It follows that

$$\limsup_{k\to\infty}\left(\frac{1}{|B_k|}\int_{B_k}\{w(P)\}^p\,dP\right)^{1/p}\leqslant c\,\varepsilon$$

except for a set of points Q_0 of harmonic measure at most ε. Since the constant c depends only on the nontangential cone at Q_0 and not on Q_0 itself, it is a simple matter to complete the proof of Theorem 1.

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RUTGERS UNIVERSITY

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A ring of analytic functions, II *

by

R. M. BROOKS (Salt Lake City, Ut.)

This paper is devoted to a further study of the commutative locally convex algebra A of all complex-valued functions which are defined and holomorphic in the open unit disc U of the complex plane, where the ring multiplication is convolution (the Hadamard product), the other operations are the usual ones, and the topology is the compact-open. Specifically, we investigate the spectra of elements of this algebra and the operations of inversion and exponentiation.

In Section 2 we give simple proofs of the two main results of [2] whose proofs (in [2]) depended on an incorrect theorem (2.3 of [2]).

Section 3 consists of one theorem which gives a means for relating convergence in the compact convergence topology of A to certain properties of the corresponding sequences of Maclaurin coefficients.

Section 4 is concerned with the spectrum of an element of A. The algebra A may be identified algebraically with a certain subalgebra $\hat{A}$ of $C(N)$, since the space N of non-negative integers (usual topology) is in a natural way homeomorphic to the space of non-zero continuous homomorphisms of A onto C (with the usual Gelfand topology). For $x\in A$, we let $\hat{x}$ be the corresponding element of $C(N)$. Then $\hat{x}(n)=x_n$, the nth Maclaurin coefficient of the holomorphic function x. We show that the spectrum $\sigma(x)$ of x is between the range $R(\hat{x})$ of $\hat{x}$ and its closure and give examples to show that in general one cannot say any more about $\sigma(x)$. Since A is not locally m-convex, the functional calculus developed for such algebras by Michael is useless here and one is induced to look at a spectrum defined for general locally convex algebras by Allan (and others) for which there is an applicable functional calculus. We identity first the set of (Allan) bounded elements of A, and show that $\mathrm{Sp}(x)=R(\hat{x})^*$, where $\mathrm{Sp}(x)$ is the spectrum defined in terms of having or not having an (Allan) bounded inverse and "*" indicates the closure in the Riemann sphere. Thus, $\mathrm{Sp}(x)$ is easily computable, once $R(\hat{x})$ is known, whereas $\sigma(x)$ is not.

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Section 5 is concerned with the group A^{-1} of invertible elements of A and with a densely defined exponential function. The mapping $x \to x^{-1}$ of A^{-1} onto itself is nowhere continuous. The exponential function is only densely defined. Its domain is computed in terms of the Gelfand transform. Exponentials need not be invertible, and the exponential function is nowhere continuous. Also, we characterize the range of the exponential function, and using this we show that

$$A^{-1} = \exp^*[\mathrm{Dom}(\exp^*) \cap -\mathrm{Dom}(\exp^*)],$$

where $\exp^*$ is the exponential function defined in terms of the convolution product "$*$".

We wish to acknowledge here an invaluable discussion with Professor C. A. McCarthy of the University of Minnesota which led to the formulation and proof of Theorem 3.1.

1. Preliminaries. In this paper U will denote the open unit disc in the complex plane C and A will denote the set of all complex-valued functions which are defined and holomorphic in U. The set A, endowed with the usual pointwise operations and the compact-open topology, is a Fréchet space. We define a ring multiplication in A by the following procedure: if $a, b \in A$, then

$$a*b(\lambda) = (2\pi i)^{-1} \int_\gamma a(\zeta) b(\zeta^{-1}\lambda)\zeta^{-1} d\zeta,$$

where $\lambda \in U$ and γ is a suitable closed curve in U which encloses 0 and λ.

With this multiplication A becomes a locally convex, complete, metrizable topological algebra with identity e, where $e(\lambda) = (1-\lambda)^{-1}$. Hence, A is a B_0-algebra (see [7]); but A is not locally m-convex. There are many ways to show this; but it is a direct consequence of Example 5.1 below, since in a locally m-convex algebra inversion is continuous.

