THEOREM 2. *Let ξ be an rlf on $\mathscr{K}$, where conditions* (M), (N) *and* (P) *are assumed for all sequences $\{M_p^j\}$. Assume further that polynomials are multipliers for each K^m. For m a positive integer and $\varepsilon > 0$ there exists an integer and r, $M \in \mathscr{A}$ and functions $\{h_a\}$, $|a| \leqslant r$, such that*

(i) $\mu(M) \geqslant 1-\varepsilon$,

(ii) $h_a(w, \cdot)$ *is continuous on F^r for all $w \in \Omega$,*

(iii) $h_a(\cdot, t)$ *is measureable for $t \in F^r$,*

(iv) *for all $w \in M$, $\varphi \in K^m$*

$$[\xi(w)](\varphi) = \sum_{|a|\leqslant r+1} (-1)^{|a-1|} \int_{F^r} h_a(w, t) D^a\varphi(t)\,dt.$$

Proof. By Theorem 1, there exist functions f_a, $|a| \leqslant r$, such that

$$\xi(w) = \sum_{|a|\leqslant r} (-1)^{|a|} D^a [M_r^m f_a(w, \cdot)],$$

where for each w, $f_a(w, \cdot)$ is essentially bounded. If Lemma 1 is applied to each $f_a(w, \cdot)$, then

$$\xi(w) = \sum_{|a|\leqslant r} (-1)^{|a|} D^{a+1} \Big(\int_0^x M_p f_a(w, \cdot) \Big)$$

and each

$$h_a(w, x) = \int_0^x M_p(t) f_a(w, t)\,dt$$

is continuous in x and measureable in w by an application of Lemma 4.

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Reçu par la Rédaction le 15. 4. 1970

STUDIA MATHEMATICA, T. XXXIX. (1971)

Holomorphy types on a Banach space

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Contents

In discussing tensor products, bilinear mappings and linear mappings on a Banach space it has been found useful to distinguish between various sorts of mappings such as the compact, nuclear, integral mappings etc. (cf. Treves [17] and Grothendieck [2]). Since n-homogeneous polynomials are nothing more than symmetric n-linear mappings and a holomorphic function on a Banach space can be looked upon as a sequence of homogeneous polynomials which satisfy certain conditions, it is not surprising that one can define various subspaces of the space of all holomorphic functions so that the resulting structure is enriched. Such is the case in Nachbin and Gupta [15], where Malgrange's approximation theorem is generalized from the finite to the infinite-dimensional case. To describe a theory for a large class of subspaces Nachbin [13] introduced the concept of holomorphy type.

Motivated by Nachbin and Gupta [15] and Nachbin [13] we describe and study in this work various topological vector spaces of holomorphic functions.

In Section 1 we recall the definition of holomorphy type and of the spaces $(\mathscr{H}_\theta(E), \mathscr{T}_\theta)$. We define α-holomorphy type and the corresponding topological vectors spaces $(H_\theta(E), T_\theta)$.

In Section 2 we get a representation for a generating family of seminorms for $(H_\theta(E), T_\theta)$ and hence show the space is complete. The bornological topology of $(H_\theta(E), T_\theta)$, t_θ, turns out to be the finest locally convex topology on $H_\theta(E)$ which induces on each space of homogeneous polynomials its original norm and for which the Taylor series converges absolutely. We also discuss the relationship between $(\mathscr{H}_\theta(E), \mathscr{T}_\theta)$ and $(H_\theta(E), T_\theta)$.

In Section 3 we introduce the concepts α-β-holomorphy type, α-β-γ-holomorphy type, Borel transform, formal power series and partial differential operators. For α-β-holomorphy types we characterize the dual spaces using the Borel transform. For α-β-γ-holomorphy types we show partial differential operators on formal power series are onto maps and the solutions can be approximated by exponential polynomial solutions.

Section 4 contains examples and counter-examples, where we take E to be a separable infinite-dimensional Hilbert space.

1. DEFINITION OF TOPOLOGIES

E will denote a complex Banach space. For m a positive integer or zero $\mathscr{P}(^mE)$ will denote the Banach space of all m-homogeneous complex-valued polynomials on E with the norm

$$\|P\| = \sup_{\|x\|\leqslant 1} |P(x)|,$$

which is called the *current norm*. A polynomial is a finite sum of homogeneous polynomials. $\mathscr{H}(E)$ will denote the vector space of all complex-valued functions on E which are holomorphic on all of E. For each $f \in \mathscr{H}(E)$ we have its Taylor series at the origin,

$$f(x) = \sum_{m=0}^{\infty} \frac{1}{m!} \hat{d}^m f(0)(x),$$

for every $x \in E$ and the corresponding differentials of order $m = 0, 1, \ldots$

$$\hat{d}^m f(0) \in \mathscr{P}(^mE).$$

In many cases (cf. [13] and [15]) it has been found necessary to consider various vector subspaces of $\mathscr{P}(^mE)$ and of $\mathscr{H}(E)$. We recall some of the more important of these.

EXAMPLE 1. If E' denotes the dual of E, we have $\varphi^m \in \mathscr{P}(^mE)$ for each $\varphi \in E'$. We denote by $\mathscr{P}_f(^mE)$ the vector space of those elements of $\mathscr{P}(^mE)$ each of which can be represented as a finite sum $\varphi_1^m + \ldots + \varphi_r^m$, where $\varphi_j \in E'$ for $j = 1, \ldots, r$. An element of $\mathscr{P}(^mE)$ is said to be *of finite type* in case it lies in $\mathscr{P}_f(^mE)$.

EXAMPLE 2. The Banach space of m-homogeneous compact polynomials is the closure of $\mathscr{P}_f(^mE)$ in $\mathscr{P}(^mE)$ with the topology induced from $\mathscr{P}(^mE)$.

EXAMPLE 3. The Banach space $\mathscr{P}_N(^mE)$ of all m-homogeneous nuclear polynomials can be characterized by the following requirements (cf. [14]):

(1) $\mathscr{P}_N(^mE)$ is a vector subspace of $\mathscr{P}(^mE)$.

(2) $\mathscr{P}_N(^mE)$ is a Banach space with respect to a norm, denoted by $\| \|_N$, and called the *nuclear norm*.

It is to be distinguished from the current norm by the following two requirements:

(3) $\mathscr{P}_f(^mE)$ is contained and dense in $\mathscr{P}_N(^mE)$ with respect to the nuclear norm.

(4) For each $P \in \mathscr{P}_f(^mE)$ its nuclear norm is equal to the infimum of the sums $\|\varphi_1\|^m + \ldots + \|\varphi_r\|^m$ for all possible representations $P = \varphi_1^m + \ldots + \varphi_r^m$.

For another space of polynomials see Section 3 (the integral polynomials). All these spaces of polynomials satisfy the following definition (cf. [13], p. 34, and [3], p. 18):

Definition 1. A *holomorphy type* θ on E is a sequence of Banach spaces $\mathscr{P}_\theta(^mE)$ for $m = 0, 1, \ldots$ the norm of each being denoted by $\| \|_\theta$ such that the following conditions hold:

(1) Each $\mathscr{P}_\theta(^mE)$ is a vector subspace of $\mathscr{P}(^mE)$.

(2) $\mathscr{P}_\theta(^0E) \approx C$ (the complex numbers).

(3) There is a real number $\sigma \geqslant 1$ such that, given any $l = 0, 1, \ldots$, $m = 0, 1, \ldots$, $l \leqslant m$, $P \in \mathscr{P}_\theta(^mE)$ and $x \in E$, we have $\hat{d}^l P_m(x) \in \mathscr{P}_\theta(^lE)$ and

$$\left\| \frac{1}{l!} \hat{d}^l P(x) \right\|_\theta \leqslant \sigma^m \|P\|_\theta \|x\|^{m-l}.$$

Definition 1 leads to the first natural definition of a topological vector space of holomorphic functions on E (cf. [13], p. 43).

Definition 2.

(a) $f \in \mathscr{H}(E)$ is said to be *of holomorphy type θ at $\xi \in E$* if:

(1) $\hat{d}^m f(\xi) \in \mathscr{P}_\theta(^mE)$ for $m = 0, 1, 2, \ldots$

(2) There are real numbers $C_1 \geqslant 0$, $C_2 \geqslant 0$ such that

$$\left\| \frac{1}{m!} \hat{d}^m f(\xi) \right\|_\theta \leqslant C_1 \cdot C_2^m \quad \text{for } m = 0, 1, 2, \ldots$$

(b) $f \in \mathscr{H}(E)$ is said to be *of holomorphy type θ* if f is of holomorphy type θ at all points of E.

(c) We denote by $\mathscr{H}_\theta(E)$ the space of all functions on E of holomorphy type θ. (We note $\mathscr{H}_\theta(E)$ is a vector subspace of $\mathscr{H}_\theta(E)$.)

(d) A semi-norm p on $\mathscr{H}_\theta(E)$ is said to be *ported* by a compact subset K of E if either of the following equivalent conditions hold:

(1) Given any real number $\varepsilon > 0$, $\exists C(\varepsilon) \geqslant 0$ such that

$$p(f) \leqslant C(\varepsilon) \sum_{m=0}^{\infty} \varepsilon^m \sup_{x\in K} \left\| \frac{\hat{d}^m f(x)}{m!} \right\|_\theta \quad \text{for every } f \in \mathscr{H}_\theta(E).$$

(2) Given any real number $\varepsilon > 0$ and any open subset V of E which contains K we can find a real number $C(\varepsilon, V) \geqslant 0$ such that

$$p(f) \leqslant C(\varepsilon, V) \sum_{m=0}^{\infty} \varepsilon^m \sup_{x\in V} \left\| \frac{\hat{d}^m f(x)}{m!} \right\|_\theta \quad \text{for every } f \in \mathscr{H}_\theta(E).$$

(e) The *natural topology* on $\mathscr{H}_\theta(E)$ is the topology generated by all semi-norms ported by compact sets and is denoted by $\mathscr{T}_\theta$.

(f) If $\mathscr{P}_\theta(^mE) = \mathscr{P}(^mE)$ we call the holomorphy type $(\mathscr{P}(^mE))_{m=0}^{\infty}$ the *current type*. In this case $\mathscr{H}_\theta(E) = \mathscr{H}(E)$ (by the definition of holomorphic function) and its topology is denoted by $\mathscr{T}_\omega$.

We recall one result which we shall use (cf. [13], p. 59).

PROPOSITION 1. *A semi-norm p on $\mathscr{H}(E)$ is ported a compact set $K \subset E$ if and only if, for every neighborhood V of K, $\exists C(V) \geqslant 0$ such that*

$$p(f) \leqslant C(V) \sup_{x\in V} |f(x)| \quad \textit{for every } f \in \mathscr{H}(E).$$

c_0^+ will denote the set of all sequences of positive real numbers which tend to zero at infinity. $\mathscr{K}(E)$ will denote the set of all compact subsets of E, and $\hat{\mathscr{K}}(E)$ will denote the set of all convex balanced compact subsets of E. We usually write $\mathscr{K}$ and $\hat{\mathscr{K}}$ when E is fixed. We say a normed space of m-homogeneous polynomials $(\mathscr{P}_\theta(^mE), \|\ \|_\theta)$ is *intrinsic* if it depends only on the t. v. s. (topological vector space) structure of E. We are given a Banach space with a fixed norm and we denote its unit ball by B_1. We shall, however, have reason to consider other norms on E (in most cases they will be equivalent to but not necessarily isometric to the given norm). These different norms on E may give rise to new norms on spaces of polynomials (e. g. examples 2 and 3). Thus if U is the unit ball of a new norm on E and $(\mathscr{P}_\theta(^mE), \|\ \|)_\theta$ is a normed space of m-homogeneous polynomials on E, we denote the new norm on $\mathscr{P}_\theta(^mE)$ by $\|\ \|_{\theta,U}$, e. g. for the nuclear polynomials we write $\|\ \|_{N,U}$ and for the current type we write $\|\ \|_U$.

The next most important topological vector space of holomorphic functions occurs in [15]. Using our above-mentioned notation we can define it as follows:

Definition 3. *$H_N(E)$ the space of nuclearly entire functions*

(a) $f \in H_N(E)$ if:

(1) $f \in \mathscr{H}(E)$.

(2) $\hat{d}^n f(0) \in \mathscr{P}_N(^nE)$ for $n = 0, 1, 2, \ldots$

(3) For each $K \in \hat{\mathscr{K}}$, $\exists \varepsilon > 0$ such that

$$\sum_{m=0}^{\infty} \left\| \frac{1}{m!} \hat{d}^m f(0) \right\|_{N, K+\varepsilon B_1} < \infty.$$

(b) A semi-norm p on $H_N(E)$ is said to be *N-ported* by $K \in \hat{\mathscr{K}}$ if, for each $\varepsilon > 0$, $\exists C(\varepsilon) > 0$ such that

$$p(f) \leqslant C(\varepsilon) \sum_{m=0}^{\infty} \left\| \frac{1}{m!} \hat{d}^m f(0) \right\|_{N, K+\varepsilon B_1} \quad \text{for all } f \in H_N(E).$$

(c) The topology on $H_N(E)$ is that generated by all semi-norms N-ported by elements of $\hat{\mathscr{K}}$.

In generalizing Definition 3 we found it first necessary to put some further conditions on the spaces of polynomials involved.

Definition 4. An *a-holomorphy type θ* is a holomorphy type which satisfies the following conditions:

(1) $\mathscr{P}_\theta(^mE)$ is an intrinsic space of polynomials for each n and the σ in Definition 1 depends only on the t. v. s. structure of E.

(2) If $\|\ \|_U$ and $\|\ \|_V$ give the original topology on E and C is a positive real number such that $CU \subset V$, then for each n we have

$$C^n \|P_n\|_{\theta,U} \leqslant \|P_n\|_{\theta,V} \quad \text{for all } P_n \in \mathscr{P}_\theta(^nE).$$

Remark. If $U \subset V$, then for each n

$$\|P_n\|_{\theta,U} \leqslant \|P_n\|_{\theta,V}, \quad P_n \in \mathscr{P}_\theta(^nE).$$

If $CU = V$, then for each n

$$C^n \|P_n\|_{\theta,U} = \|P_n\|_{\theta,V} \quad \text{for all } P_n \in \mathscr{P}_\theta(^nE).$$

Remark. It is easy to check that Examples 2 and 3 satisfy these conditions.

Definition 5.

(a) Let θ be an a-holomorphy type; then $H_\theta(E)$ is the set of all functions f on E which satisfy the following conditions:

(1) $f \in \mathscr{H}(E)$.

(2) $\hat{d}^n f(0) \in \mathscr{P}_\theta(^nE)$ for $n = 0, 1, 2, \ldots$

(3) For each $K \in \hat{\mathscr{K}}$, $\exists\, \varepsilon > 0$ such that

$$\sum_{m=0}^{\infty} \left\| \frac{\hat{d}^m f(0)}{m!} \right\|_{\theta, K+\varepsilon B_1} < \infty.$$

(b) A semi-norm p on $H_\theta(E)$ is θ-ported by $K \in \hat{\mathscr{K}}$ if, for each $\varepsilon > 0$, $\exists\, C(\varepsilon) > 0$ such that

$$p(f) \leqslant C(\varepsilon) \sum_{m=0}^{\infty} \left\| \frac{\hat{d}^m f(0)}{m!} \right\|_{\theta, K+\varepsilon B_1} \quad \text{for all } f \in H_\theta(E).$$

(c) The topology on $H_\theta(E)$ is that generated by all semi-norms ported by some element of $\hat{\mathscr{K}}$. It is denoted by T_θ.

(d) If θ is the current type, we denote the space by $H(E)$ and the topology by T_ω.

2. THE SPACE $(H_\theta(E), T_\theta)$.

In this section we obtain a different characterization of the space $(H_\theta(E), T_\theta)$ and hence show it is complete.

PROPOSITION 2. *Let*

$$f = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!} \in \mathscr{H}(E) \quad \textit{and} \quad \hat{d}^n f(0) \in \mathscr{P}_\theta(^n E)$$

for $n = 0, 1, \ldots$ Then the following three conditions are equivalent:

(1) $f \in H_\theta(E)$,

(2) *for each $K \in \hat{\mathscr{K}}$, $(a_n)_{n=0}^{\infty} \in c_0^+$, we have*

$$\sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+a_n B_1} < \infty,$$

(3) *for each $K \in \hat{\mathscr{K}}$, $(a_n)_{n=0}^{\infty} \in c_0^+$, we have*

$$\lim_{n\to\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+a_n B_1}^{1/n} = 0.$$

Proof. (1) $\Rightarrow$ (2). Let $K \in \mathscr{K}$ and $(a_n)_{n=0}^{\infty} \in c_0^+$ be chosen arbitrarily. By Definition 5 (a), $\exists\, \varepsilon > 0$ such that

$$\sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+\varepsilon B_1} < \infty.$$

Let n_0 be a positive integer such that $a_n \leqslant \varepsilon$ for $n \geqslant n_0$. By Definition 4, (1), we have

$$\left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+a_n B_1} \leqslant \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+\varepsilon B_1} \quad \text{for } n \geqslant n_0.$$

Hence

$$\sum_{n=n_0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+a_n B_1} < \infty.$$

Since

$$\sum_{n=0}^{n_0-1} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+a_n B_1} < \infty$$

we have that (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3). Given $K \in \hat{\mathscr{K}}$ and $(a_n)_{n=0}^{\infty} \in c_0^+$. Let $(\beta_n)_{n=0}^{\infty}$ be an arbitrary sequence of positive real numbers such that $d = \sup_n \beta_n^{1/n} < \infty$. Then $dK \in \hat{\mathscr{K}}$ and $(\beta_n^{1/n} a_n)_{n=0}^{\infty} \in c_0^+$. By (2) we have

$$\sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, dK+\beta_n^{1/n} a_n B_1} < \infty.$$

Since θ is an α-holomorphy type we get for each n

$$\beta_n \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+a_n B_1} = \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, \beta_n^{1/n} K+\beta_n^{1/n} a_n B_1} \leqslant \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, dK+\beta_n^{1/n} a_n B_1}.$$

Thus

$$\sum_{n=0}^{\infty} \beta_n \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+a_n B_1} < \infty$$

and

$$\limsup_{n\to\infty} \left(\beta_n \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+a_n B_1} \right)^{1/n} \leqslant 1.$$

Letting $\beta_n = c^n$ we get

$$\limsup_{n\to\infty} \left(\left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+a_n B_1} \right)^{1/n} \leqslant \frac{1}{c}.$$

Hence

$$\lim_{n\to\infty} \left(\left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+a_n B_1} \right)^{1/n} = 0$$

and (2) $\Rightarrow$ (3).

