Addendum. In a paper soon to appear in Math. Annalen, Professor S. Saxon has introduced the notion of a Baire-like locally convex space; a locally convex space E is said to be Baire-like whenever E cannot be written as the union of an increasing sequence of nowhere dense, closed, balanced, convex sets. Theorem 2.10 of the paper (entitled "Product spaces, Baire-like spaces and the strongest locally convex topology") states that every countably co-dimensional subspace of a Baire-like space is Baire-like. Our Theorem 1 therefore can be strengthened to: *$V^{\Phi_1}(\mathfrak{X})$ is a meager, uncountably codimensional linear subspace of $V^{\Phi_2}(\mathfrak{X})$*. Indeed, the S_k's are an increasing sequence of closed, convex, balanced, nowhere dense sets, $\bigcup_k S_k = V^{\Phi_1}(\mathfrak{X})$.

References

[1] S. Banach, *Theorie des operations lineaires*, New York 1932.

[2] G. Birkhoff, *Lattice Theory*, Am. Math Soc. Coll. Pub., 3rd ed., New York 1967.

[3] J. Diestel, *Some remarks on subspaces of Orlicz spaces of Lebesgue-Bochner measurable functions*, to appear.

[4] A. Grothendieck, *Sur les applications lineaires faiblement compactes d'espaces du type $C(K)$*, Canad. J. Math. 5 (1953), pp. 129–173.

[5] — *Sur certains sous-espaces vectoriels de L^p*, Canad. J. Math. 6 (1953), pp. 158–160.

[6] S. Kakutani, *Concrete representation of abstract (L)-spaces and the mean ergodic theorem*, Annals of Math. 42 (1941), pp. 523–537.

[7] — *Concrete representation of abstract (M)-spaces*, Annals of Math. 42 (1941), pp. 994–1024.

[8] S. Leader, *The theory of L^p-spaces for finitely additive set functions*, Annals of Math. 58 (1953), pp. 528–543.

[9] J. C. Oxtoby, *The category and Borel class of certain subsets of L_p*, Bull. Amer. Math, Soc. (1937), pp. 245–248.

[10] J. J. Uhl, *Orlicz spaces of finitely additive set functions*, Studia Math. 29 (1967), pp. 19–58.

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The chi function in generalized summability

by

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1. INTRODUCTION

In 1949 Wilansky, [11], introduced the conull and coregular classification of scalar summability matrices by the use of the chi functional. Yurimyae in [12] and Snyder in [8] and [9] showed that these properties can be characterized without the use of matrices.

Other authors, see [1], [4], [5], [6], and [7], have considered the topic of generalized summability and, in particular, have obtained analogues of the Silverman–Toeplitz and the Kojima–Schur conditions.

In this paper, we extend the concept of the chi function to the generalized situation, in a certain setting, and obtain an analogue of Snyder's result, Theorem 1, p. 378 of [9]. We also show that some of the usual summability methods utilizing the chi function carry over to this new setting.

2. *FK*-SPACES

Let F be a Fréchet space, i. e., a locally convex complete linear metric space. Recall, p. 217 of [10], the topology of E may be generated by a sequence of continuous seminorms, $\{p_j\}$. We shall use the following notation:

$E(s)$ is the space of all sequences in E with pointwise addition and scalar multiplication;

$E(m)$ is that subspace of $E(s)$ consisting of bounded sequences, i. e., $\{x_n\}$ is in $E(s)$ and $\{x_n | \ n \in \omega\}$ is a bounded subset of E;

$E(c)$ is that subspace of $E(m)$ consisting of convergent sequences;

$E(c_0)$ is that subspace of $E(c)$ consisting of sequences convergent to zero.

If X is any one of the above spaces, let $C_n\colon X \to E$ be defined by $C_n(x) = x_n$.

2.1. PROPOSITION. *$E(s)$ is a Fréchet space with seminorms*

$$\{P_{in} \mid i \in \omega \text{ and } n \in \omega\} \quad \text{where} \quad P_{in}(x) = P_i \circ C_n(x) = p_i(x_n).$$

Moreover, each C_k is continuous.

Proof. It is clear that $\{P_{in}\}$ gives a total collection of seminorms on $E(s)$ and so $E(s)$ is a linear metric space with the topology so generated. Let $\{x^n\}$ be a Cauchy sequence in $E(s)$. It then follows that for a fixed m, $\{x_m^n\}_{n=1}^{\infty}$ is Cauchy in E. Let $x_m = \lim_n x_m^n$ and let $x = \{x_m\}$. It is now easy to see that $\{x^n\}$ converges to x in $E(s)$.

Now let $x^n \to 0$ in $E(s)$. So for fixed i and k $P_{ik}(x^n) = p_i(x_k^n)$ converges to zero in E, i. e., $C_k(x^n)$ converges to zero.

We shall call any subspace of $E(s)$ that possesses a Fréchet topology stronger than that of $E(s)$ an FK-space. Thus the coordinate functions are continuous on any FK-space. Notice also that an FK-space is an FH subspace of $E(s)$ in the sense given by definition one page 202 of [10]. Consequently, smaller FK-spaces have stronger topologies, (Corollary 1 page 203 of the same reference).

