$\limsup_n n^{-1}\log|x_n|_\infty \leqslant 0$. We shall show that this last inequality is equivalent to the one in the statement of the lemma. Now

$$\limsup_n n^{-1}\log[\max(|\log|a_n||, \mathrm{PA}(a_n))] = \limsup_n n^{-1}\max[\log(|\log|a_n||), \log \mathrm{PA}(a_n)] = \max[\limsup_n n^{-1}\log(|\log|a_n||), \limsup_n n^{-1}\log \mathrm{PA}(a_n)].$$

Since $\limsup_n n^{-1}\log \mathrm{PA}(a_n) \leqslant 0$ we have

$$\limsup_n n^{-1}\log(x_n)_\infty \leqslant 0$$

if and only if

$$\limsup_n n^{-1}\log(|\log|a_n||) \leqslant 0.$$

Thus, if $a \in A$, then the easiest way of defining a logarithm for a yields an element x of A if and only if a satisfies the stated condition. It is clear that the constructed element x belongs to D.

THEOREM 5.8. $A^{-1} = \exp^*[D \cap (-D)]$.

Proof. Lemma 5.5 implies that the right side is contained in the left. Let $a \in A^{-1}$. Then $\lim_n n^{-1}\log|a_n| = 0$. For $\varepsilon > 0$, there exists $n_0 \in N$ such that if $n > n_0$, then $|\log|a_n|| < n\varepsilon$. But then

$$\log|\log|a_n|| < \log n\varepsilon \quad \text{and} \quad n^{-1}\log(|\log|a_n||) < n^{-1}\log n + n^{-1}\log\varepsilon.$$

The larger sequence converges to 0 as $n \to \infty$. Hence, a has a logarithm in A: $a = \exp^*(x)$. Since $\exp^*(x) \in A^{-1}$, Lemma 5.5 implies that $x \in D \cap (-D)$.

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Reçu par la Rédaction le 9. 5. 1970

STUDIA MATHEMATICA, T. XXXIX. (1971)

Factorization in Fréchet spaces*

by

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1. Introduction. Let A be a Banach algebra with a bounded left approximate identity. In 1959, P. J. Cohen [2] established that each element in A could be factored, and the utility of factorization as a tool in the study of Banach algebras and Banach A-modules has become apparent in the intervening decade. One fruitful approach (e. g., see Rieffel [11], Sentilles and Taylor [13], and Collins and Summers [4] has been to use the factorization theorem due to Hewitt [7] in conjunction with the device of introducing auxiliary essential left Banach A-modules in order to factor certain "large" subsets of a given essential left Banach A-module (for a definition, see [10]), and it is this procedure which will concern us in the sequel.

There are at least three natural auxiliary left A-modules that one can consider for a fixed left Banach A-module E; namely, the space $C(X; E)$ of all continuous functions from a topological space X into E, the space $C_b(X; E)$ of bounded functions in $C(X; E)$, and the space $C_0(X; E)$ of functions in $C(X; E)$ which vanish at infinity, each with respect to the obvious pointwise action. If X is a locally compact Hausdorff space, then $C_0(X; E)$ with the uniform topology [15] becomes an essential left Banach A-module and the above approach is valid (see Section 3). However, the situation is less obvious in the other two cases, and there seems little likelihood of realizing either $C(X; E)$ as a left *Banach* A-module or $C_b(X; E)$ as an *essential* left Banach A-module (however, see Section 4) unless X is compact. Consequently, the following question arises (see [4]): *can Hewitt's factorization theorem be extended to include the action of A on a class of locally convex spaces more general than Banach spaces?*

An affirmative answer has been announced by Ovaert [9], but it appears that a proof of Ovaert's result will entail a recasting of the technique used by Craw [5] in extending the work of Cohen to Fréchet algebras. In our context, however, it is possible to give a relatively simple

* This research was supported in part by National Science Foundation grant GP–11762.

proof based on Hewitt's theorem, and this is done in Section 2. An example is also given which shows that our result is essentially the best possible in this direction. In Section 3, we consider the auxiliary modules $C_0(X; E)$ and $C(X; E)$ in light of this factorization theorem (Theorem 2.1), and our work in this setting contains not only the known results but a somewhat surprising new one (Corollary 3.5). Finally, in Section 4, we consider the problem of factorization in the auxiliary module $C_b(X; E)$, and we show that there is an intimate connection between this and the property (WDF) introduced by Collins and Dorroh [3].