(3) ⇒ (1). Suppose (1) does not hold; then $\exists K \in \hat{\mathscr{K}}$ such that for every $\varepsilon > 0$ we have

$$\sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+\varepsilon B_1} = \infty .$$

Letting $\varepsilon = 1$ we have

$$\limsup_{n\to\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+B_1}^{1/n} \geqslant 1.$$

Choose n_1 such that

$$\left\| \frac{\hat{d}^{n_1} f(0)}{n_1!} \right\|_{\theta, K+B_1}^{1/n_1} \geqslant \tfrac{1}{2}.$$

By induction take $n_k > n_{k-1}$ such that

$$\left\| \frac{\hat{d}^{n_k} f(0)}{n_k!} \right\|_{\theta, K+1/kB_1}^{1/n_k} \geqslant \tfrac{1}{2}.$$

Define

$$a_n = \begin{cases} 1 & \text{for } n \leqslant n_1, \\ 1/k & \text{for } n_{k-1} < n \leqslant n_k. \end{cases}$$

Hence

$$\limsup_{n\to\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+a_n B_1}^{1/n} \geqslant \tfrac{1}{2} \quad \text{and} \quad (a_n)_{n=0}^{\infty} \in c_0^+ .$$

This contradicts 3.

PROPOSITION 3. *If $f \in H_\theta(E)$, then the Taylor series of f at 0 converges to f in $(H_\theta(E), T_\theta)$.*

Proof. Follows immediately from the definition of $(H_\theta(E), T_\theta)$.

PROPOSITION 4. *The topology T_θ on $H_\theta(E)$ is generated by all semi-norms of the form*

$$(*) \qquad p(f) = \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+a_n B_1},$$

where $K \in \hat{\mathscr{K}}$ and $(a_n)_{n=0}^{\infty} \in c_0^+$.

Proof. By Proposition 2, $p(f)$ is finite for all $f \in H_\theta(E)$. It is then obviously a semi-norm on $H_\theta(E)$. Given $\varepsilon > 0$ choose n_0 such that $a_n \leqslant \varepsilon$ for all $n \geqslant n_0$. As in Proposition 2 we then get

$$\sum_{n=n_0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+a_n B_1} \leqslant \sum_{n=n_0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+\varepsilon B_1} \quad \text{for all } f \in H_\theta(E).$$

For $n = 0, 1, \ldots, n_0 - 1$, $\exists \delta > 0$ such that $\delta(K + a_n B_1) \subset K + \varepsilon B_1$. Since we are dealing with an α-holomorphy type

$$\left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+a_n B_1} \leqslant \delta^{-n} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+\varepsilon B_1}$$

for all $f \in H_\theta(E)$ and $n = 0, 1, \ldots, n_0 - 1$. Thus

$$\sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+a_n B_1} \leqslant (\sup_{i \leqslant n_0} \delta^{-i}) \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+\varepsilon B_1}.$$

Hence p is continuous on $(H_\theta(E), T_\theta)$. Now let p_1 be a continuous semi-norm on $(H_\theta(E), T_\theta)$ we show p_1 is dominated by a semi-norm of the form (*). Suppose p_1 is θ-ported by $K \in \hat{\mathscr{K}}$. For every $\varepsilon > 0$ choose $C(\varepsilon) > 0$ such that

$$p_1(f) \leqslant C(\varepsilon) \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+\varepsilon B_1} \quad \text{for all } f \in H_\theta(E).$$

Hence $p_1(P_n) \leqslant C(\varepsilon) \|P_n\|_{\theta, K+\varepsilon B_1}$ for all $P_n \in P_\theta({}^nE)$. For each n and $\varepsilon > 0$ let $K_n(\varepsilon)$ be the smallest positive number or zero such that

$$p_1(P_n) \leqslant K_n(\varepsilon) \|P_n\|_{\theta, K+\varepsilon B_1} \quad \text{for all } P_n \in P_\theta({}^nE).$$

Since $K_n(\varepsilon) \leqslant C(\varepsilon)$ for all n, we get $\limsup_{n\to\infty} K_n(\varepsilon)^{1/n} \leqslant 1$. We now choose a positive integer n_1 such that $K_n(1)^{1/n} \leqslant 2$ for all $n \geqslant n_1$ and by induction we take n_k such that $n_k > n_{k-1}$ and

$$K_n\left(\frac{1}{k}\right)^{1/n} \leqslant 2 \quad \text{for } n \geqslant n_k.$$

Let

$$a_n = \begin{cases} 1 & \text{for } n < n_2, \\ \dfrac{1}{k} & \text{for } n_k \leqslant n < n_{k+1}. \end{cases}$$

Then $(a_n)_{n=0}^{\infty} \in c_0^+$ and $K_n(a_n)^{1/n} \leqslant 2$ for $n \geqslant n_1$. Hence there exists $C > 0$ such that $K_n(a_n) < C \cdot 2^n$ for all n. By Proposition 3 we have

$$p_1(f) \leqslant \sum_{n=0}^{\infty} p_1\left(\frac{\hat{d}^n f(0)}{n!}\right) \leqslant C \sum_{n=0}^{\infty} 2^n \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+a_n B_1},$$

i. e.

$$p_1(f) \leqslant C \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, 2K+2a_n B_1} \quad \text{for all } f \in H_\theta(E).$$

This gives the required domination and completes the proof.

We now get necessary and sufficient conditions on the sequence $\{P_n\}_{n=0}^{\infty}$ so that $\sum_{m=0}^{\infty} \mathscr{P}_m$ is the Taylor series expansion of an element of $\mathscr{H}(E)$.

PROPOSITION 5. *Let $P_m \in \mathscr{P}({}^mE)$ for $m = 0, 1, \ldots$ Then the following conditions are equivalent:*

(1) *$\sum_{m=0}^{\infty} P_m$ is the Taylor series expansion of an element f of $\mathscr{H}(E)$,*

(2) *for each $K \in \hat{\mathscr{K}}$ and $(a_n)_{n=0}^{\infty} \in c_0^+$, we have $\lim_{n\to\infty} \|P_m\|_{K+a_mB_1}^{1/m} = 0$,*

(3) *for each $K \in \hat{\mathscr{K}}$ and $(a_n)_{n=0}^{\infty} \in c_0^+$, we have $\sum_{m=0}^{\infty} \|P_m\|_{K+a_mB_1} < \infty$,*

(4) *for each $K \in \hat{\mathscr{K}}$, $\exists\, \delta > 0$ such that $\sum_{m=0}^{\infty} \|P_m\|_{K+\delta B_1} < \infty$.*

Proof. Since the current type is an α-holomorphy type it is easy to show $(2) \Leftrightarrow (3) \Leftrightarrow (4)$. We therefore prove $(1) \Rightarrow (2)$ and $(4) \Rightarrow (1)$.

$(1) \Rightarrow (2)$. Let $K \in \hat{\mathscr{K}}$ and $(a_n)_{n=0}^{\infty} \in c_0^+$ be given. Since

$$\|P_m\|_{K+a_mB_1} = \sup_{X\in K+a_mB_1} |P_m(X)| = \sup_{X\in K+a_mB_1} \left| \frac{1}{2\pi i} \int_{|\lambda|=\varrho} \frac{f(\lambda X)}{\lambda^{m+1}}\, d\lambda \right| \text{ for any } \varrho > 0$$

([13], p. 21) we get

$$\|P_m\|_{K+a_mB_1} \leqslant \frac{2\pi\varrho}{2\pi} \cdot \frac{1}{\varrho^{m+1}} \sup_{X\in\varrho K+\varrho a_mB_1} |f(X)|.$$

Choose $\varrho > 0$ arbitrarily. Hence $\varrho K \in \mathscr{K}$. Let V be a neighborhood of ϱK on which f is bounded (by M say). Choose n_0 such that $\varrho K + \varrho a_n B_1 \subset V$ for $n \geqslant n_0$. Then

$$\|P_m\|_{K+a_mB_1} \leqslant \frac{1}{\varrho^m} \sup_{X\in\varrho K+\varrho a_mB_1} |f(x)| \leqslant \frac{M}{\varrho^m}$$

which implies

$$\limsup_{m\to\infty} \|P_m\|_{K+a_mB_1}^{1/m} \leqslant \frac{1}{\varrho}.$$

Since $\varrho > 0$ was chosen arbitrarily we get $\limsup_{m\to\infty} \|P_m\|_{K+a_mB_1}^{1/m} = 0$.

$(4) \Rightarrow (1)$. Let x be an arbitrary point of E. Denote by X the convex balanced hull of x. By (4), $\exists\, \delta > 0$ such that $\sum_{m=0}^{\infty} \|P_m\|_{X+\delta B_1} < \infty$. Hence $\limsup_{m\to\infty} \|P_m\|_{X+\delta B_1}^{1/m} \leqslant 1$. By [11], p. 206, $\sum_{m=0}^{\infty} P_m$ is the Taylor series expansion at the origin of a function holomorphic in the interior of $X + \delta B_1$. In particular it is holomorphic at x. This completes the proof.

COROLLARY 1. $\mathscr{H}(E) = H(E)$.

COROLLARY 2. *Let $P_m \in \mathscr{P}_\theta({}^mE)$ for $m = 0, 1, \ldots$; then the following conditions are equivalent:*

(1) *$\sum_{m=0}^{\infty} P_m$ is the Taylor series expansion of an element f of $H_\theta(E)$.*

(2) *For each $K \in \hat{\mathscr{K}}$, $\exists\, \varepsilon > 0$ such that $\sum_{n=0}^{\infty} \|P_n\|_{\theta, K+\varepsilon B_1} < \infty$.*

(3) *For each $K \in \hat{\mathscr{K}}$, $(a_n)_{n=0}^{\infty} \in c_0^+$, we have $\sum_{n=0}^{\infty} \|P_n\|_{\theta, K+a_nB_1} < \infty$.*

(4) *For each $K \in \hat{\mathscr{K}}$, $(a_n)_{n=0}^{\infty} \in c_0^+$, we have $\lim_{n\to\infty} \|P_n\|_{\theta, K+a_nB_1}^{1/n} = 0$.*

This means that conditions (2) and (3) of definition (5) (a) imply condition (1).

Proof. For any α-holomorphy type θ, $\exists\, \sigma > 1$, such that for each n, $\|P_n\| \leqslant \sigma^n \|P_n\|_\theta$ for $P_n \in \mathscr{P}_\theta({}^nE)$. Hence (2), (3) and (4) hold also where θ is the current type. Proposition 5 implies $f = \sum_{n=0}^{\infty} P_n \in \mathscr{H}(E)$ and Proposition 2 then gives the required result.

PROPOSITION 6. *$(H_\theta(E), T_\theta)$ is complete.*

Proof. Let $(f_\alpha)_{\alpha\in A}$ be a Cauchy net in $(H_\theta(E), T_\theta)$. Hence, for $n = 0, 1, \ldots$, $\{\hat{d}^n f_\alpha(0)\}_{\alpha\in A}$ is a Cauchy net in the Banach space $(\mathscr{P}_\theta({}^nE), \|\,\|_\theta)$. Suppose $\hat{d}^n f_\alpha(0) \to P_n \in \mathscr{P}_\theta({}^nE)$ as $\alpha \to \infty$ for $n = 0, 1, \ldots$ Let $K \in \hat{\mathscr{K}}$ and $(a_n)_{n=0}^{\infty} \in c_0^+$. Given $\varepsilon > 0$ choose β_0 such that for $\beta_1, \beta_2 \geqslant \beta_0$ we have

$$\sum_{n=0}^{\infty} \left\| \frac{\hat{d}(f_{\beta_1} - f_{\beta_2})(0)}{n!} \right\|_{\theta, K+a_nB_1} \leqslant \varepsilon.$$

Hence for any positive integer m and $\beta_1, \beta_2 \geqslant \beta_0$ we have

$$\sum_{n=0}^{m} \left\| \frac{\hat{d}^n f_{\beta_1}(0)}{n!} - \frac{\hat{d}^n f_{\beta_2}(0)}{n!} \right\|_{\theta, K+a_nB_1} \leqslant \varepsilon.$$

Letting $\beta_1 \to \infty$ we get

$$(*) \qquad \sum_{n=0}^{m} \left\| \frac{P_n}{n!} - \frac{\hat{d} f_{\beta_2}(0)}{n!} \right\|_{\theta, K+a_nB_1} \leqslant \varepsilon \quad \text{for all } m \text{ and all } \beta_2 \geqslant \beta_0.$$

In particular, we get

$$\sum_{n=0}^{m} \left\| \frac{P_n}{n!} \right\|_{\theta, K+a_nB_1} \leqslant \sum_{n=0}^{\infty} \left\| \frac{\hat{d} f_{\beta_0}(0)}{n!} \right\|_{\theta, K+a_nB_1} + \varepsilon.$$

Thus

$$\sum_{n=0}^{\infty}\left\|\frac{P_n}{n!}\right\|_{\theta,K+a_nB_1} \leqslant \sum_{n=0}^{\infty}\left\|\frac{\hat{d}^n f_{\beta_0}(0)}{n!}\right\|_{\theta,K+a_nB_1} + \varepsilon < \infty$$

and Proposition 5 implies

$$f = \sum_{n=0}^{\infty}\frac{P_n}{n!}\in H_\theta(E).$$

(*) also gives

$$\sum_{n=0}^{m}\left\|\frac{P_n}{n!} - \frac{\hat{d}^n f_{\beta_2}(0)}{n!}\right\|_{\theta,K+a_nB_1} \leqslant \varepsilon \quad \text{for all } m \text{ and all } \beta_2 \geqslant \beta_0.$$

Hence

$$p(f-f_{\beta_2}) = \sum_{n=0}^{\infty}\left\|\frac{P_n}{n!} - \frac{\hat{d}^n f_{\beta_2}(0)}{n!}\right\|_{\theta,K+a_nB_1} \leqslant \varepsilon \quad \text{for } \beta_2 \geqslant \beta_0.$$

An application of Proposition 4 completes the proof.

The space $(H_\theta(E), t_\theta)$. We now discuss the finest locally convex topology on $H_\theta(E)$ which gives the same bounded sets as $(H_\theta(E), T_\theta)$. We denote this topology by t_θ. We get a representation for a set of semi-norms which generate the topology. We then show it is a complete barrelled space. We note it is the finest locally convex topology on $H_\theta(E)$ for which the Taylor series converges absolutely and which gives the $\|\ \|_\theta$ topology on each $\mathscr{P}_\theta(^nE)$. T_θ and t_θ induce the same topology on bounded sets and they also give the same compact sets.

Definition 6. We denote by t_θ the *finest locally convex topology on* $H_\theta(E)$ which has the same bounded sets as $(H_\theta(E), T_\theta)$.

Lemma 1. *Let*

$$f\in H_\theta(E) \quad \textit{and} \quad g_m = 2^m\cdot\sum_{n=m}^{\infty}\frac{\hat{d}^n f(0)}{n!};$$

then $g_m \to 0$ *in* $(H_\theta(E), T_\theta)$ *as* $m\to\infty$.

Proof. Let $K\in\hat{\mathscr{K}}$, $(a_n)_{n=0}^{\infty}\in c_0^+$ and $0<\varepsilon<\frac{1}{2}$ be arbitrary. By Proposition 2 we can choose n_0 such that for $n\geqslant n_0$ we have

$$\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{\theta,K+a_nB_1} \leqslant \varepsilon^n.$$

For $m\geqslant n_0$ we then have

$$p(g_m) \leqslant 2^m\sum_{n=m}^{\infty}\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{\theta,K+a_nB_1} \leqslant \frac{(2\varepsilon)^m}{1-\varepsilon}\varepsilon.$$

An application of Proposition 4 completes the proof.

Proposition 7. *If* $f\in H_\theta(E)$, *then the Taylor series of* f *at* 0 *converges to* f *in* $(H_\theta(E), t_\theta)$.

Proof. Let p be a continuous semi-norm on $(H_\theta(E), t_\theta)$. By Lemma 1 and Definition 6, $\left\{2^m\sum_{n=m}^{\infty}\frac{\hat{d}^n f(0)}{n!}\right\}_{m=0}^{\infty}$ is a bounded subset of $(H_\theta(E), T_\theta)$. Hence $\exists\, M>0$ such that

$$p\left(2^m\sum_{n=m}^{\infty}\frac{\hat{d}^n f(0)}{n!}\right)\leqslant M \quad \text{for all } m.$$

Thus

$$p\left(\sum_{n=m}^{\infty}\frac{\hat{d}^n f(0)}{n!}\right)\to 0 \quad \text{as } m\to\infty.$$

Since

$$\sum_{n=m}^{\infty}\frac{\hat{d}^n f(0)}{n!} = f - \sum_{n=0}^{m-1}\frac{\hat{d}^n f(0)}{n!},$$

this comples the proof.

Lemma 2. *Let*

$$f = \sum_{n=0}^{\infty}\frac{\hat{d}^n f(0)}{n!}\in H_\theta(E)$$

and $(\beta_n)_{n=0}^{\infty}$ *be a sequence of positive numbers such that* $d = \sup_n \beta_n^{1/n} < \infty$; *then*

$$g = \sum_{n=0}^{\infty}\beta_n\frac{\hat{d}^n f(0)}{n!}\in H_\theta(E).$$

Proof. Since

$$f\in H_\theta(E), \qquad \beta_n\frac{\hat{d}^n f(0)}{n!}\in\mathscr{P}_\theta(^nE).$$

Let $K\in\hat{\mathscr{K}}$ and $(a_n)_{n=0}^{\infty}\in c_0^+$. By Proposition 2

$$\lim_{n\to\infty}\left\|\beta_n\frac{\hat{d}^n f(0)}{n!}\right\|_{\theta,K+a_nB_1}^{1/n} \leqslant \sup_n \beta_n^{1/n}\lim_{n\to\infty}\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{\theta,K+a_nB_1}^{1/n} \leqslant d\cdot 0 = 0.$$

Hence $g\in H_\theta(E)$ by Corollary 2.