2.2 PROPOSITION. *$E(m)$ is an FK-space with seminorms given by $\{P_i\}$ where $P_i(x) = \sup_k p_i(x_k)$.*

Proof. This is clear except perhaps for the completeness. From Theorem 8.6, page 71 of [3], $E(m)$ is complete with the uniform topology. The uniform topology has a subbase for the neighborhoods of zero given by all sets of the form

$$\begin{aligned} N[\omega, \bigcap_{i=1}^{n} (p_i < \varepsilon)] &= \{f \mid f[\omega] \subset \bigcap_{i=1}^{n} (p_i < \varepsilon)\} \\ &= \{x \mid x_k \in \bigcap_{i=1}^{n} (p_i < \varepsilon) \text{ for all } k\} \\ &= \{x \mid \sup_k p_i(x_k) < \varepsilon \text{ for } i = 1, \ldots, n\} \\ &= \{x \mid P_i(x) < \varepsilon \text{ for } i = , \ldots, n\}. \end{aligned}$$

But these sets are a subbase at zero for the seminorm topology. Thus the topologies are identical.

It then follows that $E(c)$ is an FK-space with the same collection of seminorms as above. This results from the fact that $E(c)$ is a closed subspace of $E(m)$. Moreover, we have the following:

2.3. PROPOSITION. $\lim$: *$E(c) \to E$ is continuous.*

Proof. Since $E(c)$ and E are Fréchet spaces, it suffices to show that lim is a bounded function. Let B be a bounded subset of $E(c)$. Then for any fixed i there is a number M such that $P_i(x) \leqslant M$ for all x in B.

Thus

$$p_i(\lim x) = p_i(\lim_k x_k) = \lim_k p_i(x_k) \leqslant P_i(x) \leqslant M$$

for any x in B. Thus $\lim[B]$ is a bounded subset of E.

Notice, then, that $E(c_0)$ is also an FK-space with the same seminorms as $E(c)$ and $E(m)$ since $E(c_0)$ is the null space of lim.

Before introducing the concept of a conservative matrix and its associated summability space, we need some technical results.

2.4. PROPOSITION. *Let X be an FK-subspace of $E(s)$ and let F be a Fréchet space. If u: $X \to F(s)$ is a continuous linear map and if Y is an FK-subspace of $F(s)$, then $u^{-1}[Y]$ is an FK-subspace of $E(s)$ with seminorms $\{P_i\} \cup \{Q_i \circ u\}$ where $\{P_i\}$ gives the X-topology and $\{Q_i\}$ gives the Y-topology. Moreover, u is continuous from $u^{-1}[Y]$ with this topology into Y with the $\{Q_i\}$ topology.*

Proof. Similar to that of Theorem 1, page 226 of [10].

2.5. LEMMA. *Let X be a linear topological space and f a linear map from X into $E(s)$. Then f is continuous if and only if $P_{in} \circ f$ is continuous for each i and each n.*

Proof. Suppose $P_{in} \circ f$ is continuous for each i and each n. Let $x^a \to x$ in X. Then $P_{in}\big(f(x^a) - f(x)\big) \to 0$. Thus $f(x^a) \to f(x)$ in $E(s)$ and f is continuous. The other direction is even more clear.

2.6. COROLLARY. *Let g be a linear map from X, a Fréchet space, into Y, an FK-subspace of $E(s)$. Then g is continuous if and only if $P_{in} \circ g$ is continuous for all i and all n.*

Proof. Use the above lemma and Theorem 1, page 203 of [10].

We are now ready to begin our study of conservative matrices. Unless otherwise mentioned, throughout this paper the term infinite matrix will always mean a matrix $A = (A_{nk})$ where each A_{nk} is a continuous linear map from a Fréchet space E, which will be called the base domain space, into a Fréchet space F, which will be called the base range space. We will use the notation $y = Ax$ as in [6], i. e., y is the sequence given by $y_n = \sum_j A_{nj} x_j$, assuming that the series involved are convergent in F. The next result shows that matrix maps between FK-spaces are always continuous.

2.7. PROPOSITION. *Let A be an infinite matrix which takes X, an $E(s)$ FK-space, into Y, an $F(s)$ FK-space. Then A is continuous from X into Y.*

Proof. For each n and each x in X, let $T_n(x) = \lim_m \sum_{k=1}^{m} = A_{nk}(x_k) = \lim_m \sum_{k=1}^{m} A_{nk} \circ C_k(x)$. Each T_n is thus continuous by the Banach–Steinhaus closure theorem. Now note that $P_{in} \circ A(x) = p_i\big(T_n(x)\big)$ and our proposition follows from 2.6.

