§ 1. Introduction. In this note we use the conventions and notation of A. Peress [4]. A normed linear space is denoted by $E$, and $E'$ is its topological dual with the strong dual topology. We use $(f, g)$ to indicate the action of a vector $f$ in $E$ and a functional $g$ in $E'$. For $1 \leq p < \infty$, $p$ and $p'$ are the usual conjugate numbers. $L^p(X, \mu; E')$ is the Banach space of equivalent classes of strongly $p$-measurable $E'$-valued functions $K$ such that $\int |K(x)|^p d\mu < \infty$. In case $E'$ is $C$ the complex numbers, we simply write $L^p(X, \mu)$ instead of $L^p(X, \mu; C)$. All the measures in this note are countably additive and positive, and all the operators are bounded.

We aim to investigate a class of linear operators $T: E \to L^p(X, \mu)$ and their adjoint $T': L^p(X, \mu) \to E'$ such that

$$|Tf(x)| \leq \gamma(x) ||f||$$

for some $\gamma$ in $L^p(X, \mu)$. When $E$ is a reflexive Banach space or $E'$ is separable, it turns out that each of such $T$ can be represented by a unique $K$ in $L^p(X, \mu; E')$. In the following way:

$$Tf(x) = \langle f, K(x) \rangle \text{ a.e. and } T^*g = \int g(x) K(x) d\mu$$

for $f$ in $E$, $g$ in $L^p(X, \mu)$ where the integral is taken to be the Bochner integral. In this case, $T$ is an operator of type $-N_p$ and $T^*$ is of type $-N_p$ ([4], Theorem 1 and 2). They are all completely continuous operators.

Our result is quite similar to Dunford–Pettis Theorem in which they obtained a representation for any operators $T: E \to L^p(X, \mu)$ and $S: L^p(X, \mu) \to E'$ by a bounded weakly measurable $E'$-valued function under the hypotheses that $E$ is a separable Banach space (see [5], p. 499).

As a consequence of our result, we obtain an alternative proof for a theorem of Shields–Waller–Williams which asserts that if operator $T: E \to L^p(X, \mu)$ satisfies $|Tf(x)| \leq \gamma(x) ||f||$ for some $\gamma$ in $L^p(X, \mu)$, then $T$ is absolutely $p$-summing. In case $E = L^p(Y, \nu)$, our result also includes
some work of A. Peress (14), Theorem 3) which gives a characterization of Hilbert–Schmidt operators on $L^2$-spaces.

**Theorem 1.** Let $E$ be a Banach space which is either reflexive or its dual $E'$ is separable. For linear operators $T: E \to L^p(X, \mu)$, $1 < p < \infty$, the following conditions are equivalent:

(a) There is a unique $K$ in $L^p(X, \mu; E')$ such that

$$Tf(x) = \langle f, K(x) \rangle$$

for all $f$ in $L^p(X, \mu)$.

(b) $|Tf(x)| \leq \gamma(x)\|f\|$ a.e. for some $\gamma$ in $L^p(X, \mu)$.

Note. The exceptional set of measure zero may depend upon $f$.

**Theorem 2.** Let $E$ be as in Theorem 1. For linear operators $S: L^p(X, \mu) \to E', \frac{1}{p} + \frac{1}{q'} = 1, 1 < p < \infty$, the following conditions are equivalent:

(a) There is a unique $K$ in $L^p(X, \mu; E')$ such that

$$Sc = \int_X g(x)K(x)\,d\mu$$

for $g$ in $L^p(X, \mu)$.

(b) $S$ is the adjoint of a linear operator $T: E \to L^p(X, \mu)$ such that

$$|Tf(x)| \leq \gamma(x)\|f\|$$

for some $\gamma$ in $L^p(X, \mu)$.

Note. The integral in condition (a) is Bochner integral.

