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DICKINSON COLLEGE
CARLISLE, PENNSYLVANIA

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On a class of absolutely p-summing operators

by

TIN KIN WONG (Detroit, Mich.)

§ 1. Introduction. In this note we use the conventions and notation of A. Persson [4]. A normed linear space is denoted by E, and E' is its topological dual with the strong dual topology. We use $\langle f, f'\rangle$ to indicate the action of a vector f in E and a functional f' in E'. For $1 \leqslant p < \infty$, p and p' are the usual conjugate numbers. $L^p(X, \mu; E')$ is the Banach space of equivalent classes of strongly μ-measurable E'-valued functions K such that $\int_X \|K(x)\|^p d\mu < \infty$. In case E' is C — the complex numbers, we simply write $L^p(X, \mu)$ instead of $L^p(X, \mu; C)$. All the measures in this note are countably additive and positive, and all the operators are bounded.

We aim to investigate a class of linear operators $T\colon E \to L^p(X, \mu)$ and their adjoint $T^*\colon L^{p'}(X, \mu) \to E'$ such that

$$|Tf(x)| \leqslant \gamma(x)\|f\|$$

for some γ in $L^p(X, \mu)$. When E is a reflexive Banach space or E' is separable, it turns out that each of them can be represented by a unique K in $L^p(X, \mu; E')$ in the following way:

$$Tf(x) = \langle f, K(x)\rangle \text{ a. e.} \quad \text{and} \quad T^* g = \int_X g(x) K(x)\, d\mu$$

for f in E, g in $L^{p'}(X, \mu)$ where the integral is taken to be the Bochner integral. In this case, T is an operator of type $-N_p$ and T^* is of type $-N^p$ ([4], Theorem 1 and 2). They are all completely continuous operators. Our result is quite similar to Dunford–Pettis Theorem in which they obtained a representation for any operators $T\colon E \to L^\infty(X, \mu)$ and $S\colon L^1(X, \mu) \to E'$ by a bounded weakly measurable E'-valued function under the hypotheses that E is a separable Banach space (see [6], p. 469). As a consequence of our result, we obtain an alternative proof for a theorem of Shields–Wallen–Williams which asserts that if operator $T\colon E \to L^p(X, \mu)$ satisfies $|Tf(x)| \leqslant \gamma(x)\|f\|$ for some γ in $L^p(X, \mu)$, then T is absolutely p-summing. In case $E = L^{p'}(Y, \nu)$, our result also includes some work of A. Persson ([4], Theorem 3) which gives a characterization of Hilbert–Schmidt operators on L^2-spaces.

THEOREM 1. *Let E be a Banach space which is either reflexive or its dual E' is separable. For linear operators $T\colon E\to L^p(X,\mu)$, $1\leqslant p<\infty$, the following conditions are equivalent:*

(a) *There is a unique K in $L^p(X,\mu;E')$ such that*

$$Tf(x)=\langle f, K(x)\rangle \quad a.\,e.$$

(b) *$|Tf(x)|\leqslant \gamma(x)\|f\|$ a. e. for some γ in $L^p(X,\mu)$.*

Note. The exceptional set of measure zero may depend upon f.

THEOREM 2. *Let E be as in Theorem 1. For linear operators $S\colon L^{p'}(X,\mu)\to E'$, $\frac{1}{p}+\frac{1}{p'}=1$, $1\leqslant p<\infty$, the following conditions are equivalent:*

(a) *There is a unique K in $L^p(X,\mu;E')$ such that*

$$Sg=\int\limits_X g(x)K(x)\,d\mu$$

for g in $L^{p'}(X,\mu)$.

(b) *S is the adjoint of a linear operator T; $E\to L^p(X,\mu)$ such that*

$$|Tf(x)|\leqslant\gamma(x)\|f\| \quad a.\,e.$$

for some γ in $L^p(X,\mu)$.

Note. The integral in condition (a) is Bochner integral.