Each $x \in A$ determines a sequence $\{x_n\}_{n=0}^{\infty}$ of complex numbers, the coefficients of its Maclaurin expansion: $x_n = (n!)^{-1} x^{(n)}(0)$. We shall regard elements of A either as functions on U or as complex sequences $\{x_n\}_{n=0}^{\infty}$ which satisfy

$$\limsup_n |x_n|^{1/n} \leqslant 1,$$

or equivalently,

$$\limsup_n n^{-1} \log |x_n| \leqslant 0.$$

Thus, we shall write "$a = \{a_n\} \in A$". When a and b are elements of A, then $a*b = \{a_n b_n\}$, a somewhat easier situation to work with than the given multiplication.

In [2] we studied the ideal structure of the algebra A. We summarize here the relevant results from that paper. Let $\mathscr{M}$ denote the maximal ideal space of A with the hull-kernel topology, $\mathscr{M}_0$ the subspace of closed maximal ideals, and S the space of all non-zero continuous homorphisms of A to C with the relative weak*-topology. Then (1) $\mathscr{M}_0$ and S are in a natural one-to-one correspondence which is topological, (2) S and N (the non-negative integers with the discrete topology) are in a natural one-to-one correspondence $n \leftrightarrow \varphi_n$, where $\varphi_n(x) = x_n$ ($x \in A$, $n \in N$). This map is topological. Also, (3) $M_n = \{x \in A: x_n = 0\}$ is the corresponding element of $\mathscr{M}_0$, (4) $\mathscr{M}$ is homeomorphic to $\beta N (M_p \leftrightarrow p)$, and (5) every C-valued homomorphism of A is continuous.

We shall identify S and N and define the Gelfand transform $\hat{x}$ of an element x of A to be a function on N; specifically, $\hat{x}(n) = \varphi_n(x) = x_n$ for $n \in N$. Thus, the sequence of coefficients of an element of A is just its Gelfand transform, and

$$\hat{A} = \big\{\{a_n\}_{n=0}^{\infty} \in C^N: \limsup_n |a_n|^{1/n} \leqslant 1\big\}.$$

2. The maximal ideal space of A. Section 2 of [2] was devoted to a study of the maximal ideal space of A. The proofs of several of the theorems of that section were based on Theorem 2.3 of [2]. D. L. Plank has shown that this theorem is false [6]. The theorem purported to characterize the maximal ideals M_p, $p \in \beta N - N$. He has shown that the points for which the characterization holds are exactly the P-points of $\beta N - N$ ([6], Theorem 7.6). In this section we shall give proofs of Corollary 2.3.1 and Theorem 2.4 of [2] which do not depend on Theorem 2.3 of that paper, since these are the only places where Theorem 2.3 is used directly (most of the applications being of Corollary 2.3.1).

LEMMA 2.1. *For $p \in \beta N$, let $J_p = \{a \in A: \bar{a}^\beta(p) < 1\}$, where $\bar{a}$ is defined by $\bar{a}(n) = |a_n|^{1/n}$ $(n \in N)$ and $\bar{a}^\beta$ is the continuous extension of $\bar{a}$ to βN. Then*

(1) *J_p is a prime ideal in A and*

(2) $\bigcap \{J_p: p \in \beta N - N\} = \bigcap \{M_p: p \in \beta N - N\}$.

Proof. Fix $a, b \in J_p$. Let $r = \bar{a}^\beta(p)$ and $s = \bar{b}^\beta(p)$. Then each of r and s is less than one. Let $t = \max(r, s)$. There exists neighborhoods U, V, and W of p such that (i) if $n \in U \cap N$, then $\bar{a}(n) < 2^{-1}(r+1)$, (ii) if $n \in V \cap N$, then $\bar{b}(n) < 2^{-1}(s+1)$, and (iii) if $n \in W \cap N$, then $2^{1/n} < 1+\varepsilon$, where $\varepsilon < \big(3(1+t)\big)^{-1}(1-t)$.

If $n \in U \cap V \cap W \cap N$, then

$$\begin{aligned}\overline{a+b}(n) = |a_n+b_n|^{1/n} &\leqslant 2^{1/n} \max\big(\bar{a}(n), \bar{b}(n)\big) \\ &\leqslant (1+\varepsilon)\max\big(2^{-1}(r+1), 2^{-1}(s+1)\big) = (1+\varepsilon)2^{-1}(t+1) \\ &< 3^{-1}(2+t) < 1.\end{aligned}$$

Hence, $\overline{(a+b)}^{\beta}(p) < 1$, and $a+b \in J_p$. The remaining verifications of part (1) are easy. Part (2) is Lemma 20.8 of [5].