Lemma 3. *Let* p *be a continuous semi-norm on* $(H_\theta(E), t_\theta)$; *then*

$$\lim_{n\to\infty} p\left(\frac{\hat{d}^n f(0)}{n!}\right)^{1/n} = 0 \quad \textit{for each } f\in H_\theta(E).$$

Proof. Let $(\beta_n)_{n=0}^{\infty}$ be a sequence of positive numbers such that $\sup_n \beta_n^{1/n} < \infty$. By Lemma 2, Proposition 4 and Definition 6 we get $\left\{\beta_n \frac{\hat{d}^n f(0)}{n!}\right\}_{n=0}^{\infty}$ is a bounded subset of $(H_\theta(E), t_\theta)$. Taking $\beta_n = c^n$, where $c > 0$ is arbitrary, we get $\left\{p\left(c^n \frac{\hat{d}^n f(0)}{n!}\right)\right\}_{n=0}^{\infty}$ is a bounded set of real numbers.

Hence there exists M such that

$$p\left(\frac{\hat{d}^n f(0)}{n!}\right) \leqslant M/c^n.$$

Therefore

$$\limsup_{n\to\infty} p\left(\frac{\hat{d}^n f(0)}{n!}\right)^{1/n} \leqslant 1/c.$$

This completes the proof.

COROLLARY 3. *Let p be a continuous semi-norm on $(H_\theta(E), t_\theta)$ and let*

$$f = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!} \in H_\theta(E);$$

then

$$\sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n!}\right) < \infty.$$

We say a series $\sum_{n=0}^{\infty} X_n$ in a locally convex topological vector space converges absolutely if for each continuous semi-norm p on the space we have $\sum_{n=0}^{\infty} p(X_n) < \infty$.

COROLLARY 4. *The Taylor series of $f \in H_\theta(E)$ at 0 converges absolutely in $(H_\theta(E), t_\theta)$.*

LEMMA 4. *$(H_\theta(E), t_\theta)$ induces on $\mathscr{P}_\theta({}^nE)$ the topology generated by the norm $\| \; \|_\theta$.*

Proof. Since $p(f) = \left\|\frac{\hat{d}^n f(0)}{n!}\right\|_\theta$ is a continuous semi-norm on $(H_\theta(E), T_\theta)$ it is also a continuous semi-norm on $(H_\theta(E), t_\theta)$. Hence t_θ induces on $\mathscr{P}_\theta({}^nE)$ a topology stronger than or equal to the norm topology $\| \; \|_\theta$. Let $C_n = \{P_n \in \mathscr{P}_\theta({}^nE), \|P_n\|_\theta \leqslant 1\}$. C_n is easily seen to be a bounded subset of $(H_\theta(E), t_\theta)$. Hence t_θ induces on $\mathscr{P}_\theta({}^nE)$ a topology weaker than or equal to the $\| \; \|_\theta$ topology.

LEMMA 5. *Let $(f_n)_{n=0}$ be a bounded subset of $(H_\theta(E), T_\theta)$; then*

$$g = \sum_{n=0}^{\infty} \frac{\hat{d}^n f_n(0)}{n!} \in H_\theta(E).$$

Proof. Since $f_n \in \mathscr{P}_\theta(E)$, for all n, $\hat{d}^n f_n(0) \in \mathscr{P}_\theta({}^nE)$ for all n. Let $K \in \hat{\mathscr{K}}$, $(\alpha_n)_{n=0}^{\infty} \in c_0^+$ and $(\beta_n)_{n=0}^{\infty}$ be a sequence of positive numbers such that $d = \sup_n \beta_n^{1/n} < \infty$. Hence $dK \in \hat{\mathscr{K}}$ and $(\beta_n^{1/n}\alpha_n)_{n=0}^{\infty} \in c_0^+$. By hypothesis

$$\sup_n \left\{\sum_{m=0}^{\infty} \left\|\frac{\hat{d}^m f_n(0)}{n!}\right\|_{\theta, dK+\beta_m^{1/m}\alpha_m B_1}\right\} < \infty.$$

Since we are dealing with an α-holomorphy type this implies

$$\sup_n \left\{\sum_{m=0}^{\infty} \beta_m \left\|\frac{\hat{d}^m f_n(0)}{m!}\right\|_{\theta, K+\alpha_m B_1}\right\} < \infty.$$

Thus

$$\sup_n \left\{\beta_n \left\|\frac{\hat{d}^n f_n(0)}{n!}\right\|_{\theta, K+\alpha_n B_1}\right\} < \infty$$

and hence

$$\lim_{n\to\infty} \left\|\frac{\hat{d}^n f_n(0)}{n!}\right\|_{\theta, K+\alpha_n B_1}^{1/n} = 0.$$

By Proposition 2 and Corollary 2 we get

$$g = \sum_{n=0}^{\infty} \frac{\hat{d}^n f_n(0)}{n!} \in H_\theta(E).$$

PROPOSITION 8. *Let p be a semi-norm on $H_\theta(E)$ with the following properties:*

(1) *For each $n = 0, 1, \ldots$, p induces on $\mathscr{P}_\theta({}^nE)$ a topology weaker than or equal to the $\| \; \|_\theta$ topology.*

(2) *If*

$$f = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!} \in H_\theta(E),$$

then

$$\sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n!}\right) < \infty;$$

then

$$p_1(f) = \sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n!}\right)$$

is a continuous semi-norm on $(H_\theta(E), t_\theta)$.

Proof. Since t_θ is a bornological topology it suffices to show that for each bounded set F of $(H_\theta(E), t_\theta)$ we have $\sup_{f\epsilon F} p_1(f) < \infty$. By Condition (1) we get for each n that

$$(*)\qquad \sup_{f\epsilon F} p\left(\frac{\hat{d}^n f(0)}{n!}\right) < \infty.$$

Now suppose $\sup_{f\epsilon F} p_1(f) = \infty$. By $(*)$ and the definition of p_1 we thus have for each positive integer n_0

$$\sup_{f\epsilon F} \sum_{n=n_0}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n!}\right) = \infty.$$

Choose f_1 such that $\sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n f_1(0)}{n!}\right) \geqslant 2$ and take n_1 such that

$$\sum_{n=0}^{n_1} p\left(\frac{\hat{d}^n f_1(0)}{n!}\right) \geqslant 1.$$

By induction choose, for each k, f_k, such that

$$\sum_{n=n_{k-1}+1}^{\infty} p\left(\frac{\hat{d}^n f_{n_k}(0)}{n!}\right) \geqslant 2$$

and take n_k such that

$$\sum_{n_{k-1}+1}^{n_k} p\left(\frac{\hat{d}^n f_{n_k}(0)}{n!}\right) \geqslant 1 \qquad (n_k \geqslant k).$$

Let

$$g_n = \begin{cases} f_1 & \text{for } 0 \leqslant n \leqslant n_1, \\ f_k & \text{for } n_{k-1} < n \leqslant n_k \ (k \geqslant 2). \end{cases}$$

By Lemma 5

$$g = \sum_{n=0}^{\infty} \frac{\hat{d}^n g_n(0)}{n!} \epsilon H_\theta(E).$$

But

$$p_1(g) = \sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n g_n(0)}{n!}\right) = \infty$$

which contradicts (2). Hence $\sup_{f\epsilon F} p_1(f) < \infty$ and p_1 is a continuous semi-norm on $(H_\theta(E), t_\theta)$.

Proposition 9. *The topology* t_θ *on* $H_\theta(E)$ *is generated by all semi-norms which satisfy the following conditions:*

(1) $p(f) = \sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n!}\right)$ *for all* $f \epsilon H_\theta(E)$.

(2) *For each* $n = 0, 1, \ldots,$ p *induces on* $\mathscr{P}_\theta(^nE)$ *a topology weaker than or equal to the* $\|\ \|_\theta$ *topology.*

Proof. Proposition 8 says all such semi-norms are continuous. Let q be a continuous semi-norm on $(H_\theta(E), t_\theta)$. Lemma 4 gives that

$$\sum_{n=0}^{\infty} q\left(\frac{\hat{d}^n f(0)}{n!}\right) < \infty \quad \text{for each } f = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!} \epsilon H_\theta(E)$$

and Corollary 3 gives condition (2).

By Proposition 8

$$p_1(f) = \sum_{n=0}^{\infty} q\left(\frac{\hat{d}^n f(0)}{n!}\right)$$

is a continuous semi-norm on $(H_\theta(E), t_\theta)$. Proposition 7 gives $q(f) \leqslant p_1(f)$. Hence every continuous semi-norm on $(H_\theta(E), t_\theta)$ is dominated by a continuous semi-norm which satisfies our conditions. This proves the proposition.

Corollary 5. *The* t_θ *topology on* $H_\theta(E)$ *is the finest locally convex topology on* $H_\theta(E)$ *for which the Taylor series at* 0 *converges absolutely and which induces on each* $\mathscr{P}_\theta(^nE)$ *the* $\|\ \|_\theta$ *topology.*

Proof. Apply Proposition 9.

Proposition 10.

(a) $(H_\theta(E), t_\theta)$ *is a complete topological vector space.*

(b) $(H_\theta(E), t_\theta)$ *is a barrelled space.*

(c) $(H_\theta(E), T_\theta)$ *and* $(H_\theta(E), t_\theta)$ *induce the same topology on all bounded sets.*

(d) $(H_\theta(E), T_\theta)$ *and* $(H_\theta(E), t_\theta)$ *have the same compact sets.*

Proof. (a) By using the representation of semi-norms in Proposition 9 and a method similar to that used in Proposition 6 we get the required result.

(b) A complete bornological space is barrelled (cf. [5], p. 218).

(c) Since $t_\theta \geqslant T_\theta$ it suffices to show if $(f_l)_{l\in A}$ is a bounded net in $(H_\theta(E), T_\theta)$ and $f_l \to 0$ in $(H_\theta(E), T_\theta)$, then $f_l \to 0$ in $(H_\theta(E), t_\theta)$. Suppose this is not true. Then there exists p a semi-norm on $(H_\theta(E), t_\theta)$ of the form described in Proposition 9, $(f_l)_{l\in A'}$ a cofinal subnet of $(f_l)_{l\in A}$ and $\delta > 0$ such that

$$\sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n f_l(0)}{n!}\right) \geqslant \delta \quad \text{for all } l \in A'.$$

Since $f_{l'} \to 0$ in $(H_\theta(E), T_\theta)$ we get that for each n,

$$p\left(\frac{\hat{d}^n f_{l'}(0)}{n!}\right) \to 0 \quad \text{as } l' \to \infty.$$

Hence we can choose for each positive integer k, $f_k \in (f_l)_{l\in A'}$ and a positive integer n_k such that

(a) $n_k \geqslant k$,

(b) $\displaystyle\sum_{n_{k-1} < n \leqslant n_k} p\left(\frac{\hat{d}^n f_k(0)}{n!}\right) \geqslant \frac{\delta}{2}.$

Not let

$$g_n = \begin{cases} f_1, & 0 \leqslant n \leqslant n_1, \\ f_k, & n_{k-1} < n \leqslant n_k,\ k \geqslant 2, \end{cases}$$

By Lemma 5

$$g = \sum_{n=0}^{\infty} \frac{\hat{d}^n g_n(0)}{n!} \in H_\theta(E).$$

But

$$p(g) = \sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n g_n(0)}{n!}\right) = \infty$$

which is a contradiction. Hence we get the required result.

(d) Trivial by using (c) and the fact that $t_\theta \geqslant T_\theta$.

Some of the preceding results suggest further properties of sequential and bounded convergence so it is not surprising that we get the following

Definition 7. A topological vector space V is said to *satisfy Mackey's convergence criterion* (cf. [5], p. 255) if for each sequence of elements of V converging to 0, $(f_n)_{n=0}^{\infty}$ say, there exists $(\lambda_n)_{n=0}^{\infty}$ a sequence of positive numbers such that $\lim_{n\to\infty} \lambda_n = \infty$ and $\lambda_n f_n \to 0$ in V as $n \to \infty$.

Proposition 11. (a) *If $(f_l)_{l\in A}$ is a bounded net in $(H_\theta(E), T_\theta)$, then $f_l \to 0$ as $l \to \infty$ if and only if $\hat{d}^n f_l(0) \to 0$ in $\mathscr{P}_\theta(^nE)$ as $l \to \infty$ for each n.*

(b) *$(H_\theta(E), T_\theta)$ and $(H_\theta(E), t_\theta)$ both satisfy Mackey's convergence criterion.*

Proof. (a) Use a technique similar to that used in Proposition 10 (c).

(b) By Proposition 10 (c) it suffices to consider the case $(H_\theta(E), T_\theta)$. Suppose $\lim_{n\to 0} f_n = 0$. For each positive integer m let

$$p_m(f) = \sum_{n=0}^{m} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, mB_1}.$$

Let k_m be an increasing sequence of positive integers such that

$$p_m(f_n) \leqslant \frac{1}{4^m} \quad \text{for } n \geqslant k_m.$$

Take

$$\lambda_n = \begin{cases} 2^m, & k_m \leqslant n < k_{m+1}, \\ 1, & n < k_1; \end{cases}$$

then $\lambda_n \to \infty$ as $n \to \infty$. Choose $K \in \hat{\mathscr{K}}$ and $(\alpha_n)_{n=0}^{\infty} \in c_0^+$ arbitrarily. Since $(f_n)_{n=0}$ is a bounded subset of $(H_\theta(E), T_\theta)$ there exists $M > 0$ such that

$$\sup_n \left(\sum_{m=0}^{\infty} \left\| \frac{\hat{d}^m f_n(0)}{m!} \right\|_{\theta, 4K + 4\alpha_m B_1} \right) \leqslant M.$$

Hence for any s

$$\sup_n \left(\sum_{m=s}^{\infty} 4^m \left\| \frac{\hat{d}^m f_n(0)}{m!} \right\|_{\theta, K + \alpha_m B_1} \right) \leqslant M$$

which implies

$$2^s \cdot \sup_n \left(\sum_{m=s}^{\infty} \left\| \frac{\hat{d}^m f_n(0)}{m!} \right\|_{\theta, K + \alpha_m B_1} \right) \leqslant \frac{M}{2^s}.$$

Thus

$$\lim_{s\to\infty} 2^s \cdot \sup_n \sum_{m=s}^{\infty} \left\| \frac{\hat{d}^m f_n(0)}{m!} \right\|_{\theta, K + \alpha_m B_1} = 0.$$

Given $\varepsilon > 0$ choose m_0 such that $K + \alpha_n B_1 \subset m_0 B_1$ for all n and $2^{-m_0} \leqslant \varepsilon$. Then

$$\sum_{n=0}^{m} \left\| \frac{1}{n!} \hat{d}^n f(0) \right\|_{\theta, K + \alpha_n B_1} \leqslant p_m(f)$$

for all $f \in H_\theta(E)$, $m \geqslant m_0$. If $n \geqslant k_{m_0}$ we have $k_s \leqslant n < k_{s+1}$ for some $s \geqslant m_0$ and hence

$$p(\lambda_n f_n) \leqslant 2^s p_s(f_n) + 2^s \underbrace{\sum_{m=s}^{\infty} \left\| \frac{\hat{d}^m f_n(0)}{m!} \right\|_{\theta, K + a_m B_1}}_{\beta_n}$$

$$\leqslant 2^s \cdot \frac{1}{4^s} + \beta_n \leqslant \varepsilon + \beta_n.$$

Using (*) we get the required result.

By using condition (3) of Definition 1 and a method used in [15] for $H_N(E)$ one can show that $H_\theta(E)$ is a translation invariant space and that the point $0 \in E$ is not a special point with respect to the topologies T_θ or t_θ. This result is of importance if one wishes to consider convolution operators on $H_\theta(E)$ (cf. [15]).

The spaces $\mathscr{H}_\theta(E)$ and $H_\theta(E)$. We discuss the relationship between $\mathscr{H}_\theta(E)$ and $H_\theta(E)$. We show $H_\theta(E) \subset \mathscr{H}_\theta(E)$ continuously. A partial answer to the opposite inclusion is also given.

Using [11], p. 206, we can rephrase Lemma 1 of 9, [13], as follows.

LEMMA 6. *Let $P_m \in \mathscr{P}_\theta({}^m E)$ for $m = 0, 1, \ldots$ and suppose $\limsup_{m \to \infty} \|P_m\|_\theta^{1/m} = 1$; then $f = \sum_{m=0}^{\infty} P_m$ is a holomorphic function for all x such that $\|x\| < 1/\sigma$ and is of holomorphy type θ on the same set. (σ is the constant occurring in the definition of holomorphy type.)*

PROPOSITION 12. *Let θ be an a-holomorphy type; then $(H_\theta(E), T_\theta) \subset (\mathscr{H}_\theta(E), \mathscr{T}_\theta)$ continuously.*

Proof. Let $f \in H_\theta(E)$; then $f \in \mathscr{H}(E)$ so $f \in \mathscr{H}_\theta(E)$ if and only if it is of holomorphy type θ at each point x of E. Let X be the balanced convex hull of σx. By the definition of $H_\theta(E)$ there $\exists \varrho > 0$ such that

$$\sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, X + \varrho B_1} < \infty.$$

Lemma 6 and the Cauchy-Hadamard formula show $f \in \mathscr{H}_\theta(E)$. Hence $H_\theta(E) \subset \mathscr{H}_\theta(E)$. To prove the inclusion is continuous it suffices to show that every semi-norm on $(\mathscr{H}_\theta(E), \mathscr{T}_\theta)$ ported by $K \in \hat{\mathscr{K}}$ is also θ-ported by $2\sigma K$. Given $\varepsilon > 0$, choose $C(\varepsilon) > 0$ such that

$$p(f) \leqslant C(\varepsilon) \sum_{m=0}^{\infty} \varepsilon^m \sup_{x \in K} \left\| \frac{\hat{d}^m f(x)}{m!} \right\|_\theta$$

for all $f \in H_\theta(E) \subset \mathscr{H}_\theta(E)$. Let $V = 2\sigma K + 2\sigma\varepsilon B_1$ and let $\varrho = \sup_{x \in K} \|x\|_{2K + 2\varepsilon B_1}$.

