_1+g_2| \\ &\qquad\qquad \leqslant C\left(|g_1|+|g_2|+|g_1|\log^+|g_1|+|g_2|\log^+|g_2|\right), \\ &\text{(b)}\quad \int_{Q_d}\Big((1/|Q_d|)\int_{Q_d}|f|\,dY\Big)\log^+\Big((1/|Q_d|)\int_{Q_d}|f|\,dY\Big)\,dX \\ &\qquad\qquad \leqslant \int_{Q_d}|f|\log^+|f|\,dX, \\ &\text{(c)}\quad |ND_\lambda| \leqslant N^m|D_\lambda| \leqslant (N^m/\lambda)\int_{R^m} f\,dX, \end{aligned}$$

and from (2.2.14) we get

$$\text{(2.2.16)}\quad \int_{ND_\lambda}|K^{n,j}(f_2)|\,dX \leqslant C_1'\lambda^{-1}\int_{R^m} f\,dX + C_2'\int_{R^m} f\,dX + C_3'\cdot\int_{R^m} f\log^+ f\,dX.$$

This, together with (2.2.12) and (2.2.11) gives

$$\text{(2.2.17)}\quad \int_{R^m}|K^{n,j}(f_2)|\,dX \leqslant C_2''\lambda^{-1}\int_{R^m} f\,dX + C_2''\int_{R^m} f\,dX + C_3''\int_{R^m} f\log^+ f\,dX,$$

$j = 1, \ldots, L$, and the constants do not depend on f or λ.

Our next step is to represent $k_\varepsilon * f_2$ as

$$\text{(2.2.18)}\quad k_\varepsilon * f_2 = \gamma_n^{-2}(|\Sigma|/L)\sum_{j=1}^{L} K_\varepsilon^{n,j} * K^{n,j}(f_2)$$

everywhere.

For fixed $\varepsilon > 0$, $k_\varepsilon(X)$ and $K_\varepsilon^{n,j}(X)$, $j = 1, \ldots, L$, belong to L^2 and according to the fact that $\|\hat{f}_2\|_\infty < \|f_2\|_1$ and also $\|K^{n,j}(f_2)^\wedge\|_\infty < \|K^{n,j}(f_2)\|_1$ we see that both sides are in L^2. On the other hand, it is easy to verify that the Fourier transforms of both sides are identical, so that (2.2.18) is verified a.e. But since for fixed ε both sides are continuous functions, (2.2.18) follows. (The continuity of the right-hand member follows from the fact that $\|K_\varepsilon^{n,j}\|_\infty < A_\varepsilon$, and $K^{n,j}(f_2)\in L^1$).

Now using the fact that the operation $\sup_\varepsilon|K_\varepsilon^{n,j} * g|$ is weak type (1-1) $j = 1, \ldots, L$, we get that

$$\text{(2.2.19)}\quad \begin{aligned} |E(f_2^* > \lambda)| &< (1/\lambda)C\sum_{j=1}^{L}\|K^{n,j}(f_2)\|_1 \\ &\leqslant (\bar{C}_1/\lambda^2)\int_{R^m} f\,dX + (\bar{C}_2/\lambda)\int_{R^m} f\,dX + (\bar{C}_3/\lambda)\int_{R^m} f\log^+ f\,dX. \end{aligned}$$

Let us return to f_1; since $f_1\in L^2$, we have

$$\text{(2.2.20)}\quad |E(f_1^* > \lambda)| < (C_2/\lambda^2)\int_{R^m} f_1^2\,dX \leqslant [(C_2+b_m)/\lambda]\int_{R^m} f\,dX.$$

This last inequality is valid because $f_1 < \lambda$ in $R^m - D_\lambda$ and $f_1 < b_m\lambda$ in D_λ, and also

$$\text{(a)}\quad 0 \leqslant f_1 = f \text{ in } R^m - D_\lambda, \qquad \text{(b)}\quad \int_{Q_d} f_1\,dX = \int_{Q_d} f\,dX.$$

This finishes the proof of the inequalities involving f^*. The pointwise result follows from the fact that if $f\in C_0^\infty$, $k_\varepsilon * f$ converges everywhere as $\varepsilon\to 0$; which can be readily checked because in this case the following two properties are enough:

(a) $k(|X|)\in L^1$,

(b) $k_\varepsilon(|X|) = \varepsilon^{-m}k(\varepsilon^{-1}|X|)$.