For any infinite matrix A, let c_A denote the linear space of all sequences x such that Ax is in $F(c)$ and let d_A denote the linear space of all sequences x such that Ax is defined. We will call A *conservative* if c_A contains $E(c)$. Ramanujan has given necessary and sufficient conditions for A to be conservative, [6]. Using the technique to be found on pages 227 and 228 of [10], one may prove the following:

2.8. Proposition. *Let A be an infinite matrix, then d_A is FK with seminorms*

$$\{P_{in}|\ i \in \omega \text{ and } n \in \omega\} \cup \{R_{in}|\ i \in \omega \text{ and } n \in \omega\},$$

where $R_{in}(x) = \sup_r q_i\big(\sum_{j=1}^{r} A_{nj}x_j\big)$. *$c_A$ is FK with seminorms*

$$\{P_{in}\} \cup \{R_{in}\} \cup \{H_i\},$$

where $H_i(x) = \sup_k q_i\big(\sum_{j=1}^{\infty} A_{kj}x_j\big)$. *In this notation, $\{q_i\}$ is the sequence of seminorms which generates the topology on F, the base range space.*

For each x in $E(s)$, let $U_n(x)$ be the nth section of x, i. e.,

$$U_n(x) = \{x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots\}.$$

Then for each x in $E(c_0)$, $U_n(x) \to x$ in $E(c_0)$ since for any i,

$$P_i\big(x - U_n(x)\big) = \sup_k p_i\Big(C_k\big(x - U_n(x)\big)\Big) = \sup_{k \geqq n+1} p_i(x_k)$$

which is small for n sufficiently large. Of course, it follows immediately that $U_n(x - \dot{l}) + \dot{l} \to x$ for any x in $E(c)$ where $\dot{l}$ is the sequence of constant value $\lim x$. (In the remainder of this paper, $\dot{z}$ will mean the sequence of constant value z.)

The convergence of $U_n(x)$ to x is also valid in d_A, A an infinite matrix. For consider

$$P_{ik}\big(x - U_n(x)\big) = p_i\Big(C_k\big(x - U_n(x)\big)\Big) = 0 \quad \text{for} \quad n \geqslant k$$

and

$$R_{ik}\big(x - U_n(x)\big) = \sup_r q_i\Big(\sum_{j=1}^{r} A_{kj}\Big(C_j\big(x - U_n(x)\big)\Big)\Big) = \sup_r q_i\Big(\sum_{j=n+1}^{r} A_{kj}x_j\Big).$$

Since the series $\sum_{j=1}^{\infty} A_{kj}x_j$ is convergent in F, the above value is small for n sufficiently large. Thus for any x in d_A, $U_n(x) \to x$ in d_A.

Still another class of functions will be useful in the future. These are the insertion functions. For each i and for each x in E, let $I_i(x) = \{0, 0, \ldots, 0, x, 0, \ldots\}$ where x is in the ith place. Then I_i is continuous from E into $E(c_0)$ and thus into any FK-space containing $E(c_0)$. For let $x_n \to x$ in E. Then

$$P_j\big(I_i(x_n) - I_j(x)\big) = \sup_k p_j\Big(C_k\big(I_i(x_n) - I_i(x)\big)\Big) = p_j(x_n - x)$$

and the latter tends to zero as n increases. Actually, I_i is also continuous into d_A, even if A is not conservative.

Let us now turn to the question of representing the continuous linear functionals on these spaces. We shall begin with $E(c_0)$, but first consider the following. If X is any locally convex space with topology generated by the family of seminorms Φ, then a linear functional, f, on X is continuous if and only if there exists a finite subset of Φ, say $p_1, p_2, \ldots, p_n$ and a number M such that

$$|f(x)| \leqslant M \sum_{k=1}^{n} p_k(x)$$

for all x in X, see page 216 of [10]. This may be reworded as f is continuous on X if and only if f is continuous on X under the seminorm $p = \sum_{k=1}^{n} p_k$, which in turn is the case if and only if $\|f\|_p$ is finite, (page 65 of same reference), where $\|f\|_p$ is the supremum of $|f(x)|$ taken over all x such that $p(x) \leqslant 1$.

2.9. Proposition. *Let G be a continuous linear functional on $E(c_0)$. Then there exists a unique sequence $\{g_i\}$ in E^* such that $G(x) = \sum_i g_i(x_i)$ for all x in $E(c_0)$. Conversely, any such sequence for which the series is convergent for all x in $E(c_0)$ defines an element of $E(c_0)^*$. Moreover, there exists a finite collection of seminorms, $p_1, \ldots, p_m$ from the sequence defining the topology of E such that*

$$\frac{1}{m}\sum_i \|g_i\|_p \leqslant \|G\|_P \leqslant \sum_i \|g_i\|_p, \quad \textit{where} \quad p = \sum_{k=1}^{m} p_k \textit{ and } P = \sum_{k=1}^{m} P_k.$$

Proof. Let G be as above and for each n let $g_n = G \circ I_n$. Then for any x in $E(c_0)$ we have

$$G(x) = G\big(\lim_n U_n(x)\big) = \lim_n G\Big(\sum_{k=1}^{n} I_k(x_k)\Big) = \sum_{k=1}^{\infty} g_k(x_k).$$

If also $G(x) = \sum_k h_k(x_k)$, then for any p and any x in E $g_p(x) = G\big(I_p(x)\big) = h_p(x)$. Thus the representation is unique. The converse mentioned above follows from the Banach–Steinhaus closure theorem.