2. The factorization theorem. Let E be a Fréchet space which is a left A-module in the algebraic sense, and let $\{p_n\}_{n=1}^{\infty}$ be an increasing sequence of semi-norms on E such that if $U_n = \{x \in E: p_n(x) \leqslant 1\}$, then $\{U_n\}_{n=1}^{\infty}$ is a neighborhood base at zero in E. We will call E a *left Fréchet A-module* if there exists a real number $c \geqslant 1$ so that for each n

$$p_n(a\cdot x) \leqslant c\|a\| p_n(x)$$

for all $a \in A$ and $x \in E$. It is readily seen that we may assume $c = 1$ by considering

$$p_n'(x) = \max[p_n(x), \sup\{p_n(a\cdot x): \|a\| \leqslant 1\}]$$

for each natural number n. Further, E will be called *essential* if for each $x \in E$ we have that $e_a\cdot x \to x$ where $\{e_a\}$ is the left approximate identity for A. Assuming that $d = \sup\{\|e_a\|\}$, we have the following extension of the Hewitt factorization theorem ([7], p. 151).

2.1. THEOREM. *If E is an essential left Fréchet A-module, if $x \in E$, and if U is any neighborhood of x, then there exist $a \in A$, $y \in E$ with the following properties:*

(1) $x = a\cdot y$;

(2) $y \in \mathrm{cl}(A\cdot x)$;

(3) $y \in U$;

(4) $\|a\| \leqslant d$.

The proof depends on the following lemma discovered independently by Varopoulos [14] and Johnson [8] which, as is pointed out by Rieffel [11], is an immediate consequence of the Hewitt factorization theorem.

2.2. LEMMA (Varopoulos–Johnson). *If F is an essential left Banach A-module, if $f \in c_0(F)$, and if $\varepsilon > 0$, then there exist $a \in A$ and $g \in c_0(F)$ with the following properties:*

(i) $f = a\cdot g$ (*the action being defined pointwise*);

(ii) $g \in \mathrm{cl}(A\cdot f)$ (*in $c_0(F)$*);

(iii) $\|f-g\|_\infty < \varepsilon$;

(iv) $\|a\| \leqslant d$.

Proof of Theorem 2.1. Let Π_n: $E \to E/p_n^{-1}(0)$ for each natural number n, be the natural projection; define Π_{kn}' for $1 \leqslant k \leqslant n$ by $\Pi_{kn}'(\Pi_n(z)) = \Pi_k(z)$, and let E_n denote the completion of $(E/p_n^{-1}(0), \|\cdot\|_n)$ where $\|\Pi_n(z)\|_n = p_n(z)$, while Π_{kn}: $E_n \to E_k$ for $1 \leqslant k \leqslant n$ will denote the extension of Π_{kn}'. In this case, E is the projective limit $\mathrm{PL}\,\Pi_{kn}E_n$ of the sequence $\{E_n\}_{n=1}^{\infty}$ of Banach spaces with respect to the mappings $\{\Pi_{kn}: 1 \leqslant k \leqslant n\}_{n=1}^{\infty}$ ([12], p. 53). Now each E_n can be considered as a left A-module by first defining $a\cdot\Pi_n(z) = \Pi_n(a\cdot z)$ and then extending. This action is well-defined, and since E is an essential left Fréchet A-module, it is clear that each E_n is an essential left Banach A-module. So we may apply Hewitt's factorization theorem to each E_n to obtain $a_n' \in A$ and $z_n' \in E_n$ satisfying $\Pi_n(x) = a_n'\cdot z_n'$ and $\|a_n'\| \leqslant d$. Taking $a_n = n^{-1}a_n'$ and $z_n = nz_n'$ for each n, we have that $\{a_n\}_{n=1}^{\infty}$ is in $c_0(A)$; Lemma 2.2 now permits us to choose a sequence $\{b_n\}_{n=1}^{\infty}$ in $c_0(A)$ and an $a \in A$ such that $\|a\| \leqslant d$, $a_n = a\cdot b_n$ for each n, $\{b_n\}_{n=1}^{\infty} \in \mathrm{cl}(A\cdot\{a_n\}_{n=1}^{\infty})$, and $\|a_n - b_n\| < \varepsilon$ for each n where ε is a positive number (depending on U). Now define $y_n = b_n\cdot z_n$ for each n, let $\eta > 0$, fix the natural numbers k_i, $i = 1, \ldots, m$, and choose $b \in A$ so that

$$\sup\{\|b_k - b\cdot a_k\|: k = 1, 2, \ldots\} < \eta[1+\sup\{\|z_{k_i}\|_{k_i}: i = 1, \ldots, m\}]^{-1}.$$