Remark. It is not difficult to see the equivalence of Theorem 1 and Theorem 2. We shall only prove Theorem 2. Moreover, if there are $K$ and $K_1$ in $L^p(X, \mu; E')$ such that $\langle f, K(x) \rangle = \langle f, K_1(x) \rangle$ a.e. for each $f$, where the exceptional $\mu$-null set depends on $f$. Then if $g$ in $L^p(X, \mu)$, we have

$$\left\langle \int g(x)K(x)\,d\mu, f \right\rangle = \int \left\langle g(x), K(x) \right\rangle \,d\mu = \int \left\langle g(x), K_1(x) \right\rangle \,d\mu = \left\langle \int g(x)K_1(x)\,d\mu, f \right\rangle$$

for all $f$ in $E$, therefore

$$\int g(x)K(x)\,d\mu = \int g(x)K_1(x)\,d\mu$$

for all $g$ in $L^p(X, \mu)$. In particular

$$\int (K(x) - K_1(x))\,d\mu = 0$$

for all $\mu$-measurable set $A$ with $\mu(A) < \infty$. This proves the uniqueness assertion in Theorem 1 and Theorem 2.

### § 2. Some measure-theoretic preparation

Let $(X, \mu)$ be a finite measure space. Let $\tau: L^p(X, \mu) \to L^p(\Omega, \mu)$ be the Gelfand isomorphism $f \to f$, where $\Omega$ is the Gelfand space of $L^p(X, \mu)$. Let $\mu$ be the Radon measure on $\Omega$ such that $\int_X fd\mu = \int f\,d\mu$ for all $f$ in $L^p(X, \mu)$.

**Proposition 1.** For each $1 < p < \infty$, there exists a linear isometry $\tau_p$ of $L^p(X, \mu)$ onto $L^p(\Omega, \mu)$ that extends the Gelfand isomorphism $\tau$. Moreover, this family of isometries $(\tau_p; 1 < p < \infty)$ has the following properties:

(i) For $1 < p < q < \infty$, $\tau_p$ extends $\tau_q$.

(ii) If $f$, $g$ and $fg$ are all in $L^p(X, \mu)$, then

$$\tau_p(fg) = \tau_p(f)\tau_p(g)$$

and $\tau_p(f) = \tau_p(f)$

where $f$ is the complex conjugate of $f$. Hence $\tau_p(\overline{f}) = \tau_p(f)$.

(iii) $\int_X f\,d\mu = \int \tau_p(f)\,d\mu$ for $f$ in $L^p(X, \mu)$. Hence $\int f\,d\mu = \int \tau_p(f)\,d\mu$ where $A$ is the compact open subset of $\Omega$ such that $\chi_A = \chi_A$ and $\chi_A$ is the characteristic function of $A$.

Proposition 1 is well known (cf. [2] p. 18) for the existence of $\tau_p$. The properties (i)–(iii) requires only routine checking.

The following proposition plays a key role in the proof of our theorems.

**Proposition 2.** Notations as above, let $E$ be a Banach space. For $1 < p < \infty$, there exists a linear isometry $\sigma_p$ of $L^p(\Omega, \mu; E)$ onto $L^p(X, \mu; E)$ such that

$$\int \sigma_p(K)(x)\,d\mu = \int K(x)\,d\mu$$

for $K$ in $L^p(\Omega, \mu; E)$ and $A \mu$-measurable set with $\chi_A \in L^p$.

Proof. We know that the formal identity $I: L^p(\Omega, \mu; E) \to L^p(\Omega, \mu)$ is a linear isometry of $L^p(\Omega, \mu; E)$ (cf. [2], p. 18). Thus $I\circ I: L^p(X, \mu; E)$ is an isometric isomorphism. Since $\chi_A = \chi_B$ in $L^p(X, \mu)$ if and only if $A$ and $B$ represent the same element in the measure ring of $(X, \mu)$. This gives rise to an isomorphism between the measure rings of $(X, \mu)$ and $(\Omega, \mu)$. For any step function $f$, $\int f\,d\mu = \sum a_i\chi_{A_i}$, we may regard $\chi_{A_i}$ as an element in the measure ring of $(\Omega, \mu)$ with the compact open set $A_i$ as a representative. Let

$$\sigma_p K = \sum a_i \chi_{A_i}f_i$$

where $\chi_{A_i} = \chi_{A_i}$.
Then \( c_p K \) is a step function in \( L^p(X, \mu; E) \) and \([c_p K] = [K]\). Because of the measure ring isomorphism, we have

\[
c_p(K_1 + K_2) = c_p(K_1) + c_p(K_2)
\]

for \( K_1 \) and \( K_2 \) step-functions. Thus \( c_p \) is a linear isometry on the dense linear manifold of the step-functions in \( L^p(\Omega, \mu; \hat{E}) \) into \( L^p(X, \mu; E) \). Therefore \( c_p \) can be extended to a linear isometry of \( L^p(\Omega, \mu; \hat{E}) \) into \( L^p(X, \mu; E) \). Since the range is dense, hence \( c_p \) maps \( L^p(\Omega, \mu; \hat{E}) \) onto \( L^p(X, \mu; E) \).