Then $\varrho < \frac{1}{2}$. We show in fact that

$$(*) \qquad p(f) \leqslant C \cdot C(\varepsilon) \sum_{m=0}^{\infty} \left\| \frac{\hat{d}^m f(0)}{m!} \right\|_{\theta, V} \quad \text{for all } f \in H_\theta(E).$$

If

$$\sum_{m=0}^{\infty} \left\| \frac{\hat{d}^m f(0)}{m!} \right\|_{\theta, V} = \infty,$$

then (*) hold trivially.

If

$$\sum_{m=0}^{\infty} \left\| \frac{\hat{d}^m f(0)}{m!} \right\|_{\theta, V} < \infty,$$

then, [13], p. 35—36, implies

$$\left\| \frac{\hat{d}^m f(x)}{m!} \right\|_{\theta, 2K + 2\varepsilon B_1} \leqslant \sum_{n=m}^{\infty} \sigma^n \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, 2K + 2\varepsilon B_1} \|x\|_{2K + 2\varepsilon B_1}^{n-m}$$

$$\leqslant \sum_{n=m}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, V} \varrho^{n-m}$$

for all $x \in K$.

Hence

$$\sum_{m=0}^{\infty} \varepsilon^m \sup_{x \in K} \left\| \frac{\hat{d}^m f(x)}{m!} \right\|_\theta \leqslant \sum_{m=0}^{\infty} (\tfrac{1}{2})^m \sup_{x \in K} \left\| \frac{\hat{d}^m f(x)}{m!} \right\|_{\theta, 2\varepsilon B_1}$$

$$\leqslant \sum_{m=0}^{\infty} (\tfrac{1}{2})^m \sup_{x \in K} \left\| \frac{\hat{d}^m f(x)}{m!} \right\|_{\theta, 2K + 2\varepsilon B_1}$$

$$\leqslant \sum_{m=0}^{\infty} (\tfrac{1}{2})^m \sum_{n=m}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, V} \varrho^{n-m}$$

$$= \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, V} \varrho^n \sum_{m=0}^{n} \left(\frac{1}{2\varrho} \right)^m$$

$$= \sum_{n=0}^{\infty} \frac{(\frac{1}{2})^n - 2\varrho^{n+1}}{1 - 2\varrho} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, V}.$$

Since $(\frac{1}{2})^n \to 0$ as $n \to \infty$ and $\varrho < \frac{1}{2}$ we have $\sup_n \frac{(\frac{1}{2})^n - 2\varrho^{n+1}}{1 - 2\varrho} = C < \infty$.

Therefore $p(f) \leqslant C(\varepsilon)\cdot C\sum_{n=0}^{\infty}\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{\varrho,V}$, i. e. (*) holds. This completes the proof.

The space $\mathscr{H}(E)$. We study in the remainder of this section the vector space of all holomorphic functions an E. We show $\mathscr{T}_\omega$ and T_ω coincide on $\mathscr{H}(E)$, i. e. the two most natural topologies on $\mathscr{H}(E)$ coincide. We know already $\mathscr{H}_\omega(E) = \mathscr{H}(E)$ (cf. [13]) and, by Corollary 1, $H(E) = \mathscr{H}(E)$. Hence $\mathscr{H}_\omega(E) = H(E) = \mathscr{H}(E)$. Proposition 12 says $H(E) \subset \mathscr{H}(E)$ continuously so to show $(\mathscr{H}(E), \mathscr{T}_\omega) = (H(E), T_\omega)$ it remains only to show $(\mathscr{H}(E), \mathscr{T}_\omega) \subset (H(E), T_\omega)$ continuously.

Proposition 13. *$(\mathscr{H}(E), \mathscr{T}_\omega) = (H(E), T_\omega)$ algebraically and topologically.*

Proof. Let $K \in \hat{\mathscr{K}}$, $(a_n)_{n=0}^{\infty} \in c_0^+$ and

$$p(f) = \sum_{n=0}^{\infty}\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{K+a_nB_1}$$

for all $f \in \mathscr{H}(E)$. We show $p(f)$ is a continuous semi-norm on $(\mathscr{H}(E), \mathscr{T}_\omega)$. Let V be a neighborhood of K. Choose $\varepsilon > 0$ such that

$$\left(\frac{1+\varepsilon}{1+\frac{1}{2}\varepsilon}\right)K+\tfrac{1}{2}\varepsilon B_1 \subset \left(\frac{1+\varepsilon}{1+\frac{1}{2}\varepsilon}\right)K+\varepsilon B_1 \subset V.$$

Take n_0 a positive integer or zero such that $a_n \leqslant \frac{1}{2}\varepsilon$ for all $n \geqslant n_0$. Hence

$$K+a_nB_1 \subset \left(\frac{1+\varepsilon}{1+\frac{1}{2}\varepsilon}\right)K+a_nB_1 \subset V \quad \text{for all } n \geqslant n_0.$$

For $n \geqslant n_0$ and $\varrho > 0$ we have

$$\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{K+a_nB_1} = \sup_{x\in K+a_nB_1}\left|\frac{\hat{d}^n f(0)\cdot(x)}{n!}\right| = \sup_{x\in K+a_nB_1}\left|\frac{1}{2\pi i}\int_{|\lambda|=\varrho}\frac{f(\lambda x)}{\lambda^{n+1}}d\lambda\right|$$
$$\leqslant \frac{1}{\varrho^n}\sup_{x\in\varrho K+\varrho a_nB_1}|f(x)|.$$

We choose ϱ such that $1 < \varrho < \dfrac{1+\varepsilon}{1+\frac{1}{2}\varepsilon}$. This gives

$$\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{K+a_nB_1} \leqslant \frac{1}{\varrho^n}\sup_{x\in V}|f(x)| \quad \text{for } n \geqslant n_0. \tag{1}$$

For $n = 0, 1, \ldots, n_0-1$

$$\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{K+a_nB_1} \leqslant \frac{1}{\varrho_1^n}\sup_{x\in\varrho_1K+\varrho_1a_nB_1}|f(x)|, \tag{2}$$

where ϱ_1 is chosen such that

$$\varrho_1(K+a_nB_1) \subset V \quad \text{for } n = 1, 0, \ldots, n_0-1.$$

Thus

$$p(f) = \sum_{n=0}^{\infty}\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{K+a_nB_1} \leqslant \sum_{i=0}^{n_0}(\varrho_1^{-i})\sup_{x\in V}|f(x)| + \sup_{x\in V}|f(x)|\sum_{n=n_0}^{\infty}\frac{1}{\varrho^n}$$

by (1) and (2). Let $C(V) = \sum_{i=0}^{n_0}(\varrho_1^{-i}) + \sum_{n=n_0}^{\infty}(1/\varrho^n)$. An application of Proposition 1 gives the required result.

3. α-β-HOLOMORPHY TYPES AND BOREL TRANSFORMS

Definition 8. An α-holomorphy type θ is said to be an *α-β-holomorphy type* if it satisfies the following conditions:

(1) For each n, $\mathscr{P}_\theta(^nE) \supset \mathscr{P}_N(^nE)$ and $\|P_n\|_\theta \leqslant \|P_n\|_N$ for all $P_n \in \mathscr{P}_N(^nE)$.

(2) For each n, $\mathscr{P}_f(^nE)$ is dense in $(\mathscr{P}_\theta(^nE), \|\;\|_\theta)$.

Condition (1) is natural since the nuclear norm on $\mathscr{P}_f(^nE)$ (cf. [17], § 43) can be regarded as the largest norm on $\mathscr{P}_f(^nE)$. Condition (2) is necessary to define the Borel transform in a one to one way. If, however, θ is an α-holomorphy type which satisfies only condition (1) by taking the closure of $\mathscr{P}_f(^nE)$ in $(\mathscr{P}_\theta(^nE), \|\;\|_\theta)$ we get an α-β-holomorphy type.

By ([3], p. 35) $\mathscr{P}_N(^nE)' \cong \mathscr{P}(^nE')$ hence we can regard $\mathscr{P}_\theta(^nE)'$ as a subset of $\mathscr{P}(^nE')$. We denote by $\|\;\|_{\theta'}$ the dual norm on $\mathscr{P}_\theta(^nE)'$. We define $^\wedge : \mathscr{P}_\theta(^nE)' \to \mathscr{P}(^nE')$ by $n!\,\hat{T}_n(\varphi) = T_n(\varphi^n)$ and we denote the image of $\mathscr{P}(^nE)'$ under $^\wedge$ by $\mathscr{P}_{\theta'}(^nE')$. We norm $\mathscr{P}_{\theta'}(^nE')$ by $\|\;\|^{\theta'}$, where $\|\hat{T}_n\|^{\theta'} = 1/n!\,\|T_n\|_{\theta'}$. We also denote the inverse of $^\wedge$ by $^\vee$.

By condition (1) for any α-β-holomorphy type θ we have $H_N(E) \subset H_\theta(E)$.

Definition 9. A function f on E is said to be an *exponential function* if $\exists\ \varphi \in E'$ such that $f(x) = \exp(\varphi(x))$ for all $x \in E$.

We denote by W the vector space spanned by all the exponential functions. We note $W \subset H_N(E) \subset H_\theta(E)$ for any α-β-holomorphy type (cf. [3]).

Lemma 7. *If θ is an α-β-holomorphy type, then the closure of W in $(H_\theta(E), t_\theta)$ is $H_\theta(E)$. (Hence the closure of W in $(H_\theta(E), T_\theta)$ is also $H_\theta(E)$.)*

Proof. By Propositions 3 and 7 and definition 8 it suffices to show φ^n belongs to the closure of W for each $\varphi \in E'$. Let p be a semi-norm on $(H_\theta(E), t_\theta)$ of the form

$$p(f) = \sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n!}\right)$$

(see Proposition 9). For λ a non-zero complex number we have

$$p\left(\frac{\exp(\lambda\varphi)-1}{\lambda} - \varphi\right) = \sum_{n=2}^{\infty} |\lambda|^{n-1} p\left(\frac{\varphi^n}{n!}\right) = |\lambda| \cdot \sum_{n=2}^{\infty} |\lambda|^{n-2} p\left(\frac{\varphi^n}{n!}\right).$$

Since

$$\exp(\varphi) \subset H_\theta(E), \qquad \sum_{n=0}^{\infty} p\left(\frac{\varphi^n}{n!}\right) < \infty.$$

Hence

$$p\left(\frac{\exp(\lambda\varphi)-1}{\lambda} - \varphi\right) \to 0 \quad \text{as } \lambda \to 0.$$

So φ belongs to the closure of W. Now suppose φ^n belongs to the closure of W for $n \leqslant k$. For $\lambda \neq 0$ we have

$$p\left(\frac{\exp(\lambda\varphi) - \sum_{i=0}^{k} (\lambda^i \varphi^i / i!)}{\lambda^{k+1}} - \varphi^{k+1}\right) \leqslant |\lambda| \cdot \sum_{i=k+2} \frac{|\lambda|^{i-k-2}}{i!} p(\varphi^i).$$

Hence φ^{k+1} belongs to the closure of W and this completes the proof.

Definition 10. The *Borel transform* of an element T of $(H_\theta(E), t_\theta)'$ is the function $\hat{T}$ defined on E' by $\hat{T}(\varphi) = T(\exp\varphi)$.

By Lemma 7 a function F on E' can be the Borel transform of at most one element of $(H_\theta(E), t_\theta)'$. Since $(H_\theta(E), t_\theta) \subset (H_\theta(E), T_\theta)$ continuously definition 10 also defines the Borel transform of the elements of $(H_\theta(E), T_\theta)'$.

If $T_n \in \mathscr{P}_\theta({}^nE)'$, then T_n can be extended to a continuous linear functional on $(H_\theta(E), t_\theta)$ by the formula $T_n^*(f) = T_n\left(\frac{\hat{d}^n f(0)}{n!}\right)$. We note that

$$\hat{T}_n^*(\varphi) = T_n^*(\exp\varphi) = T_n\left(\frac{\varphi^n}{n!}\right) = \frac{1}{n!} T_n(\varphi^n) = \hat{T}_n(\varphi).$$

Hence the Borel transform is an extension of the mapping ^ defined previously.

PROPOSITION 14. *If $T \in (H_\theta(E), t_\theta)'$, then $\hat{T} \in \mathscr{H}(E')$ and $\hat{d}^n\hat{T}(0) \in \mathscr{P}_{\theta'}({}^nE')$.*

Proof. Denote by T_n the restriction of T to $\mathscr{P}_\theta({}^nE)$. Then

$$T_n \in \mathscr{P}_\theta({}^nE)', \qquad \hat{T}_n \in \mathscr{P}_{\theta'}({}^nE') \qquad \text{and} \qquad \hat{T}(\varphi) = \sum_{n=0}^{\infty} \frac{\hat{T}_n(\varphi^n)}{n!} = \sum_{n=0}^{\infty} \hat{T}_n(\varphi).$$

Definition 8 gives $\|T_n\|_{N'} \leqslant \|T_n\|_{\theta'}$ and by ([3], p. 35) we see that $\|\hat{T}_n\|^{N'} = \|\hat{T}_n\|$. Hence

$$\|\hat{T}_n\| = \|\hat{T}_n\|^{N'} = \frac{1}{n!}\|T_n\|_{N'} \leqslant \frac{1}{n!}\|T_n\|_{\theta'}.$$

To show $\hat{T} \in \mathscr{H}(E')$ it suffices therefore to show

$$\limsup_{n\to\infty} \|T_n\|_{\theta'}^{1/n} < \infty.$$

If not $\exists\,(n_j)_{j=1}^{\infty}$ a sequence of positive integers increasing to infinity such that

$$\|T_{n_j}\|_{\theta'}^{1/n_j} > j.$$

Choose $P_{n_j} \in \mathscr{P}_\theta({}^{n_j}E)$ such that $\|P_{n_j}\|_\theta = 1$ and $T_{n_j}(P_{n_j}) > (j/2)^{n_j}$ and let $f = \sum_{j=0}^{\infty} (2/j)^{n_j} P_{n_j}$. f belongs to $H_\theta(E)$. But

$$T(f) = \sum_{j=0}^{\infty} \left(\frac{2}{j}\right)^{n_j} T_{n_j}(P_{n_j}) \geqslant \sum_{j=0}^{\infty} \left(\frac{2}{j}\right)^{n_j} \left(\frac{j}{2}\right)^{n_j} = \infty$$

which is a contradiction. Hence $\hat{T} \in \mathscr{H}(E')$. Also $\hat{d}^n\hat{T}(0) = n!\,\hat{T}_n \in \mathscr{P}_{\theta'}({}^nE')$.

PROPOSITION 15. *Let $F \in \mathscr{H}(E')$ and $F_n = \hat{d}^n F(0) \in \mathscr{P}_{\theta'}({}^nE')$. Then F is the Borel transform of an element of $(H_\theta(E), T_\theta)'$ if and only if $\exists\, K \in \hat{\mathscr{K}}$ such that for every $\varepsilon > 0$ we have*

$$\limsup_{n\to\infty} (\|F_n\|^{\theta', K+\varepsilon B_1})^{1/n} \leqslant 1.$$

Proof. We use the identification obtained in the preceding proposition. Suppose $F = \hat{T}$, where $T \in (H_\theta(E), T_\theta)'$. Let p be a semi-norm on $H_\theta(E)$, θ-ported by $K \in \hat{\mathscr{K}}$ such that $|T(f)| \leqslant p(f)$ for all $f \in H_\theta(E)$. For each $\varepsilon > 0$, $\exists\, C(\varepsilon) > 0$ such that

$$|T(f)| \leqslant p(f) \leqslant C(\varepsilon) \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+\varepsilon B_1}$$

for all $f \in H_\theta(E)$. Now

$$T(\varphi) = \sum_{n=0}^{\infty} \frac{T_n(\varphi^n)}{n!} = \sum_{n=0}^{\infty} \hat{T}_n(\varphi) = \sum_{n=0}^{\infty} \frac{F_n(\varphi)}{n!}.$$

For $P_n \in P_\theta({}^nE)$ we get $|T(P_n)| = |T_n(P_n)| \leqslant C(\varepsilon)\|P_n\|_{\theta,K+\varepsilon B_1}$. Thus $\|T_n\|_{\theta',K+\varepsilon B_1} \leqslant C(\varepsilon)$ and this implies

$$\limsup_{n\to\infty}(\|F_n\|^{\theta',K+\varepsilon B_1})^{1/n} \leqslant \limsup_{n\to\infty} C(\varepsilon)^{1/n} = 1.$$

Hence the only if part is proved. Conversely, suppose there exists $K \in \hat{\mathscr{K}}$ such that for every $\varepsilon > 0$

$$\limsup_{n\to\infty}(\|F_n\|^{\theta',K+\varepsilon B_1})^{1/n} \leqslant 1.$$

Define T on $H_\theta(E)$ by the following formula:

$$T(f) = \sum_{n=0}^{\infty} \frac{F_n}{n!}\left(\frac{\hat{d}^n f(0)}{n!}\right).$$

Let $\varepsilon > 0$ be arbitrary, choose $C(\varepsilon) > 0$ such that $\|F_n\|^{\theta',K+\varepsilon B_1} \leqslant C(\varepsilon)2^n$ for all n. Then

$$\sum_{n=0}^{\infty}\left|\frac{F_n}{n!}\left(\frac{\hat{d}^n f(0)}{n!}\right)\right| \leqslant \sum_{n=0}^{\infty}\left\|\frac{F_n}{n!}\right\|_{\theta',K+\varepsilon B_1}\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{\theta,K+\varepsilon B_1}$$

$$= \sum_{n=0}^{\infty}\|F_n\|^{\theta',K+\varepsilon B_1}\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{\theta,K+\varepsilon B_1} \leqslant C(\varepsilon)\sum_{n=0}^{\infty}\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{\theta,2K+2\varepsilon B_1}.$$

Hence $T \in (H_\theta(E), \mathcal{T}_\theta)'$. Also we have

$$\hat{T}(\varphi) = T(e^\varphi) = \sum_{n=0}^{\infty}\frac{F_n}{n!}\left(\frac{\varphi^n}{n!}\right) = \sum_{n=0}^{\infty}\frac{1}{n!}F_n(\varphi) = F(\varphi).$$

Definition 11. If

$$F = \sum_{n=0}^{\infty}\frac{F_n}{n!} \in \mathscr{H}(E'), \qquad F_n \in \mathscr{P}_{\theta'}({}^nE') \quad \text{and} \quad \exists K \in \hat{\mathscr{K}}$$

such that for every $\varepsilon > 0$ we have

$$\limsup_{n\to\infty}(\|F_n\|^{\theta',K+\varepsilon B_1})^{1/n} \leqslant 1$$

we say F is *of θ'-compact exponential type in E*.