It remains to show the norm condition is satisfied. Since G is continuous, there exist $P_1, \ldots, P_m$, where $P_j(x) = \sup_k p_j(x_k)$, such that G is $P = \sum_1^m P_k$ continuous. For each n, let $T_n(x) = \sum_{i=1}^{n} g_i(x_i)$ for x in $E(c_0)$. Then

$$|T_n(x)| = \Big|\sum_{i=1}^{n} g_i(x_i)\Big| = \big|G\big(U_n(x)\big)\big| \leqslant \|G\|_P P\big(U_n(x)\big) \leqslant \|G\|_P P(x).$$

Thus $\|T_n\|_P \leqslant \|G\|_P$ for all n.

We will now show that $\|T_n\|_P \geqslant \frac{1}{m}\sum_{i=1}^{n}\|g_i\|_p$, where $p = \sum_{i=1}^{m} p_i$, and it will follow that $\frac{1}{m}\sum_{i=1}^{\infty}\|g_i\|_p \leqslant \|G\|_P$. (Note that each g_i is p-continuous. For let $p(x_n - x) \to 0$. Then for any i,

$$P\big(I_i(x_n) - I_i(x)\big) = \sum_{k=1}^{m} p_k(x_n - x) = p(x_n - x) \to 0.$$

Thus

$$g_i(x_n - x) = G\big(I_i(x_n) - I_i(x)\big) \to 0.)$$

For $\varepsilon > 0$ choose $x_1, \ldots, x_n$ in E such that $p(x_i) \leqslant 1$ and $|g_i(x_i)| \geqslant \|g_i\|_p - \varepsilon/n$. Let $\theta_i = \operatorname{signum} g_i(x_i)$, $y_i = \theta_i x_i$, and $y = \{y_1, y_2, \ldots, y_n, 0, 0, \ldots\}$. Then $P(y) \leqslant m$ and

$$|T_n(y)| = \sum_{i=1}^{n} |g_i(x_i)| \geqslant \sum_{i=1}^{n} \|g_i\|_p - \varepsilon.$$

Thus

$$\|T_n\|_P \geqslant \frac{1}{m}\sum_{i=1}^{n}\|g_i\|_p \quad \text{and} \quad \|G\|_P \geqslant \frac{1}{m}\sum_{i}\|g_i\|_p.$$

The other inequality follows readily and our result is established.

Due to the norm condition of the previous proposition, let us call the space of all sequences of E^* such that $\sum_i g_i(x_i)$ is convergent for x in $E(c_0)$ $l_1(E^*)$. So the proposition says that $E(c_0)^*$ is essentially $l_1(E^*)$. In the same way, $E(c)^*$ is $l_1(E^*)$ in the sense for any G in $E(c)^*$ we have

$$G(x) = g_0(\lim x) + \sum_i g_i(x_i - \lim x)$$

which follows from the fact that $x = x - \dot{l} + \dot{l}$ where $\dot{l}$ is the sequence of constant value $\lim x$. Since the series represents a continuous linear functional on $E(c_0)$, the norm condition applies, i. e.,

$$\frac{1}{m}\sum_{i=1}^{\infty}\|g_i\|_p \leqslant \|G|E(c_0)\|_P \leqslant \sum_{i=1}^{\infty}\|g_i\|_p.$$

This in turn shows that for any y in E, $\sum_{i=1}^{\infty} g_i(y)$ is convergent. Thus the formula

$$\chi(g)(y) = g_0(y) - \sum_{i=1}^{\infty} g_i(y)$$

defines a continuous linear functional on E and we may write

$$G(x) = \chi(g)(\lim x) + \sum_{i=1}^{\infty} g_i(x_i).$$

We now turn to a consideration of d_A and c_A. Since $\{U_n(x)\}$ converges to x for any x in d_A, we see that any G in d_A^* is given by

$$G(x) = \sum_i g_i(x_i)$$

where $\{g_i\}$ is a sequence in E^* such that $\sum_i g_i(x_i)$ is convergent for all x in d_A. Moreover, the representation is unique and any such sequence defines an element of d_A^*. In order to study c_A^*, we need the next result.

2.10. Proposition. *Let X be an FK-subspace of $E(s)$ and Y an FK-subspace of $F(s)$. Let $u\colon X \to F(s)$ be a continuous linear map and let f be a continuous linear functional on $u^{-1}[Y]$. Then there is an F in X^* and a $G \in Y^*$ such that $f = F + G \circ u$.*

Proof. Same as that of Theorem 5 page 230 of [10].

2.11. Proposition. *Let A be a conservative matrix. For any f in c_A^*, there exist $\{f_i\}$ and $\{g_i\}$ in E^* and F^* respectively such that $\sum_i f_i(x_i)$ is convergent for all x in d_A,* $\{g_i\}$ is in $l_1(F^*)$ and for all x in c_A

$$f(x) = \sum_i f_i(x_i) + \Big(g_0 - \sum_i g_i\Big)(\lim_A x) + \sum_i g_i(Ax)_i.$$

In this notation, $\lim_A x$ *is the limit of Ax and $(Ax)_i$ is $C_i(Ax)$.*

Proof. Use 2.10 with $X = d_A$ and $Y = F(c)$.