For \( K = \sum_{i=1}^n \lambda_i f_i \) a step-function, and \( \mathcal{A} \)-\( \mu \)-measurable set, we see that

\[
\int_{\mathcal{A}} K(\omega) d\mu = \int_{\mathcal{A}} c_pK(\omega) d\mu.
\]

Since \( \int_{\mathcal{A}} d\mu : L^p(\Omega, \mu; \hat{E}) \to E \) is a bounded operator, and since the step-functions are dense in \( L^p(\Omega, \mu; \hat{E}) \), then

\[
\int_{\mathcal{A}} K(\omega) d\mu = \int_{\mathcal{A}} c_pK(\omega) d\mu
\]

for all \( K \) in \( L^p(\Omega, \mu; \hat{E}) \). The proof is complete.

**§ 3.** The proof of Theorem 1 hinges on the implication (b) \( \Rightarrow \) (a), because condition (a) implies (b) clearly. We may assume that the set \( A = \{ \omega : \gamma(\omega) = 0 \} \) has measure zero. If \( \mu(A) \) were positive, then we can write \( L^p(X, \mu; E) \) as \( L^p(A, \mu|_A; E) \oplus L^p(\Omega \setminus A, \mu|_{\Omega \setminus A}; E) \) for any Borel space \( \Omega \), since the multiplication by \( \lambda \) is a projection (idempotent) on \( L^p(X, \mu; E) \). We use \( A^c \) to denote the complement of \( A \) in \( X \), and \( \mu|_A \) is the measure \( \mu \) restricted to \( A \). The condition \( ||\gamma|| = \gamma(\omega) ||f|| \) a.e. implies that \( T(f) = \gamma(\omega) ||f|| \) a.e. is contained in \( L^p(A^c, \mu|_{A^c}) \). Hence \( T \) factors as

\[
\begin{array}{c}
E \to L^p(A^c, \mu|_{A^c}) \to L^p(X, \mu).
\end{array}
\]

Now \( \gamma \) is in \( L^p(A^c, \mu|_{A^c}) \) and \( \gamma > 0 \) on \( A^c \).

If \( T(f) = \langle f, K_\omega(\cdot) \rangle \) for some \( K_\omega \) in \( L^p(\Omega, \mu; \hat{E}) \), using the natural injection \( i : L^p(A^c, \mu|_{A^c}; E) \to L^p(\Omega, \mu; \hat{E}) \) we obtain \( K \) in \( L^p(X, \mu; E) \) such that \( T(f) = \langle f, \gamma(\omega) \rangle \) a.e. Therefore we can and do assume that \( \gamma(\omega) > 0 \) a.e.

**Lemma 1.** Let \( E \) be a normed linear space. Let \( \gamma > 0 \) a.e. be in \( L^p(X, \mu) \). Define the finite measure \( \nu = \gamma^p \cdot \mu \). Let \( \phi : L^p(X, \nu) \to L^p(X, \mu) \) be the linear isometry \( f \to \nu f \). For operators \( T_\omega : E \to L^p(X, \mu) \) and \( T_\omega : E \to L^p(X, \mu) \) such that \( \phi T_\omega = T_\omega \), then

\[
\|T_\omega f\| \leq \gamma(\omega) \|f\| \text{ a.e. (a) if and only if } \|T_\omega f\| \leq \|f\| \text{ a.e. (b)}.
\]

**Proof.**

The hypotheses imply that \( \hat{T}_\omega = \hat{\phi} T \) is a linear operator such that \( \|\hat{T}_\omega f\| \leq \|f\| \) a.e. then there exists a bounded linear operator \( \Phi : L^p(\Omega, \mu; \hat{E}) \to L^p(\Omega, \mu; \hat{E}) \) such that \( \hat{T}_\omega f = \phi(T_\omega f) \) a.e. Therefore we can use the same argument as in Proposition 1 and 2. By a similar argument one proves the converse. This concludes the proof of Lemma 2.