In this case, we have for $i = 1, \ldots, m$ that

$$\|y_{k_i} - b\cdot\Pi_{k_i}(x)\|_{k_i} \leqslant \|b_{k_i} - b\cdot a_{k_i}\|\cdot\|z_{k_i}\|_{k_i} < \eta,$$

and so $\{y_n\}_{n=1}^{\infty}$ is in the closure of $A\cdot\{\Pi_n(x)\}_{n=1}^{\infty}$ in $\prod_{k=1}^{\infty} E_k$. But

$$A\cdot\{\Pi_n(x)\}_{n=1}^{\infty} \subseteq \mathrm{PL}\,\Pi_{kn}E_n,$$

and from this it follows that there is a $y \in E$ such that $\Pi_n(y) = y_n$ for every n. To conclude the proof it suffices to show $y \in U$, and this easily follows from the fact that

$$p_n(x-y) = \|\Pi_n(x) - y_n\|_n \leqslant \|a_n - b_n\|\cdot\|z_n\|_n.$$

As in the case for left Banach A-modules ([10], p. 453), we have the following equivalent formulations for when a left Fréchet A-module is essential.

2.3. THEOREM. *If E is a left Fréchet A-module, then the following are equivalent:*

(a) *E is essential;*

(b) *the linear span of $A\cdot E$ is dense in E;*

(c) $A\cdot E = E$.

Proof. It is evident that (c) implies (b), while the fact that (a) implies (c) is immediate from (1) of Theorem 2.1. It remains to show (b) implies (a), and this follows from a straightforward modification of the argument in the case of Banach modules.

It would be natural to ask if a version of Theorem 2.1 would hold, for example, when E is only assumed to be a complete locally convex space. As we show below, this is not the case.

Example. Let X be a locally compact Hausdorff space which is not compact, let $C_0(X)$ denote the Banach algebra of all complex valued continuous functions on X which vanish at infinity with the uniform topology, and let $C_b(X)$ be the space of all complex valued bounded continuous functions on X endowed with the strict topology β (see Buck [1]). $C_b(X)$ is a (left) $C_0(X)$-module with respect to pointwise multiplication, and it is known that $C_0(X)$ has a bounded (left) approximate identity $\{e_a\}$ with the property that $e_a f \to f(\beta)$ for every $f \in C_b(X)$ ([3], p. 159); i. e., $C_b(X)$ is essential. Moreover, $C_b(X)$ is a non-metrizable complete locally convex space ([1], p. 98), and since the semi-norms on $C_b(X)$ are given by $f \to \|\varphi f\|_\infty = \|f\|_\varphi$ for $\varphi \in C_0^+(X)$, we have that

$$\|\psi f\|_\varphi \leqslant \|\psi\|_\infty \|f\|_\varphi$$

for $\psi \in C_0(X)$ and $f \in C_b(X)$. It is clear, however, that the conclusion (1) of Theorem 2.1 fails to hold in this case.

3. Auxiliary left Fréchet A-modules. Assume E is an essential left Fréchet A-module with the sequence $\{p_n\}_{n=1}^\infty$ of semi-norms discussed in Section 2. In this section we will consider two examples of auxiliary left Fréchet A-modules and apply them to factor certain "large" sets in E. The first example will yield the analogue of the Varopoulos–Johnson result (Lemma 2.2) for Fréchet modules, while both examples will enable us to factor totally bounded subsets of E (it was the ability to factor totally bounded sets in an essential left Banach A-module which provided the impetus for the results in [4]). Finally, the second example will allow us to show that any hemi-compact subset of E can be factored through A; a new result even in the Banach module case.

For the first example, let X be a locally compact Hausdorff space and let $C_0(X; E)$ denote the space of all continuous functions from X into E which vanish at infinity endowed with the topology generated by the semi-norms

$$f \to \|f\|_n = \sup\{p_n(f(x)) \colon x \in X\}.$$

It is known, of course, that $C_0(X; E)$ is a Fréchet space, while $(a\cdot f)(x) = a\cdot f(x)$ for $x \in X$ defines an action of A on $C_0(X; E)$ with respect to which $C_0(X; E)$ is easily seen to be a left Fréchet A-module. To see that $C_0(X; E)$ is essential, fix $f \in C_0(X; E)$, a natural number k, and $\varepsilon > 0$; choose a compact set $K \subseteq X$ so that $x \in X \setminus K$ implies $p_k(f(x)) < \varepsilon/2d$.