This, together with the maximal inequalities, gives the pointwise a.e. result in the different classes.

3. Proof of Theorem 2.

3.1. Let's consider $f\in L^p$, $p > 1$, and let $\Omega(X')$ be a spherical harmonic of degree n. $K_\varepsilon(X) = \Omega(X')|X|^{-m}$ if $|X|\geqslant\varepsilon$ and zero otherwise.

The following representation holds everywhere

$$\text{(3.1.1)}\quad \int_{R^m} K_\varepsilon(X-Y)f(Y)\,dY = \varepsilon^{-m}\int_{R^m} k(\varepsilon^{-1}|X-Y|)K(f)\,dY,$$

where $k\in L^1$ and $K(k_\varepsilon) = K_\varepsilon(X)$. For fixed $\varepsilon > 0$ both sides of (3.1.1) are continuous functions. The proof of this representation follows very closely that of (2.1.2) and (2.1.3). Taking Fourier transform in (3.1.1) we see that both sides are identical provided that $f\in L^2$. Now using the fact that k_ε and K_ε belong to L^p, $1 < p < \infty$, for fixed $\varepsilon > 0$:

$$\text{(3.1.2)}\quad \begin{aligned} &|(K_\varepsilon * f)(X)| \leqslant \|K_\varepsilon\|_q\cdot\|f\|_p, \\ &\big|(k_\varepsilon * K(f))(X)\big| \leqslant \|k_\varepsilon\|_q\cdot\|K(f)\|_p \leqslant \|k_\varepsilon\|_q\cdot C_p\|f\|_p. \end{aligned}$$

Now by a density argument together with (3.1.2) we obtain the result for L^p, $p > 1$.

3.2. Consider now a radial function $\delta(|X|)$ such that

$$\text{(3.2.1)}\quad \begin{aligned} &\text{(a)}\quad \delta(|X|)\in C_0^\infty, \quad \delta(|X|)\geqslant 0, \\ &\text{(b)}\quad \operatorname{supp}\delta(|X|)\subset\{X/|X| < \tfrac{1}{4}\}, \\ &\text{(c)}\quad \int_{R^m}\delta(|X|)\,dX = 1. \end{aligned}$$

Let f belong to L^p, $p > 1$. Now we have

(3.2.2) $$\varepsilon^{-m}\int_{R^m} k(\varepsilon^{-1}|Y|)f(X-Y)\,dY$$
$$= \varepsilon^{-m}\int_{R^m}\{k(\varepsilon^{-1}|Y|)-\delta(\varepsilon^{-1}|Y|)\}f(X-Y)\,dY+$$
$$+\varepsilon^{-m}\int_{R^m}\delta(\varepsilon^{-1}|Y|)\,(f(X-Y)\,dY.$$

The first term of the right-hand member of (3.2.2) may be written as

(3.2.3) $$\int_{R^m}\big(k(|Y|)-\delta(|Y|)\big)f(X-\varepsilon Y)\,dY.$$

Without loss of generality we may assume that $\|\Omega(Y')\|_{2,\Sigma} = 1$. Now we construct a complete system of normalized spherical harmonics of degree n, $\Omega^{n,j}(X')$, $j = 1, \ldots, L$, such that $\Omega^{n,1}(X') = \Omega(X')$. That is

(3.2.4) $$\int_{\Sigma}\Omega^{n,i}(Y')\,\Omega^{n,j}(Y')\,d\sigma = \delta_{ij}.$$

According to (1.2.4)

(3.2.5) $$k(|Y|)-\delta(|Y|) = \gamma_n^{-2}(|\Sigma|/L)\sum_{j=1}^{L}K^{n,j}\big(K^{n,j}(k-\delta)\big) \quad \text{a.e.}$$

and also

(3.2.6) $$k(|Y|)-\delta(|Y|) = \gamma_n^{-2}(|\Sigma|/L)\sum_{j=1}^{L}K^{n,j}\big(K_1^{n,j}-K^{n,j}(\delta)\big),$$

since $K^{n,j}(k) = K_1^{n,j}(X)$.