**Lemma 2.** Let \( E \) be a normed linear space and let \( \{\Omega, \mu\} \) be the measure space obtained in \( \mathfrak{I} \). If \( \hat{T}_\omega : E \to L^p(\Omega, \mu; \hat{E}) \) is a linear operator such that \( \|\hat{T}_\omega f\| \leq \|f\| \) a.e. then there exists a bounded linear operator \( \Phi : L^p(\Omega, \mu; \hat{E}) \to L^p(\Omega, \mu; \hat{E}) \) such that \( \hat{T}_\omega f = \phi(T_\omega f) \) a.e. Therefore we can use the same argument as in Proposition 1 and 2. By a similar argument one proves the converse. This concludes the proof of Lemma 2.
in $E'$, and $||\Phi(a)|| \leq 1$ for $a$ in $\Omega$. Let $g$ be in $C(\Omega)$ such that $g = I^{-1}\hat{T}f$. Then for any measure set $A$ in $\Omega$

$$\int_A \hat{T}f(a)\,d\hat{\mu} = \int_A \hat{T}g(a)\,d\hat{\mu} = \int_A \langle \hat{\Phi}, \hat{\omega} \rangle \,d\hat{\mu}.$$ 

Therefore $\hat{T}f(a) = \langle \hat{f}, \hat{\Phi}(\omega) \rangle$ a.e. The proof is complete.

**Proof of Theorem 1.** As we have pointed out earlier that we only need to prove the implication (b) $\Rightarrow$ (a). In view of Lemma 1 and the remark at the beginning of §3, we may assume that $(X, \mu)$ is a finite measure space and $r = 1$ a.e. Let $(X, \mu)$ be the measure space obtained in §2. We define $\bar{T}: E \to L^p(\Omega, \mu)$ by $T = \tau_\omega T$ where $\tau_\omega$ is the linear isometry of $L^p(X, \mu)$ onto $L^p(\Omega, \mu)$ as in Proposition 2. By Lemma 2, $||\bar{T}f|| \leq ||f||$ a.e. By Lemma 3, $\bar{T}f = \langle f, \Phi(\omega) \rangle$ a.e. for a $f'$-valued function $\Phi: \Omega \to F'$ defined everywhere on $\Omega$, and the complex valued function $f = \langle f, \Phi(\omega) \rangle$ is continuous for each $f$ in $E$. Now, we suppose that $E'$ is separable. We claim that $\Phi$ is strongly measurable. We first show that $\Phi$ is weakly measurable. That is, the function $\omega \to \langle f', \Phi(\omega) \rangle$ is measurable for each $f'$ in $E'$. Indeed, for each $f'$ in the unit ball of $E'$, there exists a sequence $\langle f_n, f' \rangle_n \in E'$ converging pointwise to the function $\langle f', \Phi(\omega) \rangle$. Hence $\langle f', \Phi(\omega) \rangle$ is measurable for all $f'$ in $E'$. Using the separability of $E'$ again, we conclude that $\Phi$ is strongly measurable. Let $K = \Phi$. We have $\bar{T}f(a) = \langle f, \hat{K}(\omega) \rangle$ a.e. and $\hat{K}$ is in $L^p(\Omega, \mu; L^p(\Omega, \mu))$. In case $\bar{T}$ is reflexive, then the function $\Phi$ is weakly measurable. It follows from a theorem of Grothendieck ([3, Theorem 5, p. 104]) that there exists a strongly measurable function $\hat{K}: \Omega \to F'$ such that $\langle f, \hat{K}(\omega) \rangle = \langle f, \Phi(\omega) \rangle$ a.e., where the exceptional set of measure zero may depend upon $f$. $\hat{K}$ is essentially bounded (that is the function $\omega \to ||\hat{K}(\omega)||$ is essentially bounded). Indeed, since $\hat{K}$ is strongly $\hat{\mu}$-measurable, there are compact subsets $\Sigma_i$ of $\Omega$, such that $\hat{\mu}(\hat{\Omega} \setminus \bigcup_i \Sigma_i) = 0$ and the restriction of $\hat{K}$ to each $\Sigma_i$ is continuous. For each $f$ in $E$, there is a null set $N_f$ such that $\langle f, \hat{K}(\omega) \rangle = \langle f, \Phi(\omega) \rangle$ for all $\omega$ not in $N_f$. In particular, $\langle f, \hat{K}(\omega) \rangle = \langle f, \Phi(\omega) \rangle$ for all $\omega$ in $\bigcup_i \Sigma_i \setminus N_f$. The continuity of $\langle f, \hat{K}(\omega) \rangle$ and $\langle f, \Phi(\omega) \rangle$ on each $\Sigma_i$ implies that $\langle f, \hat{K}(\omega) \rangle = \langle f, \Phi(\omega) \rangle$ for all $\omega$ in $\bigcup_i \Sigma_i$. Therefore $\langle f, \hat{K}(\omega) \rangle = \langle f, \Phi(\omega) \rangle$ for all $\omega$ in $\bigcup_i \Sigma_i$. Hence $\hat{K}(\omega) \in L^p(\Omega, \mu)$. It is a weakly bounded subset of $E'$; hence, $\hat{K}$ is strongly bounded subset of $E'$, and $||\hat{K}(\omega)|| \leq 1$ for all $\omega$ in $\bigcup_i \Sigma_i$. This proves the essentialy bound-
where \( j \) is the natural injection which is absolutely \( p \)-summing. In fact, it is even \( p \)-integral [5]. Therefore \( T \) is \( p \)-summing. This gives an alternative proof of Corollary 1. The original proof of Shields–Wallen-Williams Theorem is very elementary.