Remark. It is not difficult to see the equivalence of Theorem 1 and Theorem 2. We shall only prove Theorem 1. Moreover, if there are K and K_1 in $L^p(X,\mu;E')$ such that $\langle f, K(x)\rangle=\langle f, K_1(x)\rangle$ a. e. for each f, where the exceptional μ-null set depends on f. Then if g in $L^{p'}(X,\mu)$, we have

$$\Big\langle f, \int g(x)K(x)\,d\mu\Big\rangle=\int\langle f, K(x)\rangle g(x)\,d\mu=\int\langle f, K_1(x)\rangle g(x)\,d\mu =\Big\langle f, \int g(x)K_1(x)\,d\mu\Big\rangle$$

for all f in E, therefore

$$\int g(x)K(x)\,d\mu=\int g(x)K_1(x)\,d\mu$$

for all g in $L^{p'}(X,\mu)$. In particular

$$\int\limits_A\big(K(x)-K_1(x)\big)\,d\mu=0$$

for all μ-measurable set A with $\mu(A)<+\infty$. Since $\|K(\cdot)-K_1(\cdot)\|^p$ is in $L^1(X,\mu)$, the support of $K-K_1$ is σ-finite. Using localization method, we see that $K(x)-K_1(x)=0$ almost everywhere. Thus $K=K_1$. This proves the uniqueness assertion in Theorem 1 and Theorem 2.

§ 2. Some measure-theoretic preparation. Let (X,μ) be a finite measure space. Let $\tau\colon L^\infty(X,\mu)\to C(\Omega)$ be the Gelfand isomorphism $f\to\hat f$, where Ω is the Gelfand space of $L^\infty(X,\mu)$. Let $\hat\mu$ be the Radon measure on Ω such that $\int\limits_X f\,d\mu=\int\limits_\Omega \hat f\,d\hat\mu$ for all f in $L^\infty(X,\mu)$.

PROPOSITION 1. *For each $1\leqslant p<\infty$, there exists a linear isometry τ_p of $L^p(X,\mu)$ onto $L^p(\Omega,\hat\mu)$ that extends the Gelfand isomorphism τ. Moreover, this family of isometries $\{\tau_p;\ 1\leqslant p<\infty\}$ has the following properties:*

(i) *For $1\leqslant p<q<\infty$, τ_p extends τ_q.*

(ii) *If f, g and fg are all in $L^p(X,\mu)$, then*

$$\tau_p(fg)=\tau_p(f)\tau_p(g) \quad\text{and}\quad \tau_p(\bar f)=\overline{\tau_p(f)}$$

where $\bar f$ is the complex conjugate of f. Hence $\tau_p(|f|)=|\tau_p(f)|$.

(iii) *$\int\limits_X f\,d\mu=\int\limits_\Omega\tau_p(f)\,d\hat\mu$ for f in $L^p(X,\mu)$. Hence $\int\limits_A f\,d\mu=\int\limits_{\hat A}\tau_p(f)\,d\hat\mu$ where $\hat A$ is the compact open subset of Ω such that $\hat\chi_A=\chi_{\hat A}$ and χ_A is the characteristic function of A.*

Proposition 1 is well known (cf. [2] p. 18 for the existence of τ_p). The properties (i)–(iii) requires only routine checking.

The following proposition plays a key role in the proof of our theorems.

PROPOSITION 2. *Notations as above, let E be a Banach space. For $1\leqslant p<\infty$, there exists a linear isometry σ_p of $L^p(\Omega,\hat\mu;E)$ onto $L^p(X,\mu;E)$ such that*

$$\int\limits_A\sigma_p(K)(x)\,d\mu=\int\limits_{\hat A}K(\omega)\,d\hat\mu$$

for K in $L^p(\Omega,\hat\mu;E)$ and A μ-measurable set with $\hat\chi_A=\chi_{\hat A}$.

Proof. We know that the formal identity $I\colon C(\Omega)\to L^\infty(\Omega,\hat\mu)$ is a linear isometry of $C(\Omega)$ onto $L^\infty(\Omega,\hat\mu)$ (cf. [2], p. 18). Thus $I\circ\tau\colon L^\infty(X,\mu)\to L^\infty(\Omega,\hat\mu)$ is an isometric isomorphism. Since $\chi_A=\chi_B$ in $L^\infty(X,\mu)$ if and only if A and B represent the same element in the measure ring of (X,μ). This gives rise to an isomorphism between the measure rings of (X,μ) and $(\Omega,\hat\mu)$. For any step-function

$$K=\sum_{i=1}^n\chi_{\hat A_i}f_i$$

in $L^p(\Omega,\hat\mu;E)$, we may regard $\hat A_i$ as an element in the measure ring of $(\Omega,\hat\mu)$ with the compact open set $\hat A_i$ as a representative. Let

$$\sigma_pK=\sum_{i=1}^n\chi_{A_i}f_i \quad\text{where}\quad \hat\chi_{Ai}=\chi_{\hat A_i}.$$