Corollary 5. *There exists a one to one correspondence between the elements of θ'-compact exponential type in E and the elements of $(H_\theta(E), \mathcal{T}_\theta)'$.*

Example 4. The nuclear type is easily seen to be on α-β-holomorphy type. N' is the current type by [3], p. 35. Thus $f \in \mathscr{H}(E')$ is of N'-compact exponential type in E if and only if $\exists K \in \hat{\mathscr{K}}$ such that for every $\varepsilon > 0$ we have

$$(*) \qquad \limsup_{n\to\infty}(\|\hat{d}^n f(0)\|^{N',K+\varepsilon B_1})^{1/n} \leqslant 1.$$

By [3], p. 35, we see this is equivalent to

$$\limsup_{n\to\infty}(\|\hat{d}^n f(0)\|_{K+\varepsilon B_1})^{1/n} \leqslant 1.$$

By using Cauchy's inequalities we see this is equivalent to the fact that for every $\varepsilon > \exists\, C'(\varepsilon) > 0$ such that

$$|f(\varphi)| \leqslant C'(\varepsilon)\exp\left(\|\varphi\|_K + \varepsilon\|\varphi\|\right)$$

i. e., f is of compact exponential type in E in the usual sense (cf. [3] and [15]).

Example 5. The compact type is an α-β-holomorphy type. Condition (1) is satisfied (cf. [3], p. 11) and by the definition of the compact type condition (2) is satisfied.

α-β-γ-holomorphy types and partial differential operators. If $T_n \in \mathscr{P}_\theta({}^nE)'$ the function

$$T_n: H_\theta(E) \to C,$$

$$f \to T_n\left(\frac{\hat{d}^n f(0)}{n!}\right)$$

defines an element of $(H_\theta(E), t_\theta)'$. Hence, for $P_m \in \mathscr{P}_\theta({}^mE)$ and $x \in E$, $T_n(\tau_{-x}P_m)$ is a complex number. The function $\gamma(T_n)(P_m)$ whose value at x is $T_n(\tau_{-x}P_m)$ is then a well-defined function on E and can easily be shown to be an element of $\mathscr{P}({}^{m-n}E)$.

Definition 12. An α-β-holomorphy type θ is an *α-β-γ-holomorphy type* if it satisfies the following conditions:

(1) $\mathscr{P}_{\theta'}(E')$ is a translation invariant space with respect to differentiation, i. e., if $P \in \mathscr{P}_{\theta'}(E')$, $\varphi \in E'$ and l is a positive integer or zero, then $\hat{d}^l P(\varphi) \in \mathscr{P}_{\theta'}({}^lE')$.

(2) If $T_n \in \mathscr{P}_\theta({}^nE)'$, then for $m = 0, 1, \ldots$ the mapping $\gamma(T_n)$ maps $\mathscr{P}_\theta({}^mE)$ into $\mathscr{P}_\theta({}^{m-n}E)$ continuously. (We take $\mathscr{P}({}^jE) = 0$ for $j < 0$.)

It is possible to replace condition (1) by the following stronger but more elegant condition:

(1') $\left(\mathscr{P}_{\theta'}({}^nE'), \|\ \|^{\theta'}\right)_{n=0}^{\infty}$ is a holomorphy type.

Condition (2) is used to get a relationship between elements of $\mathscr{P}_{\theta'}(E')$ and partial differential operators.

Example 6. The nuclear type form an α-β-γ-holomorphy type. By [3], p. 35, $\left(\mathscr{P}_{N'}({}^nE'), \|\ \|^{N'}\right) = \left(\mathscr{P}({}^nE'), \|\ \|\right)$ hence condition (1') and (1) are satisfied. Condition (2) is also proved in [3] and [15].

Definition 13. Let θ be an α-β-γ-holomorphy type.

(a) $\mathcal{F}\mathcal{P}_\theta(E) = \prod_{n=0}^{\infty} \mathcal{P}_\theta({}^nE)$ is called the *set of all θ-formal power series on E*. We define scalar multiplication and addition coordinate-wise and give $\mathcal{F}\mathcal{P}_\theta(E)$ the product topology. Hence $\mathcal{F}\mathcal{P}_\theta(E)$ is a *Fréchet space*.

(b) Q is an *n-homogeneous partial differential operator on $\mathcal{F}\mathcal{P}_\theta(E)$* if it satisfies the following conditions;

(1) $Q\colon \mathcal{F}\mathcal{P}_\theta(E) \to \mathcal{F}\mathcal{P}_\theta(E)$ is a continuous linear operator.

(2) The restriction of Q to $\mathcal{P}_\theta(E)$ is translation invariant (we identify $f \in H_\theta(E)$ with an element of $\mathcal{F}\mathcal{P}_\theta(E)$ by taking the Taylor expansion at 0), i. e., if $\xi \in E, P \in \mathcal{P}_\theta(E)$ and if $(\tau_{-\xi}P)(a) = P(a+\xi)$ for all $a \in E$, then $Q(\tau_{-\xi}P) = \tau_{-\xi}Q(P)$.

(3) $Q(\mathcal{P}_\theta({}^mE)) \subset \mathcal{P}_\theta({}^{m-n}E)$ for $m = 0, 1, \ldots$

We denote by $PD_\theta(E_n)$ the set of all n-homogeneous partial differential operators on $\mathcal{F}\mathcal{P}_\theta(E)$.

(c) Q is a *partial differential operator* if it is a finite sum of homogeneous partial differential operators on $\mathcal{F}\mathcal{P}_\theta(E)$. We denote by $PD_\theta(E)$ the set of all partial differential operators on $\mathcal{F}\mathcal{P}_\theta(E)$.

We define the Borel transform of elements of $\mathcal{F}\mathcal{P}_\theta(E)'$ in the usual way. It is easy to check that there is a one to one correspondance between $\mathcal{P}_{\theta'}(E')$ and $\mathcal{F}\mathcal{P}_\theta(E)'$.

If $Q \in PD_\theta(E)$ we denote by $\beta(Q)$ the scalar valued function defined on $\mathcal{P}_\theta(E)$ by $\beta(Q)(P) = Q(P)(0)$.

Proposition 16. (a) *$\beta(Q) \in (\mathcal{P}_\theta(E), T_\theta)'$ for all $Q \in PE_\theta(E_n)$.*

(b) *The mapping $Q \in PD_\theta(E_n) \to \widehat{\beta(Q)} \in \mathcal{P}_{\theta'}({}^nE')$ is a one to one linear onto mapping.*

Proof. (a) $\beta(Q)$ is linear since Q is linear. Let $P = \sum_{i=0}^{m} P_i$, where $P_i \in \mathcal{P}_\theta({}^iE)$, $i = 0, \ldots, m$. By condition 3 of Definition 13(b) we get

$$\beta(Q)(P) = Q(P_n)(0) = Q(P_n).$$

Now let $P_a \in \mathcal{P}_\theta(E), P_a \to 0$ in $(H_\theta(E), T_\theta)$ as $a \to \infty$. By condition 1 of Definition 13 (b) we have

$$|\beta(Q)(P_a)| = \left|Q\left(\frac{\hat{d}^n P_a(0)}{n!}\right)\right| \leqslant c \left\|\frac{\hat{d}^n P_a(0)}{n!}\right\|_\theta.$$

Hence $\beta(Q) \in (\mathcal{P}_\theta(E), T_\theta)'$.

(b) By (a) $\widehat{\beta(Q)}$ is well-defined for $Q \in PD_\theta(E_n)$. Since

$$\widehat{\beta(Q)}(\varphi) = \beta(Q)(\exp\varphi) = Q\left(\frac{\varphi^n}{n!}\right) = \frac{1}{n!}Q(\varphi^n)$$

we have

$$\widehat{\beta(Q)}(\lambda\varphi) = \frac{1}{n!}Q((\lambda\varphi)^n) = \frac{\lambda^n}{n!}Q(\varphi^n).$$

Thus $\widehat{\beta(Q)}$ is an n-homogeneous function and since $\beta(Q) \in (\mathcal{P}_\theta(E), T_\theta)'$ we get that $\widehat{\beta(Q)} \in \mathcal{P}_{\theta'}({}^nE')$. It is one to one for if $\widehat{\beta(Q_1)} = \widehat{\beta(Q_2)}$, then $\beta(Q_1) = \beta(Q_2)$. Hence for every $P \in \mathcal{P}_\theta(E)$, $\xi \in E$ we have

$$\beta(Q_1)(\tau_{-\xi}P) = \beta(Q_2)(\tau_{-\xi}P), \quad Q_1(\tau_{-\xi}P)(0) = Q_2(\tau_{-\xi}P)(0),$$

$$Q_1(P)(\xi) = Q_2(P)(\xi), \quad Q_1(P) = Q_2(P) \Rightarrow Q_1 = Q_2.$$

There remains to show that the mapping is onto. Suppose $P_n \in \mathcal{P}_{\theta'}({}^nE')$. By Definition 12, $\gamma(\check{P}_n)\colon \mathcal{P}_\theta({}^mE) \to \mathcal{P}_\theta({}^{m-n}E)$ is well-defined and continuous and has an obvious extension as a continuous linear mapping from $\mathcal{F}\mathcal{P}_\theta(E)$ into $\mathcal{F}\mathcal{P}_\theta(E)$. If $\xi, \eta \in E, P \in \mathcal{P}_\theta(E)$, then

$$\begin{aligned}(\tau_{-\xi}[\gamma(\check{P}_n)(P)])(\eta) &= \gamma(\check{P}_n)(P)(\xi+\eta) = \check{P}_n(\tau_{-\xi-\eta}P)\\ &= \check{P}_n(\tau_{-\eta}\tau_{-\xi}P) = [\gamma(\check{P}_n)(\tau_{-\xi}P)](\eta).\end{aligned}$$

Hence

$$\tau_{-\xi}[\gamma(\check{P}_n)(P)] = \gamma(\check{P}_n)(\tau_{-\xi}P).$$

Thus $\gamma(\check{P}_n)$ is an n-homogeneous partial differential operator on $\mathcal{F}\mathcal{P}_\theta(E)$ and

$$\begin{aligned}\widehat{\beta[\gamma(\check{P}_n)]}(\varphi) &= \beta[\gamma(\check{P}_n)](e^\varphi) = [\gamma(\check{P}_n)(e^\varphi)](0) = \check{P}_n(e^\varphi)\\ &= \check{P}_n\left(\frac{\varphi^n}{n!}\right) = \frac{1}{n!}\check{P}^n(\varphi^n) = P_n(\varphi).\end{aligned}$$

Hence the mapping is onto.

Corollary 6. *There is a one to one correspondence between the elements of $PD_\theta(E)$ and the elements of $\mathcal{P}_{\theta'}(E')$. This correspondence is given by the linear mapping*

$$Q \in PD_\theta(E) \to \widehat{\beta(Q)} \in \mathcal{P}_{\theta'}(E'),$$

where $\beta(Q)(P) = Q(P)(0)$ for all $P \in \mathcal{P}_\theta(E)$.

If $Q_1, Q_2 \in PD_\theta(E)$, then we denote by $Q_1 * Q_2$ the mapping from $\mathcal{F}\mathcal{P}_\theta(E)$ into $\mathcal{F}\mathcal{P}_\theta(E)$ defined by $(Q_1 * Q_2)(f) = Q_1(Q_2(f))$. $Q_1 * Q_2$ is called the *convolution of Q_1 and Q_2* and it is easily checked that $Q_1 * Q_2 \in PD_\theta(E)$. For

$x\in E$, $\varphi\in E'$ we note $\tau_{-x}\exp\varphi = \exp\varphi(x)\cdot\exp\varphi$. Hence $Q_2(\exp\varphi)(x) = Q_2(\tau_{-x}\exp\varphi)(0) = \exp\varphi(x)\cdot Q_2(\exp\varphi)(0) = \exp\varphi(x)\cdot\beta\widehat{(Q_2)}(\varphi)$ and

$$\begin{aligned}[\beta\widehat{(Q_1*Q_2)}](\varphi) &= [\beta(Q_1*Q_2)](\exp\varphi) = [(Q_1*Q_2)(\exp\varphi)](0)\\ &= [Q_1(Q_2(\exp\varphi))](0) = [Q_{1,x}(\exp\varphi(x)\cdot\beta\widehat{(Q_2)}(\varphi))](0)\\ &= \beta\widehat{(Q_2)}(\varphi)\cdot Q_1(\exp\varphi)(0) = \beta\widehat{(Q_2)}(\varphi)\cdot\beta\widehat{(Q_1)}(\varphi).\end{aligned}$$

Therefore

$$\beta\widehat{(Q_1*Q_2)} = \beta\widehat{(Q_1)}\beta\widehat{(Q_2)}.$$

Proposition 17. (a) *$\mathscr{P}_{\theta'}(E')$ is a commutative algebra under pointwise multiplication.*

(b) *$PD_\theta(E)$ is a commutative algebra under convolution.*

(c) *The mapping $PD_\theta(E)\to\mathscr{P}_{\theta'}(E')$*

$$Q\to\beta\widehat{(Q)}$$

is a 1—1 *onto linear and algebraic isomorphism.*

Lemma 8. *If $P_1, P_2\in\mathscr{P}_{\theta'}(E')$, $P_2\neq 0$ and $P_3 = \dfrac{P_1}{P_2}\in\mathscr{H}(E')$, then $P_3\in\mathscr{P}_{\theta'}(E')$.*

Proof. Using Liouville's theorem we easily get $P_3\in\mathscr{P}(E')$ (cf. [11], p. 225). By condition (1) of definition 12 it suffices to show for some $\xi\in E'$ that $\hat d^n P_3(\xi)\in\mathscr{P}_{\theta'}(^nE')$, $n = 0, 1, \ldots$ Choose $\xi\in E'$ such that $P_2(\xi)\neq 0$. Hence

$$P_3(\xi) = \frac{P_1(\xi)}{P_2(\xi)}\in\mathscr{P}_{\theta'}(^0E').$$

Now

$$P_i(x) = \sum_j\frac{\hat d^j P_i(\xi)}{j!}(x-\xi)\quad\text{for } i = 1, 2, 3.$$

By hypothesis

$$\hat d^j P_i(\xi)\in\mathscr{P}_{\theta'}(^jE')\quad\text{for all } j, i = 1, 2,$$

and

$$\hat d^j P_3(\xi)\in\mathscr{P}(^jE')\quad\text{for all } j.$$

Suppose

$$\hat d^j P_3(\xi)\in\mathscr{P}_{\theta'}({}_jE)\quad\text{for } j\leqslant k;$$

then

$$\frac{\hat d^{k+1}P_3(\xi)}{(k+1)!} = \frac{\hat d_1^{k+1}P_1(\xi) - \sum_{i=1}^{k+1}\dfrac{\hat d^i P_2(\xi)}{i!}\,\dfrac{\hat d^{k+1-i}P_3(\xi)}{(k+1-i)!}}{P_2(\xi)}.$$

Since $\mathscr{P}_{\theta'}(E')$ is an algebra by using induction we get the required result.

Definition 14. A function f on E is called an *θ-exponential polynomial* if there exists $\varphi\in E'$, $P\in\mathscr{P}_\theta(E)$ such that $f(x) = P(x)e^{\varphi(x)}$.

If Q is a partial differential operator on $\mathscr{F}\mathscr{P}_\theta(E)$, $\beta\widehat{(Q)}\in\mathscr{P}_{\theta'}(E')\subset\mathscr{P}_{N'}(E')$. Hence we can associate with it in a unique way a convolution operator $\mathcal{O}_Q$ on $(H_N(E), T_N)$ (cf. [15], p. 9). $\mathcal{O}_Q$ is in fact the restriction of Q to $H_N(E)\subset H_\theta(E)\subset\mathscr{F}\mathscr{P}(E)$. Also if $\nu\in(\mathscr{F}\mathscr{P}(E))'$, then $\hat\nu\in\mathscr{P}_{\theta'}(E')\subset\mathscr{P}_{N'}(E')$ and by Proposition 15 we can associate with it in a unique way a continuous linear functional ν_N on $(H_N(E), T_N)$. ν_N is the restriction of ν to $H_N(E)$.

Proposition 18. *Let θ be an α-β-γ-holomorphy type and let $Q\in PD_\theta(E)$; then:*

(a) *Every solution of Q can be approximated in $\mathscr{F}\mathscr{P}_\theta(E)$ by θ-exponential polynomial solutions of Q.*

(b) *If $Q\neq 0$, then Q maps $\mathscr{F}\mathscr{P}_\theta(E)$ onto itself.*

Proof. (a) If $Q = 0$, then (a) is true since the Taylor expansion converges in $\mathscr{F}\mathscr{P}_\theta(E)$. Suppose $Q\neq 0$. Let $\nu\in\mathscr{F}\mathscr{P}_\theta(E)'$ be such that $\langle\nu, f\rangle = 0$ for all θ-exponential polynomial solutions f of Q. By ([15], p. 18) we have that $\dfrac{\hat\nu_N}{\beta\widehat{(\mathcal{O}_Q)}}\in\mathscr{H}(E')$. Since $\hat\nu_N = \hat\nu$ and $\beta\widehat{(\mathcal{O}_Q)} = \beta\widehat{(Q)}$ we see that $\dfrac{\hat\nu}{\beta\widehat{(Q)}}\in\mathscr{H}(E')$. Lemma 8 implies $\dfrac{\hat\nu}{\beta\widehat{(Q)}}\in\mathscr{P}_{\theta'}(E')$. Hence, by Corollary 6, $\exists\, Q_\nu, Q_1\in PD_\theta(E)$ such that $\hat\nu = \beta\widehat{(Q_\nu)}$ and

$$\beta\widehat{(Q_1)} = \frac{\hat\nu}{\beta\widehat{(Q)}} = \frac{\beta\widehat{(Q_\nu)}}{\beta\widehat{(Q)}}.$$

Hence $Q_1*Q = Q_\nu$.