THEOREM 2.2. (COROLLARY 2.3.1 of [2].) $\bigcap\{M_p\colon p \in \beta N - N\} = A_0$, *the set of elements of A with radius of convergence greater than one.*

Proof. In view of Lemma 2.1 it suffices to show that $A_0 = \bigcap\{J_p\colon p \in \beta N - N\}$. If $a \in A_0$, then $\limsup_n \bar{a}(n) = r < 1$, and there is an integer m such that if $n > m$, then $\bar{a}(n) < 2^{-1}(r+1) = s$. Fix $p \in \beta N - N$, and let V be a neighborhood of p which misses $\{0, 1, \ldots, m\}$. Hence, if $n \in V$, then $\bar{a}(n) < s$. Since $p \in \overline{V \cap N}$ we must have $\bar{a}^{\beta}(p) \leqslant s < 1$, and $a \in J_p$.

If $a \in A - A_0$, then there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ such that $\lim_k \bar{a}(n_k) = 1$. If p is any point of $\beta N - N$ which lies in the closure of $\{n_1, n_2, \ldots\}$, then $\bar{a}^{\beta}(p) = 1$. Hence, $a \notin J_p$.

We now prove Theorem 2.4 of [2]: "The maximal ideal space of A, endowed with the hull-kernel topology, is homeomorphic to βN." Let $f\colon \mathcal{M} \to \beta N$ be the map $f(M_p) = p$. We know that f is one-to-one and onto, that $\mathcal{M}$ is compact, and that βN is a Hausdorff space. Thus, it suffices to establish the continuity of f. Recall that a base for the closed sets of $\mathcal{M}$ is $\{E(a)\colon a \in A\}$, where $E(a) = \{M \in \mathcal{M}\colon a \in M\}$, and that $g\colon \beta N \to \mathcal{M}_C \left(= \mathcal{M}(C(N))\right)$ given by $g(p) = M_p^C$ is a homeomorphism (Gelfand–Kolmogoroff Theorem; [3], Theorem 7.3). We have also, ([2], Theorem 2.2), that $M_p^C \cap A = M_p$ for each p in βN. Let $h = g \circ f\colon \mathcal{M} \to \mathcal{M}_C$. Fix $y \in C(N)$. We show that $h^{-1}(E_C(y))$ is closed in $\mathcal{M}$, where $E_C(y) = \{M_p^C \in \mathcal{M}_C\colon y \in M_p^C\}$. Let x be the characteristic function of $N - Z(y)$. Then $Z(x) = Z(y)$, so $x \in M_p^C$ if, and only if, $y \in M_p^C$. Thus, $E_C(x) = E_C(y)$. But $x \in A$, and $M_p^C \in E_C(x)$ if and only if $x \in M_p^C \cap A = M_p$. Thus,

$$h^{-1}\left(E_C(y)\right) = h^{-1}\left(E_C(x)\right) = E(x)$$

which is closed.

3. A convergence theorem. Since the convolution multiplication defined in the set A is most easily thought of in terms of the coefficient sequences we establish a criterion for compact convergence in A in terms of the corresponding sequences.

THEOREM 3.1. *Let $\{f_n(z) = \sum_{p=0}^{\infty} a_{np} z^p\}_{n=1}^{\infty}$ be a sequence in A. Then $\{f_n\}_{n=1}^{\infty}$ converges to 0 in A (compact convergence) if and only if*

(1) *for each $p \in N$ we have $\lim_n a_{np} = 0$, and*

(2) *for each r, $0 < r < 1$, $\sup_n \sum_{p=0}^{\infty} |a_{np}| r^p < \infty$.*

Proof. Necessity. Since compact convergence of $\{f_n\}$ implies the same convergence for each of the sequences $\{f_n^{(p)}\}$, $p \in N$, we have (1). To prove (2) we fix r, $0 < r < 1$, and let $s = 2^{-1}(r+1)$. For each $n \in N$ we have

$$\sum_p |a_{np}| r^p = \sum_p |a_{np}| \left(2r(r+1)^{-1}\right)^p s^p$$
$$\leqslant \Big[\sum_p \{2r(r+1)^{-1}\}^{2p}\Big]^{1/2} \Big[\sum_p |a_{np}|^2 s^{2p}\Big]^{1/2}$$

by the Cauchy–Schwarz inequality. Call the first factor C_r. The second factor is the square root of

$$\frac{1}{2\pi}\int_0^{2\pi} |f_n(se^{i\theta})|^2 d\theta$$

which is dominated by M_r, the supremum of the sequence $\{f_n\}$ on the disc of radius $2^{-1}(r+1)$. Thus, $\sup_n \sum_p |a_{np}| r^p \leqslant C_r M_r < \infty$.