Since $f(K)$ is compact, there exist $z_i \in f(K)$, $i = 1, \ldots, n$ such that if $y \in f(K)$, then $p_k(y - z_i) < \varepsilon/3d$ for some $i \in \{1, \ldots, n\}$. Choose a_0 so that $a \geqslant a_0$ implies $p_k(e_a\cdot z_i - z_i) < \varepsilon/3$ for $i = 1, \ldots, n$, and note that for $x \in K$ and $a \geqslant a_0$ we have

$$p_k\big(e_a\cdot f(x) - f(x)\big) \leqslant p_k\big(e_a\cdot f(x) - e_a\cdot z_i\big) + p_k(e_a\cdot z_i - z_i) + p_k\big(z_i - f(x)\big) < \varepsilon$$

while $x \in X \setminus K$ implies

$$p_k\big(e_a\cdot f(x) - f(x)\big) \leqslant p_k\big(e_a\cdot f(x)\big) + p_k\big(f(x)\big) < \varepsilon.$$

Thus $\|e_a\cdot f - f\|_k \leqslant \varepsilon$ whenever $a \geqslant a_0$, and we have proved the following result.

3.1. Theorem. *If E is an essential left Fréchet A-module, then $C_0(X; E)$ is an essential left Fréchet A-module.*

The preceding theorem together with Theorem 2.1 enables us to obtain the following extension of the Varopoulos–Johnson result for Banach modules.

3.2. Corollary. *Let E be an essential left Fréchet A-module and let N denote the natural numbers with the discrete topology. If $f \in C_0(N; E) = c_0(E)$ and if U is any neighborhood of f, then there exist $a \in A$ and $g \in c_0(E)$ satisfying properties* (1) *through* (4) *of Theorem* 2.1.

3.3. Corollary. *Let E be an essential left Fréchet A-module and let X be a compact subset of E. If $f \in C_0(X; E)$ and if U is any neighborhood of f, then there exist $a \in A$ and $g \in C_0(X; E)$ satisfying properties* (1) *through* (4) *of Theorem* 2.1. *In particular, if Z is a totally bounded subset of E, then there exists a totally bounded set $Y \subseteq E$ and an $a \in A$ so that $Z = a\cdot Y$.*

Proof. It suffices to verify the last statement, and this is done by choosing $X = \mathrm{cl}(Z)$ and observing that the injection mapping $i\colon X \to E$ belongs to $C_0(X; E)$. Consequently, there is a $g \in C_0(X; E)$ and an $a \in A$ such that $i = a\cdot g$, and from this it follows that $Y = g(Z)$ is the desired totally bounded set.

For the second example of an auxiliary Fréchet module, let X be a completely regular T_1-space which is also a k-space and let $\{K_n\}_{n=1}^\infty$ be an increasing sequence of compact subsets of X. Assume $\{K_n\}_{n=1}^\infty$ forms a base for compacta in X (X is hemi-compact) and let $C(X; E)$ denote the space of all continuous functions from X into E endowed with the topology generated by the semi-norms

$$f \to \|f\|_n = \sup\{p_n(f(x)) \colon x \in K_n\}.$$

$C(X; E)$ is known to be a Fréchet space, and is clearly a left Fréchet A-module with respect to the obvious pointwise action. We wish to check that $C(X; E)$ is essential, and to this end fix $f \in C(X; E)$, a natural number k, and $\varepsilon > 0$. Since $f(K_k)$ is compact, we have as in the proof of Theorem

3.1 that there exists an α_0 so that $\alpha \geqslant \alpha_0$ implies $p_k(e_\alpha \cdot f(x) - f(x)) < \varepsilon$ for every $x \in K_k$; i. e., $\|e_\alpha \cdot f - f\|_k \leqslant \varepsilon$ for $\alpha \geqslant \alpha_0$ and we have the following result.

3.4. THEOREM. *If E is an essential left Fréchet A-module, then $C(X; E)$ is an essential left Fréchet A-module.*

It is clear, of course, that Corollary 3.3 would also follow from Theorem 3.4. However, it is the following consequence of Theorem 3.4 that is intriguing, and an application of this result is considered in the next section.