Now if $h\in\mathscr{F}\mathscr{P}_\theta(E)$ and $Q(h) = 0$, then

$$\langle\nu, h\rangle = [Q_\nu(h)](0) = (Q_1*Q)(h)(0) = [Q_1(Q(h))](0) = 0.$$

An application of the Hahn-Banach theorem completes the proof.

(b) We note $\mathscr{F}\mathscr{P}_\theta(E)$ is a Fréchet space. To show Q is onto it suffices (cf. [4], p. 308) to show ${}^tQ$ is one to one and $\operatorname{Im}{}^tQ$ is closed for the weak topology on $\mathcal{FP}_\theta(E)'$ defined by $\mathcal{FP}_\theta(E)$. Since Q is a linear operator we get $\mathrm{Im}\,{}^tQ \subset (\mathrm{Ker}\,Q)^\perp$. Let $\nu \in (\mathrm{Ker}\,Q)^\perp$. By the first part of the proposition there exists $Q_1, Q_\nu \in PD_\theta(E)$ such that $Q_1 * Q = Q_\nu$ and $\beta(Q_\nu) = \nu$. Therefore

$$\begin{aligned}\langle {}^tQ\beta(Q_1), f\rangle &= \langle \beta(Q_1), Q(f)\rangle \\ &= [Q_1(Q(f)](0) = [(Q_1*Q)(f)](0) \\ &= Q_\nu(f)(0) = \beta(Q_\nu)(f) = \nu(f)\end{aligned}$$

which implies

$${}^tQ\beta(Q_1) = \nu.$$

Hence $\nu \in \mathrm{Im}\,{}^tQ$ and $\mathrm{Im}\,{}^tQ = (\mathrm{Ker}\,Q)^\perp$. We have proved

$$\begin{aligned}\mathrm{Im}\,{}^tQ &= \{\nu \in \mathcal{FP}_\theta(E)' \text{ such that } \langle \nu, f\rangle = 0 \text{ for all } f \in \mathrm{Ker}\,Q\} \\ &= \bigcap_{f\in \mathrm{Ker}\,Q} \{\nu \in \mathcal{FP}_\theta(E)' \text{ such that } \langle \nu, f\rangle = 0\}.\end{aligned}$$

Hence $\mathrm{Im}\,{}^tQ$ is the intersection of weakly closed sets and hence is weakly closed. Now suppose ${}^tQ\nu = 0$. Let $Q_\nu \in PD_\theta(E)$ be such that $\beta(Q_\nu) = \nu$. For $\xi \in E$, $f \in \mathcal{P}_\theta(E)$, we have

$$\begin{aligned}[(Q_\nu * Q)(f)](\xi) &= [(Q_\nu * Q)(\tau_{-\xi}f)](0) = [Q_\nu(Q(\tau_{-\xi}f)](0) \\ &= \beta(Q_\nu)[Q(\tau_{-\xi}f)] = \nu[Q(\tau_{-\xi}f)] = 0.\end{aligned}$$

Hence

$$Q_\nu * Q = 0$$

and this implies $\widehat{\beta(Q_\nu)}\cdot\widehat{\beta(Q)} = 0$.

Since $Q \not\equiv 0$ and $\widehat{\beta(Q_\nu)}, \widehat{\beta(Q)} \in \mathcal{H}(E')$ we get $\widehat{\beta(Q_\nu)} = \hat{\nu} = 0$.

Example 1. The compact type form an α-β-γ-holomorphy type. We have seen already that it forms an α-β-holomorphy type. For the sake of simplicity we assume E is a reflexive Banach space. Hence B_1 is $\sigma(E, E')$ compact. We denote by $\mathcal{C}(B_1)$ the set of all $\sigma(E, E')$ continuous functions on B_1 and by $(\mathcal{C}(B_1), \sigma(E, E'))$ the space $\mathcal{C}(B_1)$ with the sup-norm topology over B_1. Since the norm topology on E is stronger than the $\sigma(E, E')$ topology and the polynomials of finite type are $\sigma(E, E')$ continuous on E we can by using the restriction map imbed $\mathcal{P}_C({}^nE)$ isometrically in $(\mathcal{C}(B_1), \sigma(E, E'))$ for each n. Hence if $T_n \in (\mathcal{P}_C({}^nE), \|\ \|)'$ we can extend it by the Hahn–Banach theorem to be a continuous linear functional on $(\mathcal{C}(B_1), \sigma(E, E'))$. Thus there exists a Radon measure μ on $(B_1, \sigma(E, E'))$ such that

(*) $$T_n(P^n) = \int_{\|x\|\le 1} P_n(x)\,d\mu(x) \quad \text{for all } P_n \in \mathcal{P}_C({}^nE)$$

and $\|T_n\|_{C'} \le \|\mu\|$ where $\|\mu\|$ is the norm of μ as an element of $(\mathcal{C}(B_1), \sigma(E, E'))'$. If μ satisfies (*) we say μ represents T_n and write $\mu \sim T_n$. By the Hahn–Banach theorem

$$\|T_n\|_{C'} = \inf_{\mu\sim T_n} \|\mu\|$$

and there exists μ a Radon measure on $(B_1, \sigma(E, E'))$ such that $\mu \sim T_n$ and $\|\mu\| = \|T_n\|_{C'}$.

Definition 15. Let $P \in \mathcal{P}({}^nE')$. P is said to be an *n-homogeneous integral polynomial on E'* if $\exists\,\mu$ a Radon measure on $(B_1, \sigma(E, E'))$ such that

(*) $$P(\varphi) = \int_{\|x\|\le 1} \langle \varphi, x\rangle^n d\mu(x) \quad \text{for all } \varphi \in E'.$$

We denote by $\mathcal{P}_I({}^nE')$ the space of n-homogeneous integral polynomials on E'. We say μ a Radon measure on $(B_1, \sigma(E, E'))$ represents P (and we write $P \sim \mu$) if (*) holds.

We norm $\mathcal{P}_I({}^nE)$ by $\|P\|_I = \inf_{\mu\sim P}\|\mu\|$, where $\|\mu\|$ is the norm of μ as an element of $(\mathcal{C}(B_1), \sigma(E, E'))'$.

Lemma 9. *$(\mathcal{P}_{C'}({}^nE'), \|\ \|^{C'})$ is isometrically isomorphic to $(\mathcal{P}_I({}^nE'), \|\ \|_I)$ for each n.*

Proof. Let $T_n \in (\mathcal{P}_C({}^nE), \|\ \|_C)'$. By our previous remarks there $\exists\mu$ a Radon measure on $(B_1, (E, E'))$ such that $\mu \sim T_n$ and $\|\mu\| = \|T_n\|_{C'}$. For $\varphi \in E'$ we have

$$\hat{T}_n(\varphi) = \frac{1}{n!}T_n(\varphi^n) = \frac{1}{n!}\int_{\|x\|\le 1} \langle \varphi, x\rangle^n d\mu(x).$$

Hence

$$\hat{T}_n \in \mathcal{P}_I({}^nE') \quad \text{and} \quad \frac{\mu}{n!} \sim \hat{T}_n.$$

Therefore

$$\|\hat{T}_n\|_I \le \frac{\|\mu\|}{n!} = \frac{\|T_n\|_{C'}}{n!} = \|\hat{T}_n\|^{C'}.$$

Conversely suppose $T_n \in \mathcal{P}_I({}^nE')$ and $\mu \sim \hat{T}_n$. For $\sum_{i=0}^m \varphi_i^n \in \mathcal{P}_f({}^nE)$ we define

$$T_n^*\Big(\sum_{i=0}^m \varphi_i^n\Big) = n!\sum_{i=0}^m T_n(\varphi_i) = n!\int_{\|x\|\le 1}\sum_{i=0}^m \langle \varphi_i, x\rangle^n d\mu(x).$$

Thus

(**) $$\Big|T_n^*\Big(\sum_{i=0}^m \varphi_i^n\Big)\Big| \le n!\,\|\mu\|\cdot\Big\|\sum_{i=0}^m \varphi_i^n\Big\|$$

and so T_n^* has a unique extension T_n^* to an element of $(\mathscr{P}_C({}^nE), \|\ \|)'$. Also $T_n^*(\varphi) = \frac{1}{n!} T_n^*(\varphi^n) = \frac{1}{n!}\, n!\, T_n(\varphi) = T_n(\varphi)$. Hence $\mathscr{P}_{C'}({}^nE') \cong \mathscr{P}_I({}^nE')$ as sets. By (**) we have that $\|T_n\|_{C'} \leqslant n!\|\mu\| = n!\, \|\hat{T}_n\|_I$. Therefore $\|\hat{T}_n\|^{C'} = \|\hat{T}_n\|_I$.

PROPOSITION 19. *$(\mathscr{P}_I({}^nE'), \|.\|_I)_{n=0}^\infty$ is a holomorphy type.*

Proof. Since $(\mathscr{P}_I({}^nE'), \|\ \|_I)$ is linearly isomorphic to $(\mathscr{P}_C({}^nE)', \|\ \|_{C'})$, for $n = 0, 1, \ldots, (\mathscr{P}_I({}^nE'), \|\ \|_I)$ is a Banach space for $n = 0, 1, \ldots$ Because $\mathscr{P}_C({}^0E) \approx C$ we have $\mathscr{P}_I({}^0E') \approx C$. It suffices therefore to verify condition 3 of Definition (1). Let $P \in \mathscr{P}_I({}^nE')$ and $\mu \sim P$ be such that $\|\mu\| = \|P\|_I$. By [13], p. 18, we have for $l \leqslant n$, $\varphi, \psi \in E'$

$$\frac{\hat{d}^l P(\psi)(\varphi)}{l!} = C_l^n \int_{\|x\|\leqslant 1} \langle\psi, x\rangle^{n-l} \langle\varphi, x\rangle^l d\mu(x).$$

Now μ is a Radon measure on $(B_1, \sigma(E, E'))$ and $x \to \langle\psi, x\rangle^{n-l} C_l^n$ is an element of $\mathscr{C}(B_1)$, hence $d\mu'(x) = \langle\psi, x\rangle^{n-l} C_l^n\, d\mu(x)$ defines a Radon measure on $(B_1, \sigma(E, E'))$ and

$$\frac{\hat{d}^l P(\psi)(\varphi)}{l!} = \int_{\|x\|\leqslant 1} \langle\varphi, x\rangle^l d\mu'(x).$$

Therefore

$$\frac{\hat{d}^l P(\psi)}{l!} \in \mathscr{P}_I({}^lE') \quad \text{and} \quad \left\| \frac{1}{l!}\, \hat{d}^l P(\psi) \right\|_I \leqslant \|\mu'\| \leqslant C_l^n \|\mu\|\, \|\psi\|^{n-l}.$$

Thus

$$\left\| \frac{\hat{d}^l P(\psi)}{l!} \right\|_I \leqslant 2^n \|P\|_I \|\psi\|^{n-l}.$$

This completes the proof.

COROLLARY 7. *$(\mathscr{P}_I({}^nE'), \|\ \|_I)_{n=0}^\infty$ is an a-holomorphy type.*

Proof. Since $(\mathscr{P}_C({}^nE), \|\ \|_C)_{n=0}^\infty$ is an a-holomorphy type one need only use the relationship between $\mathscr{P}_I({}^nE')$, $\mathscr{P}_{C'}({}^nE')$ and $\mathscr{P}_C({}^nE)'$ and Proposition 19 to get the required result.

To show $(\mathscr{P}_C({}^nE), \|\ \|)_{n=0}^\infty$ is an a-β-γ-holomorphy type there remains only to verify condition (2) of definition 12. Let $T_n \in \mathscr{P}_{C'}({}^nE')$ and let $\mu \sim T_n$, $\|\mu\| = \|T_n\|_{C'}$. Suppose $P_m \in \mathscr{P}_C({}^mE)$ and A_m is the symmetric n-linear form on ${}^mE$ such that $\hat{A}_m = P_m$ (cf. [11]). Then for $\xi \in E$

$$[\gamma(T_n)(P_m)](\xi) = T_n\left[\frac{\hat{d}^n(\tau_{-\xi}P_m)}{n!}\right](0) = \int_{\|x\|\leqslant 1} \frac{\hat{d}^n(\tau_{-\xi}P_m)}{n!}(0)(x)\, d\mu(x)$$
$$= C_n^m \int_{\|x\|\leqslant 1} A_m(\xi)^{m-n}(x)^n d\mu(x).$$

We note $\gamma(T_n)$ maps $\mathscr{P}_f({}^mE)$ into $\mathscr{P}_f({}^{m-n}E)$ linearly. If $P_m \in \mathscr{P}_f({}^mE)$ we thus get

$$\sup_{\|\xi\|\leqslant 1} |[\gamma(T_n)(P_m)](\xi)| \leqslant C_n^m \left| \int_{\|x\|\leqslant 1} A_m(\xi)^{m-n}(x)^n d\mu(x) \right|$$
$$\leqslant C_n^m \|A_m\| \cdot \|\mu\| \leqslant C_n^m \frac{m^m}{m!} \|P_m\| \cdot \|T_n\|_{C}.$$

Hence $\gamma(T_n)$ maps $(\mathscr{P}_f({}^mE), \|\ \|)$ into $\mathscr{P}_f({}^{m-n}E)$ continuously and linearly. Thus it has a unique extension to a continuous linear mapping from $(\mathscr{P}_C({}^mE), \|\ \|)$ into $(\mathscr{P}_C({}^{m-n}E), \|\ \|)$. This completes the proof.

4. EXAMPLES ON $\mathscr{H}(E)$ AND $\mathscr{H}_C(E)$

The examples here try to show the difference between the finite and infinite dimensional theory.

Definition 15. (cf. [12] and [13]). (a) For $n = 0, 1, \ldots, \mathscr{T}_n$ will denote the topology on $\mathscr{H}(E)$ of uniform convergence on compact subsets of E of each derivative or order $\leqslant n$.

(b) $\mathscr{T}_\infty$ will denote the topology on $\mathscr{H}(E)$ of uniform convergence on compact subsets of E of each derivative.

In the following E will denote a complex separable infinite dimensional Hilbert space with inner product denoted by $\langle\ ,\ \rangle$. Since $E \cong E'$ isometrically we freely use x as an element either of E or E'. If A is a subset of E we denote by $\{A\}$ the vector space spanned by A and if B is a closed subspace of E we denote by $\mathrm{Proj}\, A \cap B$ the projection of A onto B. We need the following result which can easily be proved by contradiction.

LEMMA 10. *If K is a compact subset of E and $(X_n)_{n=0}^\infty$ is an orthonormal subset for E, then, given any positive real number a, $\exists\, n_0 = n_0(a)$ such that $\mathrm{Proj}\, K \cap \{X_i\} \subset \{x, \|x\| \leqslant a\}$ for $i \geqslant n_0$.*

PROPOSITION 20. *$(\mathscr{H}(E), \mathscr{T}_n)$ is not barrelled for $n = 0, 1, \ldots, \infty$.*

Proof. Let $(X_n)_{n=i_1}^\infty$ be an orthonormal basis for E. By considering the polynomials X_n^m and using Lemma 10 we get the required result for $n = 0, 1, 2, \ldots$ For $n = \infty$ we take $(a_n)_{n=0}^\infty \in C_0^+$ and consider the set

$$B = \left\{ f \in \mathscr{H}(E), \sum_{n=0}^\infty \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{a_n B_1} \leqslant 1 \right\}.$$

By applying the result for $n = 0, 1, \ldots$ we complete the proof.

We note that in the proof we actually show that $\mathscr{T}_0 \neq \mathscr{T}_1 \neq \mathscr{T}_2 \neq \ldots \neq \mathscr{T}_\infty \neq \mathscr{T}_\omega$. Since (cf. [12]) these topologies have all got the same bounded sets this implies (cf. [5], p. 226) that $(\mathscr{H}(E), \mathscr{T}_n)$ is not a bornological space for $n = 0, 1, \ldots, \infty$. We denote by $(\mathscr{C}(E), \mathscr{T}_0)$ the set of all continuous complex valued functions on E with the usual definitions of addition and scalar multiplication and with the topology of uniform convergence on compact subsets of E. For E a separable Hilbert space $(\mathscr{C}(E), \mathscr{T}_0)$ is a barrelled, bornological space (cf. [10]). Thus (cf. [12]) we have an example of a closed subspace of a barrelled bornological space which is neither barrelled nor bornological. For other examples of the above we refer to ([6], § 27 and § 28). It is also of interest to compare the proof that $(\mathscr{C}(E), \mathscr{T}_0)$ is barrelled with Proposition 21 below.

In [3] and [13] we see remarks and examples that if $f \in \mathscr{H}(E)$, then f need not be bounded on bounded sets. Our aim now is to prove that if C is any closed non compact subset of E there exists $f \in \mathscr{H}(E)$ such that $\sup_{x \in C} |f(x)| = \infty$.

PROPOSITION 21. *Let E be a complex infinite dimensional Hilbert space and C a closed non-compact subset of E; then there exists $f \in \mathscr{H}(E)$ such that* $\sup_{x\in C} |f(x)| = \infty$.

We first need the following preliminary lemma:

LEMMA 11. *Let E be a complex infinite dimensional Hilbert space. If $(\beta_n)_{n=0}^{\infty}$ is a sequence of positive real numbers and $(X_n)_{n=0}^{\infty}$ is an orthonormal subset of E, then identifying E and E' we get*

$$f = \sum_{n=0}^{\infty} \beta_n X_n^n \in \mathscr{H}(E)$$

if and only if

$$\sup_n \beta_n^{1/n} < \infty.$$

Proof. Suppose $f = \sum_{n=0}^{\infty} \beta_n X_n^n \in \mathscr{H}(E)$; then f has a positive radius of convergence about zero. Hence

$$\limsup_{n\to\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|^{1/n} < \infty.$$

Since

$$\left\| \frac{\hat{d}^n f(0)}{n!} \right\|^{1/n} = \beta_n^{1/n}$$

we get

$$\limsup_{n\to\infty} \beta_n^{1/n} < \infty.$$

To show $f \in \mathscr{H}(E)$ it suffices by Proposition 5 to show for each $K \in \hat{K}$, $(a_n)_{n=0}^{\infty} \in c_0^+$ that

$$\lim_{n\to\infty} \|\beta_n X_n^n\|_{K+a_nB_1}^{1/n} = 0,$$

i. e., to show

$$\lim_{n\to\infty} \beta_n^{1/n} \|X_n\|_{K+a_nB_1} = 0.$$

Now

$$\|X_n\|_{K+a_nB_1} = \sup_{x\in K+a_nB_1} |\langle X_n, x\rangle| \leqslant \sup_{x\in K} |\langle X_n, x\rangle| + a_n.$$

By Lemma 10, $\sup_{x\in K} |\langle X_n, x\rangle| \to 0$ as $n \to \infty$. Since $(a_n)_{n=0}^{\infty} \in c_0^+$ and $\sup_n \beta_n^{1/n} < \infty$ we get the required result.