Sufficiency. We first show that if $\sup_n \sum_p |a_{np}| r^p = M_r < \infty$, then for each pair (ε, δ) of positive real numbers with $\delta < r$ there exists $q \in N$ so that

$$\sup_n \sum_{p=q}^{\infty} |a_{np}| (r-\delta)^p < \varepsilon. \tag{3.1}$$

If $q \in N$, then for each $n \geqslant 1$ we have

$$\sum_{p=q}^{\infty} |a_{np}| (r-\delta)^p = \sum_{p=q}^{\infty} |a_{np}| r^p \left(r^{-1}(r-\delta)\right)^p. \tag{3.2}$$

But $r^{-1}(r-\delta) < 1$, so $\left(r^{-1}(r-\delta)\right)^p \leqslant \left(r^{-1}(r-\delta)\right)^q$ for each $p \geqslant q$. Thus, (3.2) is dominated by

$$\left(r^{-1}(r-\delta)\right)^q \sum_{p=q}^{\infty} |a_{np}| r^p \leqslant M_r \left(r^{-1}(r-\delta)\right)^q.$$

So, if q is chosen sufficiently large, then the conclusion follows (since M_r is independent of $n \geqslant 1$).

We now fix r, $0 < r < 1$, $\varepsilon > 0$, $\delta > 0$ so that $\delta < r$, and choose $q \in N$ so that (3.1) holds. Now

$$\sum_{p=0}^{q-1} |a_{np}| (r-\delta)^p \leqslant \sum_{p=0}^{q-1} |a_{np}|.$$

For $0 \leqslant p \leqslant q-1$ there exists N_p such that if $n \geqslant N_p$, then $|a_{np}| < q^{-1}\varepsilon$. Let $N = \max(N_0, \ldots, N_{q-1})$. Then for each $n \geqslant N$ we have

$$\sum_{p=q}^{\infty} |a_{np}| (r-\delta)^p < \varepsilon, \tag{3.3}$$

$$\sum_{p=0}^{q-1} |a_{np}| (r-\delta)^p \leqslant \sum_{p=0}^{q-1} |a_{np}| < \varepsilon. \tag{3.4}$$

Hence, for $n \geqslant N$ we have

$$\sum_{p=0}^{\infty} |a_{np}|(r-\delta)^p < 2\varepsilon. \tag{3.5}$$

Thus, for $n \geqslant N$ we have

$$\sup\{|f_n(z)|: |z| \leqslant r-\delta\} < 2\varepsilon, \tag{3.6}$$

and the sufficiency is established.

4. Spectra of elements of A. In [2] we gave a criterion for invertibility in A: $x \in A$ is *invertible* ($x \in A^{-1}$) if, and only if, (1) for each $n \in N$ we have $x_n \neq 0$, and (2) $\lim_n |x_n|^{1/n} = 1$. Hence, if $x \in A$, then the range of $\hat{x}$, $R(\hat{x})$, is contained in the spectrum $\sigma(x)$ of x ($= \{\lambda \in C: x-\lambda \notin A^{-1}\}$). We wish (1) to see how much larger than $R(\hat{x})$ the set $\sigma(x)$ can be and (2) to compare this spectrum with the one defined for general locally convex algebras by Allan [1].

LEMMA 4.1. *If $x \in A$, then $R(\hat{x}) \subseteq \sigma(x) \subseteq R(\hat{x})^-$.*

Proof. Fix $\lambda \in C - R(\hat{x})^-$. Then $\operatorname{dist}(\lambda, R(\hat{x})) = \delta > 0$ and

$$|x_n - \lambda|^{1/n} \geqslant \delta^{1/n} \quad \text{for each} \quad n \in N.$$

It is clear that $x - \lambda \in A^{-1}$.