3.5. COROLLARY. *Let E be an essential left Fréchet A-module and let X be a hemi-compact subset of E. If $f \in C(X; E)$ and if U is any neighborhood of f, then there exist $a \in A$ and $g \in C(X; E)$ satisfying properties* (1) *through* (4) *of Theorem* 2.1. *In particular, there exists a σ-compact set $Y \subseteq E$ and an $a \in A$ so that $X = a \cdot Y$.*

Proof. X is a completely regular T_1-space which is also a k-space since E is metrizable, and so the first statement is immediate from Theorem 3.4 and Theorem 2.1. To conclude the proof, we note that the injection map i: $X \to E$ is in $C(X; E)$, and hence there is a $g \in C(X; E)$ and an $a \in A$ such that $i = a \cdot g$; $Y = g(X)$ is clearly the desired σ-compact set.

4. Bounded factorization; the property (WDF).

As we mentioned earlier, one would not in general expect the auxiliary module $C_b(X; E)$ over an essential left Banach A-module E to be an essential left Banach A-module unless X is compact. However, by restricting our attention to the case $C_b(N; A)$ where N denotes the space of natural numbers with the discrete topology and $A = C_0(X)$ (here, and throughout the remainder of this section, X will denote a locally compact Hausdorff space), it is possible to sharpen this result, and this permits us to show that the property (WDF) for $(C_b(X), \beta)$ introduced by H. S. Collins and J. R. Dorroh ([3], p. 163) is equivalent to $(C_b(X), \beta)$ being a (DF) space.

4.1. THEOREM. *The following are equivalent*:

(i) $(C_b(X), \beta)$ *is a (DF) space*;

(ii) $(C_b(X), \beta)$ *has property (WDF); i. e., each uniformly bounded real sequence in $C_0(X)$ has an upper bound in $C_0(X)$*;

(iii) $C_b(N; C_0(X))$ *with the uniform topology* ([15], p. 119) *is an essential left Banach $C_0(X)$-module.*

Proof. It is known that (i) implies (ii) ([3], p. 163). Moreover, $C_b(N; C_0 X))$ with the uniform topology is known to be a Banach space, and it is easy to see that $C_b(N; C_0(X))$ is a left Banach $C_0(X)$-module with respect to the action of $C_0(X)$ on $C_b(N; C_0(X))$ defined by $(a \cdot f)(n) = a \cdot f(n)$ for $n \in N$. Now assume (ii) holds, choose $f \in C_b(N; C_0(X))$, and take $a \in C_0(X)$ so that $|f(n)| \leqslant a$ for each $n \in N$. Since $\{a\} \cup \{f(n): n \in N\}$ is a hemi-compact subset of $C_0(X)$, it follows from Corollary 3.5 that there is a $b \in C_0(X)$ and a sequence $\{b_n\}_{n=0}^\infty \subseteq C_0(X)$ for which $a = b \cdot b_0$ and $f(n) = b \cdot b_n$, $n \in N$. We can apply Corollary 3.5 again to obtain $c \in C_0(X)$ and $\{c_n\}_{n=0}^\infty \subseteq C_0(X)$ such that $b = c \cdot c_0$ and $b_n = c \cdot c_n$ for $n \in N$. In view of Theorem 2.3, if we can show the function g: $N \to C_0(X)$ defined by $g(n) = b \cdot c_n$ for $n \in N$ is in $C_b(N; C_0(X))$, then we will have that (iii) holds since $c \cdot g(n) = f(n)$ for each $n \in N$. To see this, we first observe that

$$|c \cdot g(n)| = |f(n)| \leqslant |a| = |c \cdot b_0 \cdot c_0|$$

for any $n \in N$, and since $c(x) = 0$ implies $g(n)(x) = 0$ for $x \in X$, it follows readily that

$$|g(n)(x)| \leqslant |b_0 \cdot c_0(x)|$$

for each $n \in N$ and any $x \in X$. Consequently, $\|g(n)\| \leqslant \|b_0 \cdot c_0\|$ for each $n \in N$; i. e., $g \in C_b(N; C_0(X))$.