Proof (of proposition). We first consider the case when C is not bounded. Hence C is not bounded in the weak topology on E and thus there exists $y \in E'$ such that $\sup_{x\in C} |\langle y, x\rangle| = \infty$. Since $y \in E' \subset \mathscr{H}(E)$ this proves our result when C is not bounded.

Now suppose C is a bounded subset of E. Suppose $C \subset \{X, \|X\| \leqslant M\}$. This means in particular that all coefficients of elements of C relative to any orthonormal expansion will be bounded by M. Since C is not compact there exists $\delta > 0$ and a sequence $(y_n)_{n=1}^{\infty}$ of elements of C such that $\|y_n - y_m\| \geqslant \delta$ for all $n \neq m$. Let $y_1 = a_{11}x_1$, $\|x_1\| = 1$ and $y_n = a_{1n}x_1 + \Phi_n$, where $\Phi_n \perp x_1$ for $n \geqslant 2$. Since $\sup |a_{1n}| \leqslant M$ by taking a subsequence if necessary we can suppose $\lim_{n\to\infty} a_{1n}$ exists and equals $M_1 \ldots$, (3). By (3), $\exists\, n_2$ such that $|a_{1n} - M_1| < \delta/32$ for $n \geqslant n_2$. Hence

$$\|\Phi_n - \Phi_m\| \geqslant \|y_n - y_m\| - |a_{1n} - a_{1m}| \geqslant \delta - \delta/32 = 15\delta/16.$$

and so without loss of generality we can assume

$$\|\Phi_n\| \geqslant \delta/3 \quad \text{ for } n \geqslant n_2.$$

By the method of induction we can, using the above procedure, choose for each integer k a positive integer n_k, $x_k \in E$, M_k such that $\|x_k\| = 1$, $x_k \perp x_i$ for $i < k$, and $y_{n_k} = \sum_{i=1}^{k} a_{in_k} x_i$, where

(4) $$|a_{in_k} - M_i| < \delta/32k \quad \text{ for } i < k$$

and $|a_{kn_k}| > \delta/3$.

We claim there exists N a positive integer such that

(5) $$|M_i| \leqslant \delta/32 \quad \text{ for } i \geqslant N.$$

If not choose r a positive integer greater than $(M64/\delta)^2$ and $s_1, \ldots, s_r$ which do not satisfy (5). Now

$$y_{n_{s_r}} = \sum_{i=1}^{n_{s_r}} a_{in_{s_r}} x_i.$$

Therefore

$$\|y_{n_{s_r}}\|^2 = \sum_{i=1}^{n_{s_r}} |a_{in_{s_r}}|^2$$

and

$$\|y_{n_{s_r}}\|^2 \geqslant \sum_{j=1}^{r} |a_{n_{s_j} n_{s_r}}|^2 \geqslant \sum_{j=1}^{r} (\delta/64)^2 > (M64/\delta)^2(\delta/64)^2 = M^2 \quad \text{(by (5))}.$$

This is impossible, hence our claim is true. Let $\beta_n = (16/3\delta)^n$ and $f = \sum_{n=N}^{\infty} \beta_n x_n^n$. By Lemma 11, $f \in \mathscr{H}(E)$ and for $k \geqslant N$

$$|f(y_{n_k})| = \Big|\sum_{j=N}^{\infty} \beta_j \langle x_j, y_{n_k}\rangle^j\Big| = \Big|\sum_{j=N}^{\infty} \beta_j \langle x_j, \sum_{i=1}^{k} a_{in_k} x_i\rangle^j\Big|$$

$$= \Big|\sum_{j=N}^{\infty} \beta_j \langle x_j, a_{jn_k} x_j\rangle^j\Big| > \beta_k |a_{kn_k}|^k - \sum_{j=N}^{k-1} \beta_j |a_{jn_k}|^j$$

$$> (16/3\delta)^k(\delta/3)^k - \sum_{j=N}^{k-1} (16/3\delta)^j(\delta/16)^j > (16/9)^k \quad \text{(by (5))}.$$

Hence $|f(y_{n_k})| \to \infty$ as $k \to \infty$ which gives $\sup_{x \in C} |f(x)| = \infty$.

In finishing this section we discuss some topological properties of the algebras $\mathscr{H}(E)$ and $\mathscr{H}_C(E)$. We show (cf. [9]) that $(\mathscr{H}(E), \mathscr{T}_0)$ and $(\mathscr{H}(E), \mathscr{T}_\infty)$ are both complete m-convex topological algebras. This shows that the uniqueness theorem for norms on a Banach algebra (cf. [7]) cannot be extended to complete m-convex topological algebras. We also characterize the set of continuous multiplicative linear functionals and the closed maximal ideals on the spaces $(\mathscr{H}_C(E), \mathscr{T}_0)$ and $(\mathscr{H}_C(E), \mathscr{T}_\infty)$.

Proposition 22. *$(\mathscr{H}(E), \mathscr{T}_n)$ is a complete space for each n.*

Proof. Trivial.

Proposition 23. (a) *$(\mathscr{H}(E), \mathscr{T}_0)$ is an m-convex topological algebra.*

(b) *$(\mathscr{H}(E), \mathscr{T}_\infty)$ is an m-convex topological algebra.*

Proof. Our multiplication is pointwise and it is well known that the product of two holomorphic functions is again a holomorphic function. It suffices therefore to show that in each space there exists a fundamental set of neighborhoods of zero $(U_\alpha)_{\alpha \in A}$ such that $U_\alpha \cdot U_\alpha \subset U_\alpha$.

(a) Let $U_{\varepsilon,K} = \{f, \sup_{x \in K} |f(x)| \leqslant \varepsilon\}$. Then as K ranges over $\mathscr{K}$ and ε over all positive numbers less than 1 we get a fundamental family of neighborhoods in $(\mathscr{H}(E), \mathscr{T}_0)$ which satisfies our conditions.

(b) Let $V_{K,n,\varepsilon} = \Big\{f \in \mathscr{H}(E), \sum_{i=0}^{n} \frac{1}{i!} \sup_{x \in K} \|\hat{d}^i f(x)\| \leqslant \varepsilon\Big\}$.

Then as K ranges over $\mathscr{K}$, n over all positive integers and ε over all positive numbers less than 1 we get a fundamental family of neighborhoods of zero in $(\mathscr{H}(E), \mathscr{T}_\infty)$ which satisfies our conditions. It is easy to check that $\mathscr{P}_C(E)$, the set of all compact polynomials, is an algebra when E is a Hilbert space. Hence $\mathscr{H}_C(E)$ is also an algebra and so $(\mathscr{H}_C(E), \mathscr{T}_\infty)$ and $(\mathscr{H}_C(E), \mathscr{T}_0)$ are both m-convex topological algebras. We now obtain a one to one correspondence between the points of E and the continuous multiplicative linear functions on $(\mathscr{H}_C(E), \mathscr{T}_0)$ and $(\mathscr{H}_C(E), \mathscr{T}_\infty)$. Since $(\mathscr{H}_C(E), \mathscr{T}_0)$ and $(\mathscr{H}_C(E), \mathscr{T}_\infty)$ are both semi-simple commutative translation invariant topological algebras this gives (cf. [1]) a characterization of the closed maximal ideals of the spaces.

Proposition 24. *ν is a continuous multiplicative linear functional on $(\mathscr{H}_C(E), \mathscr{T}_0)$ and on $(\mathscr{H}_C(E), \mathscr{T}_\infty)$ if and only if $\exists\, \xi \in E$ such that $\nu(f) = f(\xi)$ for all $f \in \mathscr{H}_C(E)$.*

Proof. If $\nu(f) = f(\xi)$ for all $f \in \mathscr{H}_C(E)$, then ν is obviously a continuous multiplicative linear functional on $(\mathscr{H}_C(E), \mathscr{T}_0)$ and on $(\mathscr{H}_C(E), \mathscr{T}_\infty)$. Conversely if ν is a continuous multiplicative linear functional on $(\mathscr{H}_C(E), \mathscr{T}_0)$, then ν restricted to E' is continuous. Hence $\exists\, \xi \in E$ such that $\nu(\varphi) = \langle \varphi, \xi\rangle$ for all $\varphi \in E'$. Let $P = \sum_{i=1}^{N} \varphi_i^n \in \mathscr{P}_f({}^nE)$; then

$$\nu(P) = \sum_{i=1}^{N} \nu(\varphi_i)^n = \sum_{i=1}^{N} \varphi_i(\xi)^n = P(\xi).$$

By continuity $\nu(P) = P(\xi)$ for all $P \in \mathscr{P}_C({}^nE)$. If $f \in \mathscr{H}_C(E)$, then

$$\nu(f) = \sum_{i=0}^{\infty} \nu\left(\frac{\hat{d}^n f(0)}{n!}\right) = \sum_{i=0}^{\infty} \frac{\hat{d}^n f(0)}{n!}(\xi) = f(\xi).$$

Hence $\nu(f) = f(\xi)$ for all $f \in \mathscr{H}_C(E)$. Since $\mathscr{T}_\infty \geqslant \mathscr{T}_0$ this also proves the result for $(\mathscr{H}_C(E), \mathscr{T}_\infty)$.

Examples on $\mathscr{H}_N(E)$. We now give a number of examples which emphasize the difference between the current and the nuclear type.

Definition 16. We say a holomorphy type $\mathscr{A}$ *has Cauchy inequalities* if, for each n, $\exists\, C(n)$ such that for all $f \in \mathscr{H}_{\mathscr{A}}(E)$ we have

$$\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{\mathscr{A}} \leqslant C(n) \sup_{\|x\| \leqslant 1} |f(x)|.$$

Lemma 12. *If $\mathscr{A}$ has Cauchy inequalities, then $(\mathscr{P}_{\mathscr{A}}({}^nE), \|\ \|_{\mathscr{A}})$ is isomorphic to a subspace of $(\mathscr{P}({}^nE), \|\ \|)$ for each n.*

Proof. By definition of a holomorphy type $\mathscr{P}_{\mathscr{A}}({}^nE) \subset \mathscr{P}({}^nE)$ and $\|\ \| \leqslant \sigma^n \|\ \|_{\mathscr{A}}$. But also $\|P_n\|_{\mathscr{A}} \leqslant C(n) \sup_{\|x\| \leqslant 1} |P_n(x)| = C(n)\|P_n\|$. Hence the required result.

We see later that the nuclear type has not got Cauchy inequalities. We now give a number of examples on $\mathscr{H}_N(E)$ which are a consequence of this and all of which are impossible for the current type. Our basic technique is the following lemma. We again assume E to be a complex infinite dimensional Hilbert space.

LEMMA 13. *Let $(X_n)_{n=1}^{\infty}$ be an orthonormal basis for E, $(\lambda_n)_{n=1}^{\infty}$ a sequence of complex numbers and m a positive integer $\geqslant 2$. Then $P = \sum_{n=1}^{\infty} \lambda_n X_n^m \in \mathscr{P}_N({}^mE)$ if and only if $\sum_{n=1}^{\infty} |\lambda_n| < \infty$. In this case $\|P\|_N = \sum_{n=1}^{\infty} |\lambda_n|$.*

Proof. By the definition of $\mathscr{P}_N({}^mE)$ we see $P \in \mathscr{P}_N({}^mE)$ iff $P = \sum_{j=1}^{\infty} \omega_j \beta_j^m$, where $\Sigma_j |\omega_j| < \infty$, $\beta_j \in E'$ and $\|\beta_j\| = 1$. For each j since $E \cong E'$

$$\beta_j = \sum_{i=1}^{\infty} \lambda_{i,j} X_j, \quad \text{where } \Big(\sum_{i=1}^{\infty} |\lambda_{i,j}|^2\Big)^{1/2} \leqslant 1.$$

In particular, we have $|\lambda_{i,j}| \leqslant 1$ for all i and j. Now

$$P(X_i) = \sum_{j=1}^{\infty} \lambda_j X_j^m(X_i) = \lambda_i = \sum_{j=0}^{\infty} \omega_j \langle \beta_j, X_i\rangle^m = \sum_{j=0}^{\infty} \omega_j \lambda_{i,j}^m.$$

Therefore

$$|\lambda_i| \leqslant \sum_{j=1}^{\infty} |\omega_j| \cdot |\lambda_{i,j}|^m$$

and

$$(*) \qquad \sum_{i=1}^{\infty} |\lambda_i| \leqslant \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\lambda_{i,j}|^m \cdot |\omega_j| = \sum_{j=1}^{\infty} |\omega_j| \sum_{i=1}^{\infty} |\lambda_{i,j}|^m$$
$$\leqslant \sum_{j=1}^{\infty} |\omega_j| \sum_{i=1}^{\infty} |\lambda_{i,j}|^2 \leqslant \sum_{j=1}^{\infty} |\omega_j|$$

since $m \geqslant 2$ and $\sum_{i=1}^{\infty} |\lambda_{i,j}|^2 \leqslant 1$.

Hence $P \in \mathscr{P}_N({}^mE)$ if and only if $\sum_{i=1}^{\infty} |\lambda_i| < \infty$. If $P \in \mathscr{P}_N({}^mE)$, then $\|P\|_N \leqslant \sum_{i=1}^{\infty} |\lambda_i|$ by the definition of the nuclear norm. Also $\|P\|_N = \inf \sum_{i=1}^{\infty} |\omega_i|$, where the infimum is taken over all representations of P. But $(*)$ implies $\|P\|_N \geqslant \sum_{i=1}^{\infty} |\lambda_i|$, hence we get $\|P\|_N = \sum_{i=1}^{\infty} |\lambda_i|$.

By taking $P_n = \sum_{i=1}^{n} X_1^2$ we see $\|P_n\|_N = n$ and $\|P_n\| = 1$, hence the nuclear type does not have Cauchy inequalities.

EXAMPLE 9. We give an example of f on E such that

(1) $f \in \mathscr{H}(E)$.

(2) $\hat{d}^n f(x) \in \mathscr{P}_N({}^nE)$ for $n = 0, 1, \ldots$ and all $x \in E$.

(3) $\lim_{n\to\infty} \left(\left\| \frac{\hat{d}^n f(0)}{n!} \right\|_N \right)^{1/n} = 1$.

(4) $\limsup_{n\to\infty} \left(\sup_{\|x\|\leqslant\varrho} \left\| \frac{\hat{d}^n f(x)}{n!} \right\|_N \right)^{1/n} < \infty$ for any fixed $\varrho > 0$.

Note. If we replace the nuclear type by the current type (3) and (4) are incompatible. This example shows that there are two different and natural ways to define bounded nuclear entire functions on a Banach space (cf. [3]). One can also define natural topologies on these spaces and using the technique below it is possible to show these topologies do not coincide on the nuclear polynomials and that they do not have the same dual spaces.

CONSTRUCTION. *Let $(x_n)_{n=1}^{\infty}$ be an orthonormal basis for E. We partition this basis as follows:*

$$\underbrace{x_1}_{1!}\ \underbrace{x_2, x_3}_{2!}\ \underbrace{x_4, x_5, x_6, x_7, x_8, x_9}_{3!} \ldots \underbrace{\qquad}_{n!}$$

i. e., the n^{th}-partition class σ_n has $n!$ elements. Let $P_m = \frac{1}{m!} \sum_{i_m \in \sigma_m} x_{i_m}^m$ and $f = \sum_{m=1}^{\infty} P_m$. Then $P_m \in \mathscr{P}_N({}^mE)$ and by Lemma (13) $\|P_m\|_N = 1$.

Proof. We first show $f \in \mathscr{H}(E)$. It suffices to show $\lim_{n\to\infty} \|P_m\|^{1/m} = 0$. Suppose $x = \sum_{i=1}^{\infty} \lambda_i x_i \in E$. Then

$$P_m(x) = \frac{1}{m!} \sum_{i_m \in \sigma_m} \langle x_{i_m}, x\rangle^m = \frac{1}{m!} \sum_{i_m \in \sigma_m} \lambda_{i_m}^m.$$

For $m \geqslant 2$, $\|x\| \leqslant 1$ we have

$$|P_m(x)| \leqslant \frac{1}{m!} \sum_{i_m \in \sigma_m} |\lambda_{i_m}|^m \leqslant \frac{1}{m!} \sum_{i_m \in \sigma_m} |\lambda_{i_m}|^2 \leqslant \frac{1}{m!}.$$

Hence

$$\|P_m\| = \sup_{\|X\|\leqslant 1} |P_m(X)| \leqslant \frac{1}{m!}.$$

Since $\lim_{m\to\infty} \left(\frac{1}{m!}\right)^{1/m} = 0$, we get $f \in \mathscr{H}(E)$. Now $\frac{\hat{d}^n f(0)}{n!} = P_n$ and $\|P_n\|_N = 1$ for all n hence $\lim_{m\to\infty} \|P_n\|_N^{1/n} = 1$, i. e., (3) is true.