EXAMPLE 4.2. We give three examples:

(a) $\sigma(x) = R(\hat{x})$,

(b) $\sigma(x) = R(\hat{x})^-$,

and

(c) $\sigma(x)$ is neither extreme.

(a) x is defined by $x_n = (n+1)^{-1}$ for $n \in N$. Here, $R(\hat{x})^- = R(\hat{x}) \cup \{0\}$. It is easily verified that $x \in A^{-1}$ $(0 \notin \sigma(x))$.

(b) x is defined by $x_n = 2^{-n}$ for $n \in N$. Again, $R(\hat{x})^- = R(\hat{x}) \cup \{0\}$. But, in this case $0 \in \sigma(x)$.

(c) x is defined by $x_n = 1 + (n+1)^{-1} = (n+1)^{-1}(n+2)$ for even $n \in N$ and $x_n = 2^{-n}$ for odd $n \in N$. Here, $R(\hat{x})^- = R(\hat{x}) \cup \{0, 1\}$. Since $\liminf_n |x_n|^{1/n} = 2^{-1}$, $0 \in \sigma(x)$. But $\liminf_n |x_n - 1|^{1/n} = 1$. So $1 \notin \sigma(x)$.

In these examples it was still easy to compute $\sigma(x)$ even though it was neither $R(\hat{x})$ nor $R(\hat{x})^-$. One can construct examples where $R(\hat{x})$ consists of the points in C with rational real and imaginary parts, $R(\hat{x})^- = C$, and $\sigma(x)$ can be almost anything in between. The difficulty of computing $\sigma(x)$ except in the simplest examples is one reason for considering the spectrum defined by Allan (and others) for locally convex algebras. Another reason is that there is a functional calculus for such algebras which uses this spectrum. Michael's functional calculus (which uses the spectrum $\sigma(x)$) is useless here because the algebra A is not locally m-convex and Michael's development requires the inverse limit decomposition peculiar to those algebras (see [4]).

An element $x \in A$ is said to be (*Allan*) *bounded* ($x \in A_b$) provided there exists $\lambda \in (0, \infty)$ such that $\{(\lambda^{-1}x)^n\}_{n=1}^{\infty}$ is bounded in A (in the TVS sense). The infimum of all such λ is denoted $\beta(x)$ and is the same as $\inf\{\lambda > 0: (\lambda^{-1}x)^n \to 0 \ (n \to \infty)\}$ ([1], Proposition 2.14). Let C^* be the one-point compactification of C. If $\lambda \in C^*$, then $\lambda \in \operatorname{Sp}(x)$ (Allan's spectrum) if and only if (1) $\lambda = \infty$ and $x \notin A_b$, or (2) $\lambda \neq \infty$ and $x - \lambda$ has no inverse belonging to A_b.

The first problem is to identify the (Allan) bounded elements of A.

THEOREM 4.3. *If $x \in A$, then $x \in A_b$ if and only if $\sigma(x)$ is a bounded subset of C (equivalently, $R(\hat{x})$ or $R(\hat{x})^-$ is bounded in C).*

Proof. If $x \in A_b$, then $\operatorname{Sp}(x)$ is bounded in C ([1], Theorem 3.12), and $\sigma(x) \subseteq \operatorname{Sp}(x)$.

Conversely, suppose $\sigma(x)$ is bounded in C, and is contained in the disc about 0 of radius M. Fix $\lambda > M$. Then $(\lambda^{-1}x)^n \to 0$ $(n \to \infty)$. We use the criterion of Theorem 3.1. For $p \in N$ we have $\big((\lambda^{-1}x)^n\big)_p = \lambda^{-n}x_p^n$. Since for each $p \in N$ the complex number x_p belongs to $\sigma(x)$ we must have $|\lambda^{-1}x_p| < 1$. Thus, $\lim_n [(\lambda^{-1}x)_p]^n \to 0$. Also, for each $n \in N$ we have

$$\sum_{p=0}^{\infty} |(\lambda^{-1}x_p)^n|\, r^p < \sum_{p=0}^{\infty} r^p < \infty.$$

THEOREM 4.4. *If $x \in A$, then $\operatorname{Sp}(x)$ is the closure of $R(\hat{x})$ in C^*.*

Proof. In this proof "$^-$" will denote closure in C and "$*$" will denote closure in C^*. We know already:

$$R(\hat{x}) \subseteq \sigma(x) \subseteq R(\hat{x})^- \subseteq R(\hat{x})^* \subseteq R(\hat{x})^- \cup \{\infty\}, \quad \text{and} \quad \sigma(x) \subseteq \operatorname{Sp}(x).$$

Also, $\operatorname{Sp}(x) = \operatorname{Sp}(x)^*$, since A is complete ([1], Corollary 3.9). Hence, $R(\hat{x})^* \subseteq \operatorname{Sp}(x)$.