$(C_b(X), \beta)$ always has a countable base for bounded sets ([1], p. 98), and it therefore suffices to verify the following condition in order to show $(C_b(X), \beta)$ is a (DF) space: namely if H_n is a β-equicontinuous subset of the topological dual $(C_b(X), \beta)^*$ for each $n \in N$ and if $H = \bigcup_{n=1}^\infty H_n$ is strongly bounded, then H is β-equicontinuous. Now $(C_b(X), \beta)^*$ is the space $M_b(X)$ of bounded Radon measures on X ([1], p. 99), $M_b(X)$ with the variation norm is an essential left Banach $C_0(X)$-module with respect to the obvious action ([4], p. 730), and from the Riesz–Markov theorem together with the fact that $C_0(X)$ is barrelled we have that the $\beta(M_b(X), C_b(X))$ bounded set H is variation norm bounded. Moreover, since each H_n is β-equicontinuous, it follows from results in [4] and Theorem 2.1 that for each $n \in N$ there exist $a_n \in C_0(X)$ and $B_n \subseteq M_b(X)$ for which $\|a_n\| \leqslant 1$, $\|a_n \cdot \mu - \mu\| \leqslant 1$ for $\mu \in B_n$, and $H_n = a_n \cdot B_n$, and thus the function f: $N \to C_0(X)$ defined by $f(n) = a_n$ is in $C_b(N; C_0(X))$. Assuming (iii) is valid, choose $a \in C_0(X)$ and $g \in C_b(N; C_0(X))$ such that $f = a \cdot g$, and then let $B = \bigcup_{n=1}^\infty g(n) \cdot B_n$. Now B is variation norm bounded since $a_n \cdot \nu \in H$ and

$$\|g(n) \cdot \nu\| \leqslant \|g(n)\| \cdot \|\nu\| \leqslant \|g\| (1 + \|a_n \cdot \nu\|)$$

for $n \in N$ and $\nu \in B_n$, and hence $a \cdot B$ is β-equicontinuous ([6], p. 330). Consequently, the fact that $H \subseteq a \cdot B$ completes the proof.

It is known, of course, that $(C_b(X), \beta)$ can be a (DF) space without X being compact ([3], p. 163), and hence the above theorem tells us that $C_b(N; C_0(X))$ with the uniform topology can be an essential left Banach $C_0(X)$-module when X is not compact. Moreover, Theorem 4.1 completely answers the question posed by Collins and Dorroh [3] concerning the relation between (DF) and (WDF) for $(C_b(X), \beta)$.

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Reçu par la Rédaction le 21. 5. 1970

STUDIA MATHEMATICA, T. XXXIX. (1971)

A theorem on kernel in the theory of operator-valued distributions

by

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1. Introduction. Let $S(\boldsymbol{R}^n)$ denote the topological vector space of test functions for tempered distributions introduced by L. Schwartz [3]. For any two functions $\varphi \in S(\boldsymbol{R}^n)$, $\psi \in S(\boldsymbol{R}^m)$ we put

$$(\varphi\otimes\psi)(x, y) \overset{\text{df}}{=} \varphi(x)\psi(y),$$

where $x\in\boldsymbol{R}^n$, $y\in\boldsymbol{R}^m$ and $(x, y)\in\boldsymbol{R}^{n+m}$. It is known that this formula defines a continuous bilinear mapping

$$\otimes\colon S(\boldsymbol{R}^n)\times S(\boldsymbol{R}^m)\to S(\boldsymbol{R}^{n+m}).$$

Let L be a topological vector space. Any continuous linear mapping

$$\boldsymbol{A}\colon S(\boldsymbol{R}^n)\to L$$

is called a L-valued distribution defined on $\boldsymbol{R}^n$. For the special case $L = \boldsymbol{C}^1$ this definition coincides with the definition of tempered distributions given by L. Schwartz. The second special case $L = L(D)$, where D is a dense linear subset of a Hilbert space H and $L(D)$ denotes the $*$- algebra of operators acting in D (the strict definition of $L(D)$ is given below), is of great importance in the quantum field theory [4]. $L(D)$-valued distributions are often called operator-valued distributions.

We say that the topological vector space L satisfies the theorem on kernel if for any separately continuous bilinear mapping

$$\boldsymbol{B}\colon S(\boldsymbol{R}^n)\times S(\boldsymbol{R}^m)\to L$$

there exists a continuous linear mapping

$$\overset{\otimes}{\boldsymbol{B}}\colon S(\boldsymbol{R}^{n+m})\to L$$

such that $\boldsymbol{B}(\varphi, \psi) = \overset{\otimes}{\boldsymbol{B}}(\varphi\otimes\psi)$ for any $\varphi\in S(\boldsymbol{R}^n)$ and $\psi\in S(\boldsymbol{R}^m)$.