Since $K^{n,j}(\delta)(X) = \Omega^{n,j}(-X')\,\varphi(|X|)$, $j = 1, \ldots, L$ (see Lemma 1.3, (1.3.7)), we have

(3.2.7) $$\int_{N<|Y|<M}\{K_1^{n,j}(Y)-K^{n,j}(\delta)(Y)\}\,dY = 0$$

for every pair N, M such that $0 \leqslant N \leqslant M < \infty$.

3.3. LEMMA. *Call $\Psi_k^{n,j}$ to the function defined in the following way:*

If $k \geqslant 1$:

$$\Psi_k^{n,j} = \begin{cases} K_1^{n,j}(X)-K^{n,j}(\delta)(X) & \text{if } 2^k \leqslant |X| < 2^{k+1},\\ 0 & \text{otherwise.}\end{cases}$$

If $k = 0$;

$$\Psi_0^{n,j} = \begin{cases} K_1^{n,j}(X)-K^{n,j}(\delta)(X) & \text{if } |X| \leqslant 2,\\ 0 & \text{otherwise.}\end{cases}$$

Then $K^{n,j}(\Psi_k^{n,j})$ verifies the following estimates:

(i) *If $|X| \geqslant 2^{k+3}$,*

$$|K^{n,j}(\Psi_k^{n,j})(X)| \leqslant C_0 2^{-km}(1+2^{-(m+1)k}|X|^{m+1})^{-1}\|\Psi_k^{n,j}\|_1.$$

(ii) *If $|X| < 2^{k+3}$ and $p > 1$,*

$$\int_{0<|X|<2^{k+3}}|K^{n,j}(\Psi_k^{n,j})|^p\,dX \leqslant C_p^p\|\Psi_k^{n,j}\|_p^p.$$

(iii) *If $2^k < |X| < 2^{k+1}$,*

$$|\Psi_k^{n,j}(X)| < C_1(1+|X|^{m+1})^{-1}.$$

The constants C_0, C_p, C_1 do not depend on k or j.

Proof. Suppose $|X| \geqslant 2^{k+3}$; then the following integral is absolutely convergent:

(3.3.1) $$\Big|\int_{R^m}K^{n,j}(X-Y)\,\Psi_k^{n,j}(Y)\,dY\Big|$$
$$= \Big|\int_{R^m}\big(K^{n,j}(X-Y)-K(X)\big)\,\Psi^{n,j}(Y)\,dY\Big|$$
$$\leqslant \int_{R^m}|K^{n,j}(X-Y)-K^{n,j}(X)|\,|\Psi_k^{n,j}(Y)|\,dY.$$

This follows from the fact that $\Psi_k^{n,j}(Y)$ has mean value zero (see (3.2.7)). On the other hand,

(3.3.2) $$|K^{n,j}(X-Y)-K^{n,j}(X)| \leqslant C'2^{k+1}|X|^{-(m+1)}$$

if $2^k < |Y| < 2^{k+1}$ and $|X| \geqslant 2^{k+3}$.

This follows from the homogeneity of $K^{n,j}$ and the smoothness of $\Omega^{n,j}(X')$. C' can be found to be independent of j also, by taking the maximum of all C'_j (corresponding to j, $1 \leqslant j \leqslant L$). On the other hand,

(3.3.3) $$2^k|X|^{-(m+1)} \leqslant C''2^{-km}(1+2^{-k(m+1)}|X|^{m+1})^{-1}$$

if $|X| \geqslant 2^k$ for a suitable constant C''.

Now (3.3.1), (3.3.2) and (3.3.3) give (i).

(ii) follows from the continuity of the mapping $K^{n,j}(f)$ from L^p into L^p for $p > 1$, $j = 1, \ldots, L$.

(iii) follows from the fact that

$$|K_1^{n,j}(X)-K^{n,j}(\delta)(X)| < C|X|^{-(m+1)} \quad \text{if } |X| \geqslant 2.$$

If $|X| \leqslant 2$, $|K_1^{n,j}(X)| < D_1$ and also $|K^{n,j}(\delta)(X)| \leqslant D_2$ since $\delta \in C_0^\infty$. This gives (iii) for a suitable constant C_1.