Fix $\lambda \in C^* - R(\hat{x})^*$. If $\lambda = \infty$, then $\lambda \notin R(\hat{x})^*$ implies that $R(\hat{x})^-$ is bounded in C. By Theorem 4.3 $x \in A_b$ and $\infty \notin \operatorname{Sp}(x)$. If $\lambda \in C$, then $(x-\lambda)^{-1}$ exists in A and

$$\sigma\big((x-\lambda)^{-1}\big) = \{(\mu-\lambda)^{-1}: \mu \in \sigma(x)\}.$$

This set is clearly bounded in C so $(x-\lambda)^{-1} \in A_b$ and $\lambda \notin \operatorname{Sp}(x)$. Thus, $\operatorname{Sp}(x) = R(\hat{x})^*$. Also, $R(\hat{x})^* = R(\hat{x})^-$ if and only if $R(\hat{x})$ is bounded. In the other case, $R(\hat{x})^* = R(\hat{x})^- \cup \{\infty\}$.

5. Inversion and exponentiation. We show here that the mapping $x \to x^{-1}$ of A^{-1} onto A^{-1} is not continuous, that the exponential function $\exp^*$ is only densely defined and is nowhere continuous, that $\exp^*(x)$ need not be invertible ($x \in A$), but that A^{-1} is the range of a suitable restriction of the exponential function.

EXAMPLE 5.1. The map $x \to x^{-1}$ is not continuous at the identity of A^{-1} (hence, is nowhere continuous since A^{-1} is a group under multiplication). We define a sequence $\{x_n\}_{n=1}^{\infty}$ in A^{-1} by $(x_n)_p = 1$ if $p \neq n$ and $(x_n)_n = n^{-n}$. Then $(x_n - e)_p = 0$ if $p \neq n$ and $(x_n - e)_n = n^{-n} - 1$. This sequence clearly converges to 0 in A. But $(x_n^{-1} - e)_p = 0$ if $p \neq n$ and $(x_n^{-1} - e)_n = n^n - 1$. This sequence fails to converge in A.

Let $x \in A$. We define $\exp^*(x)$ to be the series $\sum_{n=0}^{\infty} (n!)^{-1} x^n$ when it converges, where "x^n" is the nth convolution power in A.

The following lemma is an immediate consequence of the application of the continuous homomorphisms φ_n $(n \in \boldsymbol{N})$.

LEMMA 5.2. *If* $x \in A$ *and if* $\sum_{k=0}^{\infty} (k!)^{-1} x^k$ *converges in* A *(to* $\exp^*(x)$*), then* $(\exp^* x)_n = \exp(x_n)$ *for each* $n \in \boldsymbol{N}$.

LEMMA 5.3. *Let* $D = \{x \in A\colon \limsup_n n^{-1} \operatorname{Re} x_n \leqslant 0\}$. *If* $x \in A$, *then* $\exp^*(x)$ *exists (in* A*) if and only if* $x \in D$.

Proof. If $x \in A$, we can formally define a power series y by setting $y_n = \exp(x_n)$ for $n \in \boldsymbol{N}$. This series corresponds to an element of A if and only if $\limsup_n n^{-1} \log|y_n| \leqslant 0$ (equivalently, $\limsup_n n^{-1} \operatorname{Re} x_n \leqslant 0$). If $\exp^*(x)$ exists, then it must be given by the sequence $\{y_n\}$ defined above and $x \in D$. The converse is clear.

We note that all the elements of A which have bounded spectra (more generally, all elements x such that $\operatorname{Re}\sigma(x)$ is bounded above) belong to D. Since this includes all elements of A with radius of convergence greater than one, the function $\exp^*$ is densely defined.

That the existence of $\exp^*(x)$ is not completely determined by $\sigma(x)$ or $\operatorname{Sp}(x)$ is shown by the following example.