We prove (2) and (4) together. Since we can differentiate term by term we have for any $x \in E$

$$(*)\qquad \frac{\hat{d}^n f(x)}{n!} = \frac{1}{n!}\sum_{m=0}^{\infty} \hat{d}^n P_m(x)$$

$$= \frac{1}{n!}\sum_{m\geq n}\frac{1}{m!}\sum_{i_m \epsilon \sigma_m} x_{i_m}^n \langle x_{i_m}, x\rangle^{m-n}\; m\ldots(m-n+1).$$

Now each $\sum_{i_m \epsilon \sigma_m} x_{i_m}^n \langle x_{i_m}, x\rangle^{m-n} m\ldots(m-n+1) \epsilon \mathscr{P}_f({}^nE)$, so $\frac{\hat{d}^n f(x)}{n!} = \sum_{m=0}^{\infty} q_m(x)$, where $q_m(x) \epsilon \mathscr{P}_N({}^nE)$ for all m. Therefore to show $\frac{\hat{d}^n f(x)}{n!} \epsilon \mathscr{P}_N({}^nE)$ it suffices to show

$$\sum_{m=0}^{\infty} \|q_m(x)\|_N < \infty$$

and this we actually show in the proof of (4). Let $x = \varrho\sum_{i=1}^{\infty}\lambda_i x_i \epsilon E$, where $\sum_{i=1}^{\infty}|\lambda_i|^2 \leqslant 1$. By (*) we get

$$\frac{\hat{d}^n f(x)}{n!} = \frac{1}{n!}\overbrace{\sum_{i_n \epsilon \sigma_n} x_{i_n}^n}^{(a)} + \overbrace{\sum_{i_{n+1}\epsilon \sigma_{n+1}} \varrho\lambda_{i_{n+1}} x_{i_{n+1}}^n \frac{1}{n!}}^{(b)} + \overbrace{\sum_{m\geq n+2}\frac{1}{m!}\sum_{i_m \epsilon \sigma_m} C_n^m \lambda_{i_m}^{m-n}\varrho^{m-n} x_{i_m}^n}^{(c)}.$$

We consider (a), (b) and (c) separately:

$$(a)\qquad \frac{1}{n!}\sum_{i_n \epsilon \sigma_n} x_{i_n}^n = P_n,\quad \text{hence}\quad \Big\|\frac{1}{n!}\sum_{i_n \epsilon \sigma_n} x_{i_n}^n\Big\|_N = \|P_n\|_N = 1,$$

$$(b)\qquad \Big\|\frac{1}{n!}\sum_{i_{n+1}\epsilon \sigma_{n+1}} \varrho\lambda_{i_{n+1}} x_{i_{n+1}}\Big\|_N \leqslant \varrho\frac{1}{n!}\sum_{i_{n+1}\epsilon \sigma_{n+1}} |\lambda_{i_{n+1}}|$$

$$\leqslant \varrho\frac{1}{n!}\Big(\sum_{i_{n+1}\epsilon \sigma_{n+1}} |\lambda_{i_{n+1}}|^2\Big)^{1/2}\Big(\sum_{i_{n+1}\epsilon \sigma_{n+1}} 1\Big)^{1/2}.$$

By the Cauchy-Schwarz inequality and since σ_{n+1} has $(n+1)!$ terms we get

$$\|(b)\|_N \leqslant \varrho\frac{1}{n!}\big((n+1)!\big)^{1/2} = \varrho\frac{(n+1)^{1/2}}{(n!)^{1/2}},$$

$$(c)\qquad \Big\|\sum_{m\geq n+2}\frac{1}{m!}\sum_{i_m \epsilon \sigma_m} x_{i_m}^n \lambda_{i_m}^{m-n}\varrho^{m-n}\binom{m}{n}\Big\|$$

$$\leqslant \sum_{m\geq n+2}\frac{1}{m!}\sum_{i_m \epsilon \sigma_m} C_n^m \varrho^{m-n}|\lambda_{i_m}|^{m-n}\qquad \text{(by Lemma 12)}$$

$$\leqslant \sum_{m\geq n+2}\frac{1}{m!}\sum_{i_m \epsilon \sigma_m} C_n^m \varrho^{m-n}|\lambda_{i_m}|^2\qquad (\text{since } m \geqslant n+2)$$

$$\leqslant \sum_{m\geq n+2} C_n^m\varrho^{m-n}\frac{1}{m!} = \sum_{m\geq n+2}\frac{1}{n!}\frac{\varrho^{m-n}}{(m-n)!} \leqslant \frac{1}{n!}e^{\varrho}.$$

Hence

$$\frac{\hat{d}^n f(x)}{n!} \epsilon \mathscr{P}_N({}^nE)$$

and

$$\Big\|\frac{\hat{d}^n f(x)}{n!}\Big\|_N \leqslant 1+\varrho\frac{(n+1)^{1/2}}{(n!)^{1/2}}+\frac{e^{\varrho}}{n!}$$

for $n = 0, 1, \ldots$ and all x, $\|x\| \leqslant \varrho$. Therefore

$$\sup_{\|x\|\leqslant\varrho}\Big\|\frac{\hat{d}^n f(x)}{n!}\Big\|_N \leqslant 1+\varrho\frac{(n+1)^{1/2}}{(n!)^{1/2}}+\frac{e^{\varrho}}{n!}$$

and

$$\limsup_{n\to\infty}\Big(\sup_{\|x\|\leqslant\varrho}\Big\|\frac{\hat{d}^n f(x)}{n!}\Big\|_N^{1/n} = 1 < \infty,$$

i. e., (2) and (4) hold. This completes the proof.

Our next example shows that the conditions in the definition of $\mathscr{H}_N(E)$ are independent.

Example 10. We give an example of f on E such that

(1) $f \epsilon \mathscr{H}(E)$.

(2) $\hat{d}^n f(x) \epsilon \mathscr{P}_N({}^nE)$ for $n = 0, 1, 2 \ldots$ and $x \epsilon E$.

(3) $\lim_{n\to\infty}\Big\|\frac{\hat{d}^n f(0)}{n!}\Big\|_N^{1/n} = \infty.$

Construction. *Let* $(x_n)_{n=1}^{\infty}$, σ_m, P_m *be as in Example* 9 *and take*

$$f = \sum_{m=1}^{\infty} m^{m/2} P_m.$$

Then

$$(*)\qquad \lim_{m\to\infty}\|m^{m/2}P_m\|^{1/m} = \lim_{m\to\infty} m^{1/2}\Big(\frac{1}{m!}\Big)^{1/m} = 0.$$

Hence $f \in \mathscr{H}(E)$. Since $\|P_m\|_N = 1$, (3) holds. By (*) f has infinite radius of convergence hence we can differentiate term by term to get

$$\frac{\hat{d}^n f(x)}{n!} = \sum_{m \geqslant n} q_m,$$

where

$$q_m = \frac{m^{m/2}}{m!} \sum_{i_m \in \sigma_m} C_n^m x_{i_m}^n \langle x_{i_m}, x\rangle^{m-n}.$$

Since $q_m \in \mathscr{P}_f({}^nE)$ for all m to show $\frac{\hat{d}^n f(x)}{n!} \in \mathscr{P}_N({}^nE)$ it suffices to prove

$$\sum_{m \geqslant n+2} \|q_m\|_N < \infty.$$

By Lemma 13 for $m \geqslant n+2$

$$\|q_m\|_N = \frac{m^{m/2}}{m!} \sum_{i_m \in \sigma_m} |\langle x_{i_m}, x\rangle|^{m-n} C_n^m.$$

Let

$$x = \varrho \sum_{i=1}^{\infty} \lambda_i x_i, \quad \text{where } \sum_{i=1}^{\infty} |\lambda_i|^2 \leqslant 1.$$

Hence

$$\|q_m\|_N = \frac{m^{m/2}}{m!} \sum_{i_m \in \sigma_m} |\lambda_{i_m}|^{m-n} \varrho^{m-n} C_n^m \leqslant \frac{m^{m/2}}{m!} C_n^m \sum_{i_m \in \sigma_m} \varrho^{m-n} |\lambda_{i_m}|^2$$
$$\leqslant \frac{m^{m/2} \varrho^{m-n}}{n!(m-n)!}.$$

Since

$$\lim_{m\to\infty} \left(\frac{m^{m/2}\varrho^{m-n}}{n!(m-n)!}\right)^{1/m} = 0$$

we have

$$\sum_{m \geqslant n+2} \|q_m\|_N < \infty.$$

Finally we give example to show that $f \in \mathscr{H}(E)$ can be of nuclear type at one point of E but not at another point of E. We first need the following lemmas:

LEMMA 14. *Let $E = C^2$; then $\mathscr{P}_N({}^nE) = \mathscr{P}({}^nE)$ for $n = 0, 1, \ldots$ and $\exists\, C_1 > 0$ such that for each n we have $\|P_n\|_N \leqslant C_1^n \|P_n\|$ all $P_n \in \mathscr{P}({}^nE)$.*

LEMMA 15. *The nuclear polynomials $\mathscr{P}_N(E)$ on a separable Hilbert space form an algebra and $\exists\, C > 0$ such that for each m and n positive integers or zero*

$$\|P_n P_m\|_N \leqslant C^{m+n} \|P_n\|_N \|P_m\|_N, \quad \text{where } P_n \in \mathscr{P}_N({}^nE),\ P_m \in \mathscr{P}_N({}^mE).$$

Proof. To show $\mathscr{P}_N(E)$ is an algebra it suffices to show $P_n \cdot P_m \in \mathscr{P}_N({}^{n+m}E)$ if $P_n \in \mathscr{P}_N({}^nE)$ and $P_m \in \mathscr{P}_N({}^mE)$. Let x and $y \in E = E'$. Then $x^n \in \mathscr{P}_N({}^nE)$, $y^m \in \mathscr{P}_N({}^mE)$ and $x^n y^m \in \mathscr{P}({}^{n+n}E)$. By restriction $x^n \cdot y^m \in \mathscr{P}({}^{n+m}\{x, y\}) = \mathscr{P}_N({}^{n+m}\{x, y\})$ and so there exist $(\varphi_i)_{i=0}^{\infty} \in \{x, y\}'$ such that

$$x^n \cdot y^m = \sum_{i=0}^{\infty} \varphi_i^{n+m}, \quad \text{where } \sum_{i=0}^{\infty} \|\varphi_i\|^{n+m} \leqslant \|x^n \cdot y^m\|_N + \varepsilon.$$

By Lemma 14 $\|x^n \cdot y^m\|_N \leqslant C^{n+m} \|x\|^n \cdot \|y\|^m$. We can extend by the inner product formula $\varphi_i \in \{x \cdot y\}'$ to E'. Hence $x^n \cdot y^m = \sum_i \varphi_i^{n+m}$ as elements of $\mathscr{P}({}^{n+m}E)$, $x^n \cdot y^m \in \mathscr{P}_N({}^{n+m}E)$ and

$$\|x^n \cdot y^m\|_N \leqslant \sum_i \|\varphi_i\|^{n+m} \leqslant C^{n+m} \|x\|^n \|y\|^m + \varepsilon.$$

Thus

$$(*) \qquad \|x^n \cdot y^m\|_N \leqslant C^{n+m} \|x\|^n \cdot \|y\|^m.$$

Now let $P_n \in \mathscr{P}_N({}^nE)$, $P_m \in \mathscr{P}_N({}^mE)$. Given $\varepsilon > 0$ choose a representation of P_n, $\sum_i \varphi_i^n$, and a representation of P_m, $\sum_j \psi_j^m$, such that

$$(**) \qquad \begin{aligned} \|P_n\|_N &\leqslant \sum_i \|\varphi_i\|^n \leqslant \|P_n\|_N + \varepsilon, \\ \|P_m\|_N &\leqslant \sum_j \|\psi_j\|^m \leqslant \|P_m\|_N + \varepsilon. \end{aligned}$$

where φ_i, $\Phi_j \in E'$ for all i, j. Since the series in (**) are absolutely convergent we get $P_n P_m \in \mathscr{P}({}^{n+m}E)$ and $P_n P_m = \sum_{i,j} \varphi_i^n \psi_j^m$. (*) implies $\varphi_i^n \psi_j^m \in \mathscr{P}_N({}^{n+m}E)$ for all i, j and

$$\|\varphi_i^n \psi_j^m\|_N \leqslant C^{n+m} \|\varphi_i\|^n \cdot \|\psi_j\|^m.$$

We thus have

$$\sum_{i,j} \|\varphi_i^n \psi_j^m\|_N \leqslant C^{n+m} \sum_{i,j} \|\varphi_i\|^n \cdot \|\psi_j\|^m$$
$$= C^{n+m} \sum_i \|\varphi_i\|^n \cdot \sum_j \|\psi_j\|^m \leqslant C^{n+m} (\|P_n\|_N + \varepsilon)(\|P_m\|_N + \varepsilon).$$

Since $\mathscr{P}_N({}^{n+m}E)$ is a Banach space and $\varepsilon > 0$ was arbitrary we get $P_n \cdot P_m \in \mathscr{P}_N({}^{n+m}E)$ and $\|P_n P_m\|_N \leqslant C^{n+m} \|P_n\|_N \cdot \|P_m\|_N$.

EXAMPLE 11. We give an example of f on E such that

(1) $f \in \mathscr{H}(E)$,

(2) $\hat{d}^n f(0) \in \mathscr{P}_N({}^nE)$ for $n = 0, 1, \ldots,$

(3) $\left\| \frac{\hat{d}^n f(0)}{n!} \right\|_N \leqslant C^n$ for all n and for some $C > 0$,

(4) $f \notin \mathscr{H}_N(E)$.

CONSTRUCTION. *Let $x_1, y_1, y_2, \ldots$ be an orthonormal basis for E. We partition $(y_n)_{n=1}^{\infty}$ such that the n^{th}-partition class σ_n has $n!$ elements* (as in Examples 9 and 10). *Let*

$$P_n = \frac{1}{n!} x_1^{n-2} \sum_{i_n \in \sigma_n} y_i^2 \quad \textit{for } n \geqslant 2 \textit{ and } f = \sum_{n=2}^{\infty} P_n.$$

Proof.

$$\|P_n\| = \sup_{\|x\|<1} |P_n(x)| \leqslant \sup_{\|x\|\leqslant 1} \left| \frac{1}{n!} \sum_{i \in \sigma_n} y_i^2(x) \right|.$$

Suppose $x = \sum_{i=1}^{\infty} \lambda_i y_i + \lambda_0 x_1$, where $\left(\sum_{i=0}^{\infty} |\lambda_i|^2\right)^{1/2} \leqslant 1$. Then

$$\sup_{\|x\|<1} |P_n(x)| \leqslant \sup_{\sum_0^\infty |\lambda_i|^2 \leqslant 1} \left| \frac{1}{n!} \sum_{i=1}^{\infty} |\lambda_i|^2 \right| \leqslant \frac{1}{n!}$$

which implies

$$\lim_{n\to\infty} \|P_n\|^{1/n} = 0.$$

Hence

$$f \in \mathscr{H}(E).$$

By Lemma 15 $P_n \in \mathscr{P}_N({}^nE)$ and

$$\|P_n\|_N \leqslant C^n \left\| \frac{1}{n!} \sum_{i_n \in \sigma_n} y_i^2 \right\|_N \|x_1\|_N^{n-2}.$$

Hence (2) and (3) are true. Since f has infinite radius of convergence

$$\hat{d}^n f(x) = \sum_{m \geqslant n} \hat{d}^n P_m(x).$$

To show (4) is true we show $\hat{d}^2 f(x_1) \notin \mathscr{P}_N({}^2E)$. By the usual procedure one gets

$$\hat{d}^2 f(x_1) = \sum_{n \geqslant 2} \frac{2}{m!} \sum_{i \in \sigma_n} y_i^2.$$

Since the y_i's are all orthogonal Lemma 13 implies $\hat{d}^2 f(x_1) \notin \mathscr{P}_N({}^2E)$. Hence $f \notin \mathscr{H}_N(E)$. This completes the proof.

In this section we only considered functions defined on a separable Hilbert space. This makes the geometric reasoning behind the proofs very apparent and should help towards giving one an intuitive feeling for holomorphic functions on infinite dimensional space.

Since writing this paper we have been able to extend Proposition 21 to a large class of Banach spaces which includes all separable and all reflexive Banach spaces. Also we have been able to show that $(H(l_\infty), T_\omega)$ is not bornological.

The results in this paper are taken from my doctoral thesis at the University of Maryland, 1969. I wish to thank my thesis advisor, Professor Leopoldo Nachbin, for introducing me to this subject and for his constant encouragement and guidance at all times. I also wish to thank Professor John Horvath for his advice and encouragement.

I wrote my thesis while on leave of absence from the University of Maryland at Instituto de Matemática Pura e Aplicada, Rio de Janeiro. During that time I was supported by the Conselho Nacional de Pesquisas of Brazil and the National University of Ireland.

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Reçu par la Rédaction le 1. 7. 1970

STUDIA MATHEMATICA, T. XXXIX. (1971)

Differentiation in locally convex spaces

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In this paper we shall present an approach to a theory of differentiation in a few important classes of locally convex spaces. Differentiation in Banach spaces is one of the most useful tools of non-linear analysis (cf. [7], [8] and [10]). Our main goal is a natural generalization of this theory to a greater class of locally convex spaces. We give one of the possible realizations of this program. The other approaches can be found in reviews [1], [2], [15] and papers [3], [4], [9], [11], [14], [18] and [20]. The main idea of the differentiation is an approximation of a given map by a linear map. This approximation can be defined in many ways. The definition which is used in this paper can be found also in [14] and [18]. It is valid for an arbitrary locally convex space. However, it is not possible in the general case to obtain the mean value theorem and some results connected with it. Therefore, we consider only two classes of locally convex spaces: metrizable, quasi-normable spaces and *DF-S* spaces (*DF* spaces which are also Schwartz spaces).

We obtain the mean value theorem, the Taylor formula with the estimation of the remainder, and theorems on partial differentiability. The class of spaces which we consider in this paper includes normable spaces, Fréchet–Schwartz spaces and their duals (e. g. $\mathscr{S}$, $\mathscr{S}'$, $\mathscr{E}$, $\mathscr{E}'$).

The results of this paper are very close to those of paper [16], which dealt with a different model of the theory for *F-S* spaces.

1. Notation. We shall consider topological locally convex spaces over the field of real or complex numbers, called in this paper *locally convex spaces*. We shall assume the Hausdorf axiom. Let E be a locally convex space; $\mathscr{N}(E)$ denotes the set of all closed, absolutely convex neighbourhoods of zero in E. If $U \in \mathscr{N}(E)$, then $\|\cdot\|_U$ is the seminorm generated by U and $E_U := E/N(U)$, where $N(U) = \{e \in E;\ \|e\|_U = 0\}$. The symbol $\hat{E}$ will stand for the completion of E. $\mathscr{B}(E)$ will denote the set of all closed, absolutely convex bounded sets of E.

If E is a locally convex space, then E'_s (E'_b) is the dual space to E endowed with the weak (strong) topology. Let F be a locally convex