3.4. Estimate for $\int_{R^m} |k(|Y|) - \delta(|Y|)|\, |f(X - \varepsilon Y)|\, dY$.

It can be readily checked that

$$(3.4.1)\qquad |k(|Y|) - \delta(|Y|)| \leqslant \gamma_n^2(|\Sigma|/L) \sum_{j=1}^{L} \sum_{k=0}^{\infty} |K^{n,j}(\Psi_k^{n,j})|.$$

Then

$$(3.4.2)\qquad \int_{R^m} |k(|Y|) - \delta(|Y|)|\, |f(X-\varepsilon Y)|\, dY \leqslant C \sum_{j=1}^{L} \sum_{k=0}^{\infty} \int_{R^m} |K^{n,j}(\Psi_k^{n,j})|\, |f(X-\varepsilon Y)|\, dY.$$

Taking into account Lemma 3.3, we have

$$(3.4.3)\qquad \sum_{j=1}^{L} \int_{R^m} |K^{n,j}(\Psi_k^{n,j})|\, |f(X-\varepsilon Y)|\, dY \leqslant C' \sum_{j=1}^{L} \|\Psi_k^{n,j}\|_1 \int_{R^m} 2^{-km}(1+2^{-k(m+1)}|Y|^{m+1})^{-1} |f(X-\varepsilon Y)|\, dY + C_q'' \sum_{j=1}^{L} \|\Psi_k^{n,j}\|_q \Big(\int_{0<|X|<2^{k+3}} |f(X-\varepsilon Y)|^p\, dY\Big)^{1/p},$$

where $1/p + 1/q = 1$.

On one hand, we have, for $k \neq 0$,

$$(3.4.4)\qquad |\Psi_k^{n,j}(X)|^q \leqslant \{C_1/(1+|X|^{m+1})\}^q$$

if $2^k < |X| < 2^{k+1}$.

Therefore

$$\|\Psi_k^{n,j}\|_q \leqslant C_q'''\, 2^{-k(m+1)}\, 2^{(k+1)m/q}.$$

On the other hand,

$$(3.4.5)\qquad \Big(\int_{0\leqslant|Y|\leqslant 2^{k+3}} |f(X-\varepsilon Y)|^p\, dY\Big)^{1/p} = 2^{km/p}\, 2^{3m/p} \Big\{ (\varepsilon 2^{k+3})^{-m} \int_{0<|Y|<\varepsilon 2^{k+3}} |f(X-Y)|^p\, dY \Big\}^{1/p}.$$

If $k = 0$, we have simply

$$(3.4.6)\qquad C_q'' \|\Psi_0^{n,j}\|_q \leqslant C_q'''.$$

Finally, according to (3.4.3)-(3.4.6),

$$(3.4.7)\qquad \int_{R^m} |k(|Y|) - \delta(|Y|)|\, |f(X-\varepsilon Y)|\, dY \leqslant C_{p,q} \sum_{k=0}^{\infty} a_k \Big\{ (\varepsilon 2^{k+3})^{-m} \int_{0<|Y|<\varepsilon 2^{k+3}} |f(X-Y)|^p\, dY \Big\}^{1/p} + b_k D^{-1} \int_{R^m} (\varepsilon 2^k)^{-m} \big(1 + (\varepsilon 2^k)^{-(m+1)} |Y|^{m+1}\big)^{-1} |f(X-Y)|\, dY,$$

where $a_k < \text{const}\cdot L \cdot 2^{-k}$; $b_k < \text{const}\cdot D \cdot \sum_{j=1}^{L} \|\Psi_k^{n,j}\|_1$; $D = \int_{R^m} (1+|Y|^{m+1})^{-1}\, dY$ and $C_{p,q}$ chosen so that

$$(3.4.8)\qquad \sum_{k=0}^{\infty} a_k + b_k = 1.$$

Now we have, according to the convexity of u^p, $p > 1$,

$$(3.4.9)\qquad \Big\{ \int_{R^m} |k(|Y|) - \delta(|Y|)|\, |f(X-\varepsilon Y)|\, dY \Big\}^p \leqslant C_{p,q}^p \sum_{k=0}^{\infty} a_k (\varepsilon 2^{k+3})^{-m} \int_{|Y|<\varepsilon 2^{k+3}} |f(X-Y)|^p\, dY + b_k \Big\{ D^{-1} \int_{R^m} (\varepsilon 2^k)^{-m} \big(1+(\varepsilon 2^k)^{-m-1}|Y|^{m+1}\big)^{-1} |f(X-Y)|\, dY \Big\}^p$$
$$\leqslant C_{p,q}^p \sum_{k=0}^{\infty} a_k (\varepsilon 2^{k+3})^{-m} \int_{|Y|<\varepsilon 2^{k+3}} |f(X-Y)|^p\, dY + b_k D^{-1} \int_{R^m} (\varepsilon 2^k)^{-m} \big(1+(\varepsilon 2^k)^{-m-1}|Y|^{m+1}\big)^{-1} |f(X-Y)|^p\, dY.$$