Then $\sigma_p K$ is a step function in $L^p(X, \mu; E)$ and $\|\sigma_p K\| = \|K\|$. Because of the measure ring isomorphism, we have

$$\sigma_p(K_1+K_2) = \sigma_p(K_1)+\sigma_p(K_2)$$

for K_1 and K_2 step-functions. Thus σ_p is a linear isometry on the dense linear manifold of the step-functions in $L^p(\Omega, \hat{\mu}; E)$ into $L^p(X, \mu; E)$. Therefore σ_p can be extended to a linear isometry of $L^p(\Omega, \hat{\mu}; E)$ into $L^p(X, \hat{\mu}; E)$. Since the range is dense, hence, σ_p maps $L^p(\Omega, \hat{\mu}; E)$ onto $L^p(X, \mu; E)$. For $K = \sum_{i=1}^{n} \chi_{\hat{A}_i} f_i$ a step-function, and A μ-measurable set, we see that

$$\int_{\hat{A}} K(\omega)\,d\hat{\mu} = \int_{A} \sigma_p K(x)\,d\mu.$$

Since $\int_{\hat{A}}(\cdot)\,d\hat{\mu}$: $L^p(\Omega, \hat{\mu}; E) \to E$ is a bounded operator, and since the step-functions are dense in $L^p(\Omega, \hat{\mu}; E)$, then

$$\int_{\hat{A}} K(\omega)\,d\hat{\mu} = \int_{A} \sigma_p K(x)\,d\mu$$

for all K in $L^p(\Omega, \hat{\mu}; E)$. The proof is complete.

§ 3. The proof of Theorem 1 hinges on the implication (b) ⇒ (a), because condition (a) implies (b) clearly. We may assume that the set $A = \{x \in X;\ \gamma(x) = 0\}$ has measure zero. If $\mu(A)$ were positive, then we can write $L^p(X, \mu; E)$ as $L^p(A^c, \mu|_{A^c}; E) \oplus L^p(A, \mu|_A; E)$ for any Banach space E, since the multiplication by χ_A is a projection (idempotent) on $L^p(X, \mu; E)$. We use A^c to denote the complement of A in X, and $\mu|_{A^c}$ is the measure μ restricted to A^c. The condition $|Tf(x)| \leqslant \gamma(x)\|f\|$ a. e. implies that $T(E)$ is contained in $L^p(A^c, \mu|_{A^c})$. Hence T factors as

$$E \xrightarrow{T_1} L^p(A^c, \mu|_{A^c}) \xrightarrow{i} L^p(X, \mu).$$

Now γ is in $L^p(A^c, \mu|_{A^c})$ and $\gamma > 0$ on A^c,

$$|T_1 f(x)| \leqslant \gamma(x)\|f\|.$$

If $T_1 f(x) = \langle f, K_1(x)\rangle$ for some K_1 in $L^p(A^c, \mu|_{A^c}; E')$, using the natural injection i of $L^p(A^c, \mu|_{A^c}; E')$ into $L^p(X, \mu; E')$ we obtain K in $L^p(X, \mu; E')$ such that $Tf(x) = \langle f, K(x)\rangle$ a. e.. Therefore we can and do assume that $\gamma(x) > 0$ a. e.

LEMMA 1. *Let E be a normed linear space. Let $\gamma > 0$ a. e. be in $L^p(X, \mu)$. Define the finite measure $\nu = \gamma^p \cdot \mu$. Let φ: $L^p(X, \nu) \to L^p(X, \mu)$ be the linear isometry $f \to \gamma f$. For operators T_μ: $E \to L^p(X, \mu)$ and T_ν: $E \to L^p(X, \mu)$ such that $\varphi T_\nu = T_\mu$, then*

(i) *$|T_\mu f(x)| \leqslant \gamma(x)\|f\|$ a. e. (μ) if and only if $|T_\nu f(x)| \leqslant \|f\|$ a. e. (ν).*

(ii) *$T_\mu f(x) = \langle f, K_\mu(x)\rangle$ a. e. for some K_μ in $L^p(X, \mu; E')$ if and only if $T_\nu f(x) = \langle f, K_\nu(x)\rangle$ a. e. for some K_ν in $L^p(X\ \nu; E')$.*

The proof is trivial, we omit it.