3. THE CHI FUNCTION AND CONULL SPACES

Let A be a conservative matrix where $A\colon E(c) \to F(c)$. Ramanujan has shown, in [6], that the columns of A are pointwise convergent over E. Let us call the continuous linear functions so defined A_k, i. e., for each x in E, $A_k(x) = \lim_n A_{nk}(x)$. He also asserts that for any x in $E(c)$ we have

$$\lim_A x = \lim_A \dot{l} + \sum_k A_k(x_k - l)$$

where $l = \lim x$. This is, of course, tantalizingly close to the usual formula

$$\lim_A x = \chi \lim x + \sum_k a_k x_k,$$

see page 93 of [10], where A is a conservative scalar matrix, x is a convergent sequence of scalars and $\chi = \lim_n \sum_k a_{nk} - \sum_k a_k$. We will now see that, in a certain sense, we can obtain the second formula.

3.1. Lemma. *Let A be a conservative matrix and assume that F, the base range space, is weakly sequentially complete. Then for any x in E, $\sum_k A_k(x)$ is convergent in $w(F, F^*)$, i. e., in F under its weak topology. Moreover, the function so defined is continuous from E into F, F with its metric topology.*

Proof. Fix x in E and f in F^*. Consider the matrix

$$B = \begin{bmatrix} f(A_{11}(x)) & f(A_{12}(x)) & \dots \\ f(A_{21}(x)) & f(A_{22}(x)) & \dots \\ \dots & \dots & \dots \end{bmatrix}.$$

An easy check shows that B is a conservative scalar matrix. Thus for any k, $\lim_n f(A_{nk}(x)) = f(A_k(x))$ exists and $\sum_k f(A_k(x))$ exists. So by our completeness assumption, $\sum_k A_k(x)$ is convergent in $w(F, F^*)$. The continuity of the function so obtained will follow from the next result which is a generalization of Proposition 1.4, page 201 of [1].

3.2. Proposition. *Let $\{U_n\}$ be a sequence of continuous linear operators from a Fréchet space X to a Fréchet space Y and let $U(x) = w-\lim_n U_n(x)$ for every x in X. Then $\{U_n\}$ is equicontinuous and U is continuous.*

Proof. Since $\{U_n\}$ is weakly pointwise convergent, it is pointwise bounded. Thus $\{U_n\}$ is equicontinuous, (18.7 page 171 of [3]). Then $\{U\} \cup \{U_n\}$ is equicontinuous into $w(Y, Y^*)$ by 8.12 of the same reference. In particular, U is continuous into $w(Y, Y^*)$. Thus U takes bounded subsets of X into weakly bounded subsets of Y. Since Y is locally convex, U preserves bounded sets and so U is continuous, (Theorem 4 page 188 of [10]).

The next example, which was suggested by Professor Albert Wilansky, shows that the weak sequential completeness hypothesis in 3.1 may not be omitted.

3.3. Example. Let E and F be c, the Banach space of convergent scalar sequences and let $z^k = (-1)^k \delta^k$ for each positive integer k, where δ^k is the sequence of all zeroes except the kth coordinate, which is one. Then for any m, $\|\sum_{k=1}^m z^k\| = 1$ while $\sum_k z^k$ is not weakly convergent. For the sake of this example only, let us use the following notation

$$P_k \otimes x(y) = y_k x$$

where x and y are in c. Now consider the following matrix:

$$A = \begin{bmatrix} P_1 \otimes z^1 & 0 & 0 & 0 & 0 & \dots \\ P_1 \otimes z^1 & -P_1 \otimes z^1 & 0 & 0 & 0 & \dots \\ P_1 \otimes z^1 & P_1 \otimes z^2 & -P_1 \otimes \sum_{k=1}^{2} z^k & 0 & 0 & \dots \\ P_1 \otimes z^1 & P_1 \otimes z^2 & P_1 \otimes z^3 & -P_1 \otimes \sum_{k=1}^{3} z_k & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

A simple check shows that A satisfies the requirements to be conservative, see [6] or [1]. Let $\dot{1}$ be the constant sequence of ones. Then

$$\sum_k A_k(\dot{1}) = \sum_k P_1 \otimes z^k(\dot{1}) = \sum_k z^k$$

which is not weakly convergent.

3.4. Definition. For A conservative, F weakly sequentially complete, and x in E, let $\chi(A)(x) = \lim_A \dot{x} - \sum_k A_k(x)$.

It follows from 3.1 that $\chi(A)$ is a continuous linear map from E into F, and we now have the promised formula.

3.5. Theorem. *For A and F as above, and x in $E(c)$, with $\lim x = l$, we may write*

$$\lim_A x = \chi(A)(l) + \sum_k A_k(x_k).$$

Proof. Clear.

Let us now turn our attention to the relationship between this function and the conull-coregular dichotomy.