EXAMPLE 5.4. We define two elements x, y of A such that (1) $\sigma(x) = \sigma(y)$, (2) $\operatorname{Sp}(x) = \operatorname{Sp}(y)$, but (3) $\exp^*(x)$ exists while $\exp^*(y)$ fails to exist. The elements x, y are defined by

(a) $x_k = 0$ if $k \notin \{2^0, 2^1, 2^2, \ldots\}$, $x_{2^n} = n$ for $n \in \boldsymbol{N}$,

(b) $y_n = n$ for $n \in \boldsymbol{N}$.

Clearly $R(\hat{x}) = R(\hat{y}) = \boldsymbol{N}$; $\sigma(x) = \sigma(y) = \boldsymbol{N}$; and $\operatorname{Sp}(x) = \operatorname{Sp}(y) = \boldsymbol{N} \cup \{\infty\}$. However,

$$\lim_k k^{-1} x_k \leqslant \lim_n 2^{-n} x_{2^n} = 0.$$

Hence, $x \in D$; but $y \notin D$.

LEMMA 5.5. *If* $x \in D$ *and if* $\exp^*(x) \in A^{-1}$, *then* $\big(\exp^*(x)\big)^{-1} = \exp^*(-x)$. *Thus,* $\exp^*(x) \in A^{-1}$ *if and only if* $x \in D \cap (-D)$, *this requirement being equivalent to* $\lim_n n^{-1} \operatorname{Re} x_n = 0$.

Proof. Let x be as hypothesized. Then, for each $n \in \boldsymbol{N}$ we have

$$\big[(\exp^*(x))^{-1}\big]_n = [\exp^*(x)_n]^{-1} = \exp(-x_n).$$

The element $x = \{-n\}_{n=0}^{\infty}$ of A is in D, but not in $-D$. Hence, not every exponential is invertible.

THEOREM 5.6. *The mapping* $x \to \exp^*(x)$ *is discontinuous at every point of* D.

Proof. Fix $a \in D$. We define a sequence $\{x_n\}_{n=1}^{\infty}$ in D by $x_{np} = 0$ if $p \neq n$ and $x_{nn} = -a_n + n\log n$. Then, $\{x_n\}$ converges to 0 in A. Part (1) of Theorem 3.1 is obviously satisfied; and for each $r \in (0, 1)$ we have that

$$\sup_n \sum_p |x_{np}| r^p = \sup_n |-a_n + n\log n| r^n$$

is finite, since $\{-a_n + n\log n\}_{n=1}^{\infty}$ is the coefficient sequence of an element of A (where we let the 0th coefficient be 0).

However, if $r \in (0, 1)$ we have

$$\begin{aligned}
\sup_n \sum_p \big|(\exp^*(a)\exp^*(x_n))_p - (\exp^*(a))_p\big| r^p
&= \sup_n |\exp(a_n + x_{nn}) - \exp(a_n)| r^n \\
&= \sup_n |\exp(a_n + x_{nn}) r^n - \exp(a_n) r^n|. \\
&= \|\{\exp(a_n + x_{nn}) r^n\}_{n=1}^{\infty} - \{\exp(a_n) r^n\}_{n=1}^{\infty}\|_\infty \\
&\geqslant \big|\|\{\exp(a_n + x_{nn}) r^n\}\|_\infty - \|\{\exp(a_n) r^n\}\|_\infty\big|.
\end{aligned}$$

We consider the two terms separately. In the first term evaluation of the x_{nn}'s yields $\|\{n^n r^n\}_{n=1}^{\infty}\|_\infty$, which is clearly $+\infty$. The second term is $\sup_n \exp(\operatorname{Re} a_n) r^n$ which is finite, since $\sum_n \exp(a_n) r^n$ is absolutely summable for $r \in (0, 1)$. Thus, $\{\exp^*(a + x_n)\}$ fails to converge to $\exp^*(a)$, while $\{a + x_n\}$ converges to a.