Calling $\Lambda|f|^p(X) = \sup_{\varepsilon} \varepsilon^{-m} \int_{|Y|<\varepsilon} |f(X-Y)|^p\, dY$ and according to the fact that

$$(3.4.10)\qquad \Lambda|f|^p(X) \geqslant D^{-1} \int_{R^m} (\varepsilon 2^k)^{-m} \big(1+(\varepsilon 2^k)^{-m-1}|Y|^{m+1}\big)^{-1} |f(X-Y)|^p\, dY$$

(since it is essentially a Poisson integral), we have that the last member of inequality (3.4.9) is dominated by

$$C_{p,q}^p \Lambda|f|^p(X) \quad \text{since} \sum_{k=0}^{\infty} a_k + b_k = 1,$$

On the other hand, since $\delta(|X|)\in C_0^\infty$, $\int_{R^m}\delta\,dX = 1$, $\delta \geqslant 0$, we have

(3.4.11) $$\Big(\varepsilon^{-m}\int_{R^m}\delta(\varepsilon^{-1}|Y|)|f(X-Y)|\,dY\Big)^p \leqslant \varepsilon^{-m}\int_{R^m}\delta(\varepsilon^{-1}|Y|)|f(X-Y)|^p\,dY \leqslant CA|f|^p(X).$$

Finally we get

(3.4.12) $$\Big(\sup_\varepsilon\Big|\int_{R^m}\varepsilon^{-m}k(|Y|)f(X-Y)\,dY\Big|\Big)^p \leqslant 2^p(C_{p,q}^p+C)A|f|^p(X).$$

This, together with representation (3.1.1), gives part (ii) of Theorem 2.

3.5. Suppose that $\lim_{\varepsilon\to 0}\varepsilon^{-m}\int_{|Y|<\varepsilon}|f(X)-f(X-Y)|^p\,dY = 0$. Then

$$M_X = \sup_\varepsilon \varepsilon^{-m}\int_{|Y|<\varepsilon}|f(X)-f(X-Y)|^p\,dY$$

is finite, since the ratio is bounded as $\varepsilon\to\infty$ when $f\in L^p$ (which is our case).

Consider

(3.5.1) $$\int_{R^m}k_\varepsilon(|Y|)f(X-Y)\,dY - f(X) = \int_{R^m}k_\varepsilon(|Y|)\{f(X-Y)-f(X)\}\,dY.$$

Now according to the estimates of the preceding Section 3.4, we have

(3.5.2) $$\Big(\Big|\int_{R^m}k_\varepsilon(|Y|)\{f(X-Y)-f(X)\}\,dY\Big|\Big)^p \leqslant 2^pC\varepsilon^{-m}\int_{R^m}\delta(\varepsilon^{-1}|Y|)|f(X-Y)-f(X)|^p\,dY +$$
$$+2^pC_{p,q}^p\sum_{k=0}^{\infty}a_k(\varepsilon 2^{k+3})^{-m}\int_{|Y|<\varepsilon 2^{k+3}}|f(X)-f(X-Y)|^p\,dY +$$
$$+2^pC_{p,q}^p\sum_{k=0}^{\infty}b_kD^{-1}\int_{R^m}(\varepsilon 2^k)^{-m}\big(1+(\varepsilon 2^k)^{-m-1}|Y|^{m+1}\big)^{-1}|f(X)-f(X-Y)|^p\,dY.$$

Given $\eta>0$ there exists k_0 such that

(3.5.3) $$2^pC_{p,q}^p\sum_{k=k_0}^{\infty}a_k+b_k < (\eta/M_X).$$

Let's observe that according to the properties of the Poisson kernel, for each k we have

(3.5.4) $$\lim_{\varepsilon\to 0}D^{-1}\int_{R^m}(\varepsilon 2^k)^{-m}\big(1+(\varepsilon 2^k)^{-m-1}|Y|^{m+1}\big)^{-1}|f(X)-f(X-Y)|^p\,dY = 0,$$

provided that

$$\lim_{\varepsilon\to 0}\varepsilon^{-m}\int_{|Y|<\varepsilon}|f(X)-f(X-Y)|^p\,dY = 0.$$