LEMMA 2. *Let E be a normed linear space. Let (X, μ) be a finite measure space. Let (Ω, μ) be the measure space obtained in § 2. For linear operators T: $E \to L^p(X, \mu)$ and $\hat{T}$: $E \to L^p(\Omega, \hat{\mu})$ such that $\hat{T} = \tau_p T$, where τ_p is the isometric extension of the Gelfand isomorphism τ of $L^\infty(X, \mu)$ onto $C(\Omega)$, then*

(i) *$|\hat{T}f(\omega)| \leqslant \|f\|$ a. e. ($\hat{\mu}$) if and only if $|Tf(x)| \leqslant \|f\|$ a. e. (μ).*

(ii) *$\hat{T}f(\omega) = \langle f, \hat{K}(\omega)\rangle$ a. e. for some $\hat{K}$ in $L^p(\Omega\ \hat{\mu}; E')$ if and only if $Tf(x) = \langle f, K(x)\rangle$ a. e. for some K in $L^p(X, \mu; E')$.*

Proof.

$$\int_{\hat{A}} |\hat{T}f(\omega)|\,d\hat{\mu} = \int_{\hat{A}} |\tau_p Tf(\omega)|\,d\hat{\mu} = \int_{\hat{A}} \tau_p(|Tf|)(\omega)\,d\hat{\mu} = \int_{A} |Tf(x)|\,d\mu$$

and $\mu(A) = \hat{\mu}(\hat{A})$. We have used Proposition 1 in this argument. It follows that $|\hat{T}f(\omega)| \leqslant \|f\|$ a. e. ($\hat{\mu}$) if and only if $|Tf(x) \leqslant \|f\|$ a. e. (μ). This proves (i). Now we prove (ii). If $\hat{T}f(\omega) = \langle f, \hat{K}(\omega)\rangle$ a. e. ($\hat{\mu}$) for some $\hat{K}$ in $L^p(\Omega, \hat{\mu}; E')$. Let $K = \sigma_p \hat{K}$ where σ_p is the linear isometry of $L^p(\Omega, \mu; E')$ onto $L^p(X, \mu; E')$ (see Proposition 2). If A is any μ-measurable set, then

$$\int_{A} Tf(x)\,d\mu = \int_{A} \tau_p^{-1}\hat{T}f(x)\,d\mu = \int_{\hat{A}} \hat{T}f(\omega)\,d\hat{\mu} = \int_{\hat{A}} \langle f, \hat{K}(\omega)\rangle\,d\hat{\mu}$$
$$= \int_{A} \langle f, \sigma_p \hat{K}(x)\rangle\,d\mu = \int_{A} \langle f, K(x)\rangle\,d\mu.$$

Hence $Tf(x) = \langle f, K(x)\rangle$ a. e. The above argument makes use of Proposition 1 and 2. By a similar argument one proves the converse. This concludes the proof of Lemma 2.

LEMMA 3. *Let E be a normed linear space and let $(\Omega, \hat{\mu})$ be the measure space obtained in § 2. If $\hat{T}$: $E \to L^p(\Omega, \hat{\mu})$ is a linear operator such that $|\hat{T}f(\omega)| \leqslant \|f\|$ a. e., then there exists a bounded function Φ: $\Omega \to E'$ defined everywhere on Ω, such that for each f in E, the complex valued function $\omega \to \langle f, \Phi(x)\rangle$ is continuous and $\hat{T}f(\omega) = \langle f, \Phi(\omega)\rangle$ a. e.*