The following corollary follows immediately from Theorem 1 and Theorem 2 and 4 (Theorem 1 and 2).

**Corollary 2.** Let \( E \) be the Banach space such that either \( E \) is reflexive or \( E' \) is separable. For operator \( T : E \to L^p(X, \mu), 1 \leq p < \infty \), such that \( |Tf(x)| \leq \gamma(x)||f|| \) a.e. with some \( \gamma \) in \( L^p(X, \mu) \). Then \( T \) is an operator of type \( N_p \) and its adjoint \( T' \) is of type \( N_q' \).

**Corollary 3.** Let \((X, \mu)\) and \((X', \mu')\) be \( \sigma \)-finite measure spaces. For operator \( T : L^p(X, \nu) \to L^q(X', \mu), 1 \leq p < \infty, 1 \leq q < \infty \), the following conditions are equivalent:

(i) \( \left| \frac{f(x)}{\gamma(x)} \right| \leq \frac{\gamma(x)}{\gamma(y)} \) a.e. \((x)\) for some \( \gamma \) in \( L^p(X, \mu) \).

(ii) \( T \mu = \int \left\{ \int f(x)\gamma(y) \lambda (y) \right\} \, d\mu(x) \) a.e. \((\mu)\) for a unique \( \mu \times \nu \)-measurable scalar function \( K(x, y) \) such that

\[
\int_X \left( \int_{X'} |K(x, y)|^q \lambda (y) \right)^{\frac{1}{q}} \, d\mu(x) < +\infty.
\]

(In case \( q = 1 \), the condition on the kernel \( K(x, y) \) is modified in an obvious way.)

**Proof.** The implication (ii) \( \Rightarrow \) (i) is clear. We now prove that condition (i) implies (ii). Without loss of generality, we may assume that \((X, \mu)\) is finite and \( \gamma = 1 \) a.e. By Theorem 1, there is a unique \( K \) in \( L^p(X, \mu; L^q(X', \nu)) \) such that \( \int_1 \mu' = (\int_1 K(x) \, d\mu) \) a.e. \((\mu)\). Because \((X', \mu')\) is a finite measure space, \( K \) is a Bochner integrable function. Since \((X, \mu)\) and \((X', \mu')\) are \( \sigma \)-finite, there exists a unique (up to the set of measure zero) \( \mu \times \nu \)-measurable \( K \) such that \( K(x, \cdot) = K(x) \) a.e. \((\mu)\) [1], Theorem 17, p. 196). Therefore

\[
T \mu = \int \left\{ \int f(x)K(x, y) \lambda (y) \right\} \, d\mu(x),
\]

and

\[
\int_X \left( \int_{X'} |K(x, y)|^q \lambda (y) \right)^{\frac{1}{q}} \, d\mu(x)
\]

is finite. This completes the proof.

**Remark.** For \( 1 < p < +\infty, q = p' \), Corollary 3 gives the equivalence of conditions (c) and (d) of Persson's Theorem ([4], Theorem 3). In case \( p = q = 2 \), it gives a characterization of Hilbert–Schmidt operators in the \( L^2 \)-spaces.

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**References**


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