3.6. Definition. Let X be an FK-subspace of $E(s)$. We will call X *conservative* if X contains $E(c)$. If, in addition, $\{U_n(\dot{x})\}$ converges weakly to $\dot{x}$ for all x in E, we will call X *conull*. If A is a conservative matrix, then A is conull if and only if c_A is conull, otherwise A is *coregular*.

3.7. Proposition. *Let X and Y be FK-subspaces of $E(s)$.*

(i) *If $X \subset Y$ and X is conull, then Y is conull.*

(ii) *If $X \subset Y$, Y is conull, X is closed in Y and X is conservative, then X is conull.*

(iii) *If $\{X_k\}$ is a sequence of conull spaces, then $\bigcap_k X_k$ is conull.*

Proof. (i) and (ii) are clear. To see (iii) first notice that the intersection is an FK-space, by Theorem 3 page 205 of [10], with the supremum topology. Let f be a continuous linear functional on this space. Then the absolute of f is a continuous seminorm and there exists an m such that $|f|$ is continuous on $\bigcap X_k$ under the relative topology of $\bigcap_{k=1}^{m} X_k$. Thus for any x in E, $|f(\dot{x}-U_n(\dot{x}))|$ converges to zero.

Recalling 3.1 and 3.3, we shall now always assume that the base range space is Fréchet and is weakly sequentially complete. Moreover, the symbol $\sum_k A_k(x_k)$ will naturally mean the convergence is in the weak topology.

3.8. Definition. For conservative A and f in c_A^*, let $\chi(f) = \chi(g)\circ\chi(A)$ where $f = h+g\circ A$, as in 2.11. It follows that $\chi(f)$ is in E^*.

3.9. Lemma. *Let A be conservative, then for any x in E and f in c_A^**

$$\lim_p f(\dot{x}-U_p(\dot{x})) = \chi(f)(x).$$

Proof. Let $f = h+g\circ A$ where h is in d_A^* and g is in $F(c)^* = l_1(F^*)$. Then $h(\dot{x}-U_r(\dot{x}))$ tends to zero as r increases, see the discussion after 2.8. Now suppose g corresponds to the sequence $g_0, g_1, g_2, \ldots$ in $l_1(F^*)$. Then

$$g\circ A(\dot{x}-U_r(\dot{x})) = \chi(g)\big(\lim_A(\dot{x}-U_r(\dot{x}))\big)+\sum_i g_i\Big(\sum_{j=r+1}^{\infty} A_{ij}x\Big)$$

$$= \chi(g)\Big(\chi(A)(x)+\sum_{k=r+1}^{\infty} A_k(x)\Big)+\sum_i g_i\Big(\sum_{j=r+1}^{\infty} A_{ij}(x)\Big).$$

Since the series $\sum_k A_k(x)$ is weakly convergent, the term $\chi(g)\big(\sum_{k=r+1}^{\infty} A_k(x)\big)$ goes to zero as r tends to infinity. Let $q = \sum_{k=1}^{m} q_i$ correspond to $\{g_i\}_i^\infty$ as in 2.9. Since $\{0, x\}$ is a bounded subset of E, there is a number K such that $q\big(\sum_{j=r+1}^{\infty} A_{ij}(x)\big) \leqslant K$ for all r and all i, see Theorem 1 page 366 of [6]. Choose Q such that $\sum_{i=Q+1}^{\infty} \|g_i\|_q \leqslant \varepsilon/K$ Thus

$$\Big|\sum_i g_i\Big(\sum_{j=r+1}^{\infty} A_{ij}(x)\Big)\Big| \leqslant \Big|\sum_{i=1}^{Q} g_i\Big(\sum_{j=r+1}^{\infty} A_{ij}(x)\Big)\Big| + \sum_{i=Q+1}^{\infty}\|g_i\|_q K$$

$$\leqslant \Big|\sum_{i=1}^{Q} g_i\Big(\sum_{j=r+1}^{\infty} A_{ij}(x)\Big)\Big| + \varepsilon.$$

For any i between 1 and Q, $\sum_{j=r+1}^{\infty} A_{ij}(x)$ converges to zero as r tends to infinity and our result follows.

We now obtain the analogue of Snyder's theorem.

3.10. Theorem. *A conservative matrix A is conull if and only if $\chi(A) = 0$.*

Proof. Let A be a conull matrix and let x be in E. Consider

$$\lim_A(\dot{x}-U_n(\dot{x})) = \chi(A)(x)+\sum_{k=n+1}^{\infty} A_k(x).$$

Since $\lim_A$ is continuous from c_A into $F(c)$, it is continuous when each of these spaces is given its weak topology, confer problem 31 page 243 of [10]. By letting n tend to infinity, we obtain $\chi(A)(x) = 0$ for any x in E.

On the other hand, if $\chi(A) = 0$, we see by the lemma that for any x in E, $\{\dot{x}-U_n(\dot{x})\}$ converges to zero in the weak topology of c_A. Thus A is conull.

4. SOME SUMMABILITY RESULTS

As in the usual situation, the question of associativity is quite often crucial. We thus begin with a consideration of this problem.