Proof. The hypotheses implies that $\hat{T}f$ is in $L^\infty(\Omega, \hat{\mu})$ for all f in E, and that $\|\hat{T}f\|_\infty \leqslant \|f\|$. Let $\hat{T}_1$: $E \to L^\infty(\Omega, \hat{\mu})$ be defined by $\hat{T}_1 f = Tf$. Then $\|\hat{T}_1\| \leqslant 1$. For each ω in Ω, let δ_ω: $C(\Omega) \to C$ be the evaluation functional. We know that the formal identity mapping I: $C(\Omega) \to L^\infty(\Omega, \hat{\mu})$ is a linear isometry of $C(\Omega)$ onto $L^\infty(\Omega, \hat{\mu})$. Then $\Phi(\omega) = \delta_\omega I^{-1}\hat{T}_1$ is in E', and $\|\Phi(\omega)\| \leqslant 1$ for ω in Ω. Let g be in $C(\Omega)$ such that $g = I^{-1}\hat{T}_1 f$. Then for any measure set $\hat{A}$ in Ω

$$\int_{\hat{A}} \hat{T}f(\omega)\,d\hat{\mu} = \int_{\hat{A}} \hat{T}_1 f(\omega)\,d\hat{\mu} = \int_{\hat{A}} g(\omega)\,d\hat{\mu} = \int_{\hat{A}} \langle f, \Phi(\omega)\rangle\,d\hat{\mu}.$$

Therefore $\hat{T}f(\omega) = \langle f, \Phi(\omega)\rangle$ a. e. The proof is complete.

Proof of Theorem 1. As we have pointed out earlier that we only need to prove the implication (b) $\Rightarrow$ (a). In view of Lemma 1 and the remark at the beginning of § 3, we may assume that (X, μ) is a finite measure space and $\gamma = 1$ a. e. Let $(\Omega, \hat{\mu})$ be the measure space obtained in § 2. We define $\hat{T}\colon E \to L^p(\Omega, \hat{\mu})$ by $\hat{T} = \tau_p T$ where τ_p is the linear isometry of $L^p(X, \mu)$ onto $L^p(\Omega, \hat{\mu})$ as in Proposition 2. By Lemma 2, $|\hat{T}f(\omega)| \leqslant \|f\|$ a. e. By Lemma 3, $\hat{T}f(\omega) = \langle f, \Phi(\omega)\rangle$ a. e. for a E'-valued function $\Phi\colon \Omega \to E'$ defined everywhere on Ω, and the complex valued function $\omega \to \langle f, \Phi(\omega)\rangle$ is continuous for each f in E. Now, we suppose that E' is separable. We claim that Φ is strongly measurable. We first show that Φ is weakly measurable. That is, the function $\omega \to \langle f'', \Phi(\omega)\rangle$ is measurable for each f'' in E''. Indeed, for each f'' in the unit ball of E'', there exists a sequence $\{f_n\}_{n=1}^{\infty}$ in the unit ball of E such that $\langle f_n, f'\rangle \to \langle f'', f'\rangle$ for all f' in E'. Hence the sequence of continuous functions $\{\langle f_n, \Phi(\cdot)\rangle\}_{n=1}^{\infty}$ converges pointwise to the function $\langle f'', \Phi(\cdot)\rangle$. Hence $\langle f'', \Phi(\cdot)\rangle$ is measurable for all f'' in E''. Using the separability of E' again, we conclude that Φ is strongly measurable. Let $\hat{K} = \Phi$. We have $\hat{T}f(\omega) = \langle f, \hat{K}(\omega)\rangle$ a. e. and $\hat{K}$ is in $L^p(\Omega, \hat{\mu}; E')$. In case that E is reflexive, then the function Φ is weakly measurable. It follows from a theorem of Grothendieck ([3], Theorem 5, p. 104) that there exists a strongly measurable function $\hat{K}\colon \Omega \to E'$ such that $\langle f, \hat{K}(\omega)\rangle = \langle f, \Phi(\omega)\rangle$ a. e., where the exceptional set of measure zero may depend upon f. $\hat{K}$ is essentially bounded (that is the function $\omega \to \|\hat{K}(\omega)\|$ is essentially bounded). Indeed, since $\hat{K}$ is strongly $\hat{\mu}$-measurable, there are compact subsets Σ_j of Ω, such that $\hat{\mu}(\Omega \setminus \bigcup_{j=1}^{\infty} \Sigma_j) = 0$ and the restriction of $\hat{K}$ to each Σ_j is continuous. For each f in E, there is a $\hat{\mu}$-null set N_f such that $\langle f, \hat{K}(\omega)\rangle = \langle f, \Phi(\omega)\rangle$ for all ω not in N_f. In particular $\langle f, \hat{K}(\omega)\rangle = \langle f, \Phi(\omega)\rangle$ for all ω in $\Sigma_j \cap N_f^c$. The continuity of $\langle f, \hat{K}(\cdot)\rangle$ and $\langle f, \Phi(\cdot)\rangle$ on each Σ_j implies that $\langle f, \hat{K}(\omega)\rangle = \langle f, \Phi(\omega)\rangle$ for all ω in Σ_j. Therefore $\langle f, \hat{K}(\omega)\rangle = \langle f, \Phi(\omega)\rangle$ for all ω in $\bigcup_{j=1}^{\infty} \Sigma_j$. Hence $\hat{K}(\bigcup_{j=1}^{\infty} \Sigma_j)$ is a weakly bounded subset of E', hence is a strongly bounded subset of E', and $\|\hat{K}(\omega)\| \leqslant 1$ for all ω in $\bigcup_{j=1}^{\infty} \Sigma_j$ with $\hat{\mu}(\Omega \setminus \bigcup_{j=1}^{\infty} \Sigma_j) = 0$. This proves the essentially boundedness of $\hat{K}$. So $\hat{K}$ is in $L^p(\Omega, \hat{\mu}; E')$, and $\hat{T}f(\omega) = \langle f, \hat{K}(\omega)\rangle$ a. e. We summarize what have been already established as follows: If E is reflexive or E' is separable, then $\hat{T}f(\omega) = \langle f, \hat{K}(\omega)\rangle$ a. e. for some $\hat{K}$ in $L^p(\Omega, \hat{\mu}; E')$. By Lemma 2 again, we have $Tf(x) = \langle f, K(x)\rangle$ for some K in $L^p(X, \hat{\mu}; E')$. The proof of Theorem 1 is complete, as the uniqueness of K has been proved in the remark of § 1.