LEMMA 5.7. *An element* a *of* A *has a logarithm in* A *if and only if*

$$\limsup_n n^{-1} \log\big(|\log|a_n||\big) \leqslant 0.$$

Proof. We fix $a \in A$ and define x by $x_n = \log|a_n| + i\operatorname{PA}(a_n)$, where $\operatorname{PA}(1)$ is the principal argument, which we take in $(0, 2\pi]$. For a complex number λ we let $|\lambda|$ denote the usual absolute value in $\boldsymbol{C} = \boldsymbol{R}^2$, and we denote by $|\lambda|_\infty$ the l^∞-norm on $\boldsymbol{R}^2$. We have $|\lambda|_\infty \leqslant |\lambda| \leqslant \sqrt{2}|\lambda|_\infty$. Thus, $\limsup_n |x_n|^{1/n} \leqslant 1$ if and only if, $\limsup_n |x_n|_\infty^{1/n} \leqslant 1$ if and only if

$\limsup_n n^{-1}\log|x_n|_\infty \leqslant 0$. We shall show that this last inequality is equivalent to the one in the statement of the lemma. Now

$$\limsup_n n^{-1}\log[\max(|\log|a_n||, \mathrm{PA}(a_n))] = \limsup_n n^{-1}\max[\log(|\log|a_n||), \log \mathrm{PA}(a_n)] = \max[\limsup_n n^{-1}\log(|\log|a_n||), \limsup_n n^{-1}\log \mathrm{PA}(a_n)].$$

Since $\limsup_n n^{-1}\log \mathrm{PA}(a_n) \leqslant 0$ we have

$$\limsup_n n^{-1}\log(x_n)_\infty \leqslant 0$$

if and only if

$$\limsup_n n^{-1}\log(|\log|a_n||) \leqslant 0.$$

Thus, if $a \in A$, then the easiest way of defining a logarithm for a yields an element x of A if and only if a satisfies the stated condition. It is clear that the constructed element x belongs to D.

THEOREM 5.8. $A^{-1} = \exp^*[D\cap(-D)]$.

Proof. Lemma 5.5 implies that the right side is contained in the left. Let $a \in A^{-1}$. Then $\lim_n n^{-1}\log|a_n| = 0$. For $\varepsilon > 0$, there exists $n_0 \in N$ such that if $n > n_0$, then $|\log|a_n|| < n\varepsilon$. But then

$$\log|\log|a_n|| < \log n\varepsilon \quad \text{and} \quad n^{-1}\log(|\log|a_n||) < n^{-1}\log n + n^{-1}\log\varepsilon.$$

The larger sequence converges to 0 as $n\to\infty$. Hence, a has a logarithm in A: $a = \exp^*(x)$. Since $\exp^*(x)\in A^{-1}$, Lemma 5.5 implies that $x \in D\cap(-D)$.

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Factorization in Fréchet spaces*

by

W. H. SUMMERS (Fayetteville, Ark.)

1. Introduction. Let A be a Banach algebra with a bounded left approximate identity. In 1959, P. J. Cohen [2] established that each element in A could be factored, and the utility of factorization as a tool in the study of Banach algebras and Banach A-modules has become apparent in the intervening decade. One fruitful approach (e. g., see Rieffel [11], Sentilles and Taylor [13], and Collins and Summers [4] has been to use the factorization theorem due to Hewitt [7] in conjunction with the device of introducing auxiliary essential left Banach A-modules in order to factor certain "large" subsets of a given essential left Banach A-module (for a definition, see [10]), and it is this procedure which will concern us in the sequel.

There are at least three natural auxiliary left A-modules that one can consider for a fixed left Banach A-module E; namely, the space $C(X;E)$ of all continuous functions from a topological space X into E, the space $C_b(X;E)$ of bounded functions in $C(X;E)$, and the space $C_0(X;E)$ of functions in $C(X;E)$ which vanish at infinity, each with respect to the obvious pointwise action. If X is a locally compact Hausdorff space, then $C_0(X;E)$ with the uniform topology [15] becomes an essential left Banach A-module and the above approach is valid (see Section 3). However, the situation is less obvious in the other two cases, and there seems little likelihood of realizing either $C(X;E)$ as a left *Banach* A-module or $C_b(X;E)$ as an *essential* left Banach A-module (however, see Section 4) unless X is compact. Consequently, the following question arises (see [4]): *can Hewitt's factorization theorem be extended to include the action of A on a class of locally convex spaces more general than Banach spaces?*

An affirmative answer has been announced by Ovaert [9], but it appears that a proof of Ovaert's result will entail a recasting of the technique used by Craw [5] in extending the work of Cohen to Fréchet algebras. In our context, however, it is possible to give a relatively simple

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