4.1. Proposition. *Let A and B be infinite matrices of continuous linear operators such that $A\colon E(c)\to F(c)$ and $B\colon F(c)\to G(m)$, where E, F, and G are Fréchet spaces (not necessarily wsc). Then for any x in $E(m)$ such that $(BA)(x)$ and $B(Ax)$ are both defined, we have $(BA)(x) = B(Ax)$.*

Proof. For any n, we will show that $[(BA)(x)]_n = [B(Ax)]_n$ in the weak topology on G and the proposition will follow.

Fix f in G^*. Then for z in $F(c)$, the function $z\to\sum_j f\circ B_{nj}(z_j)$ is continuous on $F(c)$ and so on $F(c_0)$. It follows from 2.9 that we can find $q = \sum_{k=1}^{m} q_k$ such that $\sum_j \|f\circ B_{nj}\|_q$ is finite.

Now consider the double sequence $\Big\{\sum_{j=1}^{M}\sum_{k=1}^{N} f\circ B_{nk}\circ A_{kj}(x_j)\Big\}$. Since x is in $E(m)$, the set of its coordinates is a bounded subset of E and it follows from Theorem 1 page 366 of [6] that there is a number K such that $q\big(\sum_{j=1}^{M} A_{kj}(x_j)\big) \leqslant K$ for all M and all k. Choose Q such that $T > S > Q$ implies that

$$\sum_{k=S}^{T} \|f\circ B_{nk}\|_q < \varepsilon/K.$$

Then

$$\Big|\sum_{k=S}^{T}\sum_{j=1}^{M} f\circ B_{nk}\circ A_{kj}(x_j)\Big| \leqslant \sum_{k=S}^{T}\|f\circ B_{nk}\|_q K < \varepsilon.$$

Thus our double series is Cauchy in N uniformly with respect to M and the equality of the iterated limits follows from the Moore theorem, see page 28 of [2].

Before continuing we pause to note that if A is a conservative matrix such that $\chi(A)$ is invertible, i. e., one-to-one and onto, then the closed graph theorem shows that $\chi(A)^{-1}$ is continuous.

4.2. PROPOSITION. *If A is coregular and $\chi(A)$ is invertible, then the closure of $E(c)$ in c_A contains $c_A \cap E(m)$.*

Proof. Let f be an element of c_A^* that vanishes on $E(c)$. Recall that f may be written as in 2.11, i. e.,

$$f(x) = \sum_i f_i(x_i) + \chi(g)(\lim_A x) + \sum_i g_i(Ax)_i.$$

If x is in $E(c)$, this may be rewritten as

$$f(x) = \sum_i f_i(x_i) + \chi(f)(l) + \chi(g)\Big(\sum_k A_k(x_k)\Big) + \sum_i g_i(Ax)_i,$$

where $l = \lim x$ and $\chi(f) = \chi(g)\circ\chi(A)$.

Since $\chi(A)$ is invertible, $\chi(g) = \chi(f)\circ\chi(A)^{-1}$. But from 3.9 $\chi(f) = 0$ and so $\chi(g) = 0$. Thus for any x in c_A,

$$f(x) = \sum_i f_i(x_i) + \sum_i g_i(Ax)_i.$$

If x is in $c_A \cap E(m)$, then an application of 4.1 using G as the scalars and B as the diagonal matrix $\{g_i\}$ shows that f may be written as

$$f(x) = \sum_i \beta_i(x_i),$$

where each β_i is in E^*. Fix z in E, then for any p

$$0 = f\big(I_p(z)\big) = \beta_p(z).$$

Thus f vanishes on $c_A \cap E(m)$ and our result follows.

4.3. COROLLARY. *A as above. If $E(c)$ is closed in c_A, then A sums no bounded divergent sequences.*

We now obtain an analogue of a theorem due to Copping.

4.4. THEOREM. *Let A be conservative with $\chi(A)$ invertible. Suppose A has a left inverse B which carries $F(c)$ into $E(m)$. Then A sums no bounded divergent sequences.*

Proof. Let $x^n \to x$ in c_A where each x^n is in $E(c)$. Then $B(Ax^n) \to B(Ax)$ in $E(m)$, i. e., $x^n \to B(Ax)$ in $E(m)$. But $E(c)$ is closed in $E(m)$. Thus $x^n \to B(Ax)$ in $E(c)$ and so also in c_A. Thus $B(Ax) = x$ is in $E(c)$. Now use 4.3.

As a last example of this technique, we would like to consider a theorem due to Mazur. This, however, demands some preliminaries.

4.5. DEFINITION. Let A be conservative, $A\colon c_A \to F(c)$. Call A *reversible* if A is one-to-one and onto as a mapping from c_A onto $F(c)$.

4.6. PROPOSITION. *If A is reversible, the most general continuous linear functional on c_A is*

$$f(x) = g(Ax) = \chi(g)(\lim_A x) + \sum_i g_i(Ax)_i,$$

where $g = \{g_0, g_1, \ldots\}$ is in $l_1(F^*)$.

Proof. This follows from the fact that A is a congruence.