For theory of absolutely p-summing operators and operators of type N_p and type N^p, the reader is referred to [4] and [5].

Corollary 1 (Shields–Wallen–Williams [7]). *Let E be a normed linear space. If the linear operator $T\colon E \to L^p(X, \mu)$, $1 \leqslant p < \infty$, is such that $|Tf(x)| \leqslant \gamma(x)\|f\|$ a. e., for some γ in $L^p(X, \mu)$, then T is absolutely p-summing.*

Proof. Case 1. $\gamma > 0$ a. e. Consider the operator $\hat{T}\colon E \to L^p(\Omega, \hat{\nu})$, where $\nu = \gamma^p \mu$ and $\hat{T}$ factors as $E \xrightarrow{T} L^p(X, \mu) \xrightarrow{\varphi^{-1}} L^p(X, \nu) \xrightarrow{\tau_p} L^p(\Omega, \hat{\nu})$. Recall that φ is the isometry of $L^p(X, \nu)$ onto $L^p(X, \mu)$, $f \to \gamma f$ and τ_p is the extension of the Gelfand isomorphism of $L^\infty(X, \nu)$ onto $C(\Omega)$. (See Lemma 1 and Proposition 1.) Then, by Lemma 3 $\hat{T}f(\omega) = \langle f, \Phi(\omega)\rangle$ a. e. with $\|\Phi(\omega)\| \leqslant 1$ for all ω in Ω. If $\{f_j\}_{j=1}^{n}$ is any finite sequence in E, then

$$\sum_{j=1}^{n} |\hat{T}f(\omega)|^p = \sum_{j=1}^{n} |\langle f_j, \Phi(\omega)\rangle|^p \text{ a. e.}$$

But

$$\sum_{j=1}^{n} |\langle f_j, \Phi(\omega)\rangle|^p \leqslant \sup_{\|f'\| \leqslant 1} \sum_{j=1}^{n} |\langle f_j, f'\rangle|^p$$

for all ω. Integrating, we have

$$\sum_{j=1}^{n} \|Tf_j\|^p \leqslant \hat{\nu}(\Omega) \sup_{\|f'\| \leqslant 1} \Big\{ \sum_{j=1}^{n} |\langle f_j, f'\rangle|^p \Big\}.$$

Hence, $\hat{T}$ is absolutely p-summing. It follows that $T = \varphi\tau_p^{-1}\hat{T}$ is also absolutely p-summing.