In the same form for δ_ε and for each k we have

(3.5.5)
(1) $$\varepsilon^{-m}\int_{R^m}\delta(\varepsilon^{-1}|Y|)|f(X-Y)-f(X)|^p\,dY\to 0 \quad\text{as } \varepsilon\to 0,$$
(2) $$(\varepsilon 2^{k+3})^{-m}\int_{|Y|<\varepsilon 2^{k+3}}|f(X-Y)-f(X)|^p\,dY\to 0 \quad\text{as } \varepsilon\to 0.$$

Therefore

(3.5.6) $$\overline{\lim}\Big(\Big|\int_{R^m}k_\varepsilon(|Y|)\{f(X-Y)-f(X)\}\,dY\Big|\Big)^p \leqslant \overline{\lim}\,2^pC\varepsilon^{-m}\int_{R^m}\delta(\varepsilon^{-1}|Y|)|f(X-Y)-f(X)|^p\,dY +$$
$$+2^pC_{p,q}^p\sum_{k=0}^{k_0}a_k\,\overline{\lim}\,(\varepsilon 2^{k+3})^{-m}\int_{|Y|<\varepsilon 2^{k+3}}|f(X-Y)-f(X)|^p\,dY +$$
$$+2^pC_{p,q}^p\sum_{k=0}^{k_0}b_k\,\overline{\lim}\,D^{-1}\int_{R^m}(\varepsilon 2^k)^{-m}|f(X-Y)-f(X)|^p\times$$
$$\times\big(1+(\varepsilon 2^k)^{-m-1}|Y|^{m+1}\big)^{-1}\,dY +$$
$$+2^pC_{p,q}^p\sum_{k=k_0}^{\infty}a_k\sup_\varepsilon(\varepsilon 2^{k+3})^{-m}\int_{|Y|<\varepsilon 2^{k+3}}|f(X-Y)-f(X)|^p\,dY +$$
$$+2^pC_{p,q}^p\sum_{k=k_0}^{\infty}b_k\sup_\varepsilon D^{-1}\int_{R^m}(\varepsilon 2^k)^{-m}|f(X-Y)-f(X)|^p\times$$
$$\times\big(1+(\varepsilon 2^k)^{-m-1}|Y|^{m+1}\big)^{-1}\,dY$$
$$\leqslant 2^p\eta.$$

This and representation (3.3.1) give part (i) of Theorem 2.

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Reçu par la Rédaction le 18. 2. 1970

STUDIA MATHEMATICA, T. XXXIX. (1971)

Polynomials and multilinear mappings in topological vector spaces

by

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Introduction. In our forthcoming papers we want to give a unified presentation of a theory of analytic functions defined in open subsets of a complex or real topological vector space E with values in a locally convex topological vector space F. As a special case we will get the theory developed in [1] and [8] for Banach spaces.

In this paper we gather the most important facts about polynomials and multilinear mappings which are basic for the further developement of the theory. Polynomials are the simplest analytic functions which are used to bild up locally any other analytic mapping (by expanding it into a series of homogeneous polynomials). Therefore their properties should be examined first.

We hope that some of the results of this paper may be interesting for functional analysis apart from their application in the theory of analytic functions.

In Section 1 we give several necessary and sufficient conditions for a polynomial or multilinear mapping to be continuous (Theorem 1). Also here the Banach-Steinhaus theorem is extended to homogeneous polynomials of fixed degree $k \geqslant 1$ (Theorem 2).

The natural domains of existence of analytic mappings are domains in complex spaces. It appears that in order to prove some facts concerning real analytic functions it is convenient to complexify the given function at first. The problem of complexification of real vector spaces, multilinear mappings and polynomials is treated in Section 2.

Section 3 is devoted to existence of Gateaux differentials and their continuity. This is of first importance when we ask whether a given function may be locally developed into a series of homogeneous polynomials.

While preparing this paper we have been much inspired by [1], [8] and [9].

1. Polynomials and multilinear mappings. In the sequel K denotes either the field of complex numbers $\boldsymbol{C}$ or the field of real numbers $\boldsymbol{R}$. Letters E, F, G will denote vector spaces over K. If the field is not strictly