Case 2. General case. If the set $A = \{x \in X;\ \gamma(x) = 0\}$ has positive measure then T factors as

$$E \xrightarrow{T_1} L^p(A^c, \mu|_{A^c}) \xrightarrow{i} L^p(X, \mu),$$

where $T_1 f = Tf$ and i is the natural embedding, A^c is the complement of A. Then $|T_1 f(x)| \leqslant \gamma(x)\|f\|$ a. e., $\gamma > 0$ on A^c and γ in $L^p(A^c, \mu|_{A^c})$. Then by Case 1 T_1 is absolutely p-summing. Hence $T = i \cdot T_1$ is also absolutely p-summing. This completes the proof.

Note. In the proof of Case 1 $\hat{T}$ admits a factorization $\hat{T} = j\hat{T}_1$

$$E \xrightarrow{\hat{T}_1} L^\infty(\Omega, \hat{\nu}) \xrightarrow{j} L^p(\Omega, \hat{\nu})$$

where j is the natural injection which is absolutely p-summing. In fact, it is even p-integral [5]. Therefore $\hat{T}$ is absolutely p-summing. This gives an alternative proof of Corollary 1. The original proof of Shields–Wallen–Williams Theorem is very elementary.

The following corollary follows immediately from Theorem 1 and Theorem 2 and [4] (Theorem 1 and 2).

COROLLARY 2. *Let E be the Banach space such that either E is reflexive or E' is separable. For operator $T\colon E \to L^p(X, \mu)$, $1 \leqslant p < \infty$, such that $|Tf(x)| \leqslant \gamma(x)\|f\|$ a. e. with some γ in $L^p(X, \mu)$. Then T is an operator of type N_p, and its adjoint T^* is of type N^p.*

COROLLARY 3. *Let (Y, ν) and (X, μ) be σ-finite measure spaces. For operators $T\colon L^q(Y, \nu) \to L^p(X, \mu)$, $1 \leqslant p, q < \infty$, the following conditions are equivalent:*

(i) *$|Tf(x)| \leqslant \gamma(x)\|f\|$ a. e. (μ) for some γ in $L^p(X, \mu)$.*

(ii) *$Tf(x) = \int_Y K(x, y) f(y)\, d\nu(y)$ a. e. (μ) for a unique $\mu\times\nu$-measurable scalar function $K(x, y)$ such that*

$$\int_X \Big\{ \int_Y |K(x, y)|^{q'} d\nu(y)\Big\}^{p/q'} d\mu(x) < +\infty.$$

(In case $q = 1$, the condition on the kernel $K(x, y)$ is modified in an obvious way.)

Proof. The implication (ii) $\Rightarrow$ (i) is clear. We now prove that condition (i) implies (ii). Without loss of generality we may assume that (X, μ) is finite and $\gamma = 1$ a. e. By Theorem 1, there is a unique K in $L^p\big(X, \mu; L^{q'}(Y, \nu)\big)$ such that $Tf(x) = \langle f, K(x)\rangle$ a. e. (μ). Because (X, μ) is finite measure space, K is also a Bochner integrable function. Since (Y, ν) and (X, μ) are σ-finite, there exists a unique (up to set of measure zero) $\mu\times\nu$-measurable K such that $\overline{K(x, \cdot)} = K(x)$ a. e. (μ) ([1], Theorem 17, p. 198). Therefore

$$Tf(x) = \int_Y K(x, y) f(y)\, d\nu(y) \quad \text{a. e. } (\mu),$$

and

$$\int_X \Big\{\int_Y |K(x, y)|^{q'} d\nu(y)\Big\}^{p/q'} d\mu(x)$$

is finite. This completes the proof.

Remark. For $1 < p < +\infty$, $q = p'$, Corollary 3 gives the equivalence of conditions (c) and (d) of Persson's Theorem ([4], Theorem 3). In case $p = q = 2$, it gives a characterization of Hilbert–Schmidt operators in the L^2-spaces.

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DEPARTMENT OF MATHEMATICS
WAYNE STATE UNIVERSITY, DETROIT, MICHIGAN

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