4.7. PROPOSITION. *Let A be conservative and let t be in $l_1(F^*)$. Then $(tA)(x)$ is defined for each x in $c_A \cap E(m)$ and $(tA)(x) = t(Ax)$.*

Proof. An approach similar to that used in 4.1 shows that the net

$$\Big\{\sum_{j=1}^{n}\sum_{k=1}^{m} t_k A_{kj}(x_j)\Big\}$$

over $\omega\times\omega$ is Cauchy so that $(tA)(x)$ is defined. An application of 4.1 then gives the equality of $(tA)(x)$ and $t(Ax)$.

4.8. DEFINITION. Let A be conservative. We will say A is of *type M* if for any t in $l_1(F^*)$, t must be zero whenever tA vanishes on $E(c)$.

4.9. DEFINITION. Call a conservative matrix A *perfect* if $E(c)$ is dense in c_A.

Notice that if A is perfect and c_B contains c_A with $\lim_B = \lim_A$ on $E(c)$, then $\lim_B = \lim_A$ on c_A. This, of course, follows from the fact that $\lim_B$ is continuous on c_A (with the c_A-topology).

We now obtain the Mazur theorem.

4.10. THEOREM. *Let A be reversible, coregular and $\chi(A)$ invertible. Then A is perfect if and only if A is of type M.*

Proof. Let A be perfect and let t be an element of $l_1(F^*)$ such that tA vanishes on $E(c)$. For x in c_A, let $f(x) = t(Ax)$. Then f is continuous and is zero on $E(c)$. Since A is perfect, f is zero on c_A. Fix y in F; then there is an x in c_A such that $Ax = I_p(y)$. Thus

$$0 = t(Ax) = t\big(I_p(y)\big) = t_p(y)$$

and so $t = 0$.

Now suppose A is of type M. Let f be an element of c_A^* that vanishes on $E(c)$. Recall that for any x in c_A

$$f(x) = \chi(g)(\lim_A x) + \sum_i g_i(Ax)_i.$$

An argument similar to that in 4.2 shows that $\chi(g) = 0$. So for any x in $E(c)$

$$0 = \sum_i g_i(Ax)_i,$$

and since A is of type M, we have $g_i = 0$ for $i = 1, 2, \ldots$ Thus f is zero on c_A and A is perfect.

5. THE IDEAL OF CONULL MATRICES

Ramanujan has shown that if E is a Banach space, then the set of all triangular (i. e., $A_{nk} = 0$ for $k > n$) conservative matrices carrying $E(c)$ into $E(c)$ is a Banach algebra with identity (page 372 of [6]). Let us call this space $T(E)$. We shall also assume that E is weakly sequentially complete. Using $L(E)$ to denote the Banach algebra of all continuous linear transformations from E into E with the usual norm, we see that

$$\chi\colon T(E) \to L(E).$$

THEOREM. 5.1. *χ is continuous, linear, multiplicative, and onto.*

Proof. The linearity of χ is clear. To see that χ is onto fix F in $L(E)$ and let A be the diagonal matrix with all entries on the main diagonal F. Then A is clearly in $T(E)$ and $\chi(A) = F$.

The norm in $T(E)$ is given by

$$\|A\| = \underset{\substack{\|x_k\| \\ m,n=1,2\ldots}}{\text{l.u.b}} \Big\|\sum_{k=1}^{m} A_{nk}x_k\Big\|.$$

For A in $T(E)$, $\chi(A) = \lim\circ A\circ I - \sum_k A_k$, where I takes E into $E(c)$ by $I(x) = \dot{x}$. It follows that $\|I\| = 1$. Let $G(x) = \sum_k A_k(x)$ and recall that G is continuous, 3.1. Moreover, it is clear that $\|G\| \leqslant \|A\|$. We then have

$$\|\chi(A)\| = \|\lim\circ A\circ I - G\| \leqslant \|\lim\|\,\|A\| + \|A\| \leqslant \|A\|(\|\lim\|+1).$$

Thus $\|\chi\|$ is finite and χ is continuous.

It remains to show that χ is multiplicative. Let A and B be elements of $T(E)$ and $C = AB$. Recall that for any x in $E(c)$ we have

$$\lim_C x = \chi(C)(\lim x) + \sum_k C_k(x_k).$$

Letting $x = I_p(z)$ where z is in E, we see that

$$\begin{aligned} C_p(z) &= \lim_C \big(I_p(z)\big) \\ &= \lim_A \Big(B\big(I_p(z)\big)\Big) \\ &= \chi(A)\big(\lim_B I_p(z)\big) + \sum_k A_k\Big(B\big(I_p(z)\big)\Big)_k \\ &= \chi(A)\Big(\chi(B)\big(\lim I_p(z)\big) + B_p(z)\Big) + \sum_k A_k\big(B_{kp}(z)\big) \\ &= \chi(A)\big(B_p(z)\big) + \sum_k A_k B_{kp}(z). \end{aligned}$$

Fix x in E and consider

$$\begin{aligned} \chi(C)(x) &= \lim_C \dot{x} - \sum_p C_p(x) \\ &= \lim_A (B\dot{x}) - \sum_p C_p(x) \\ &= \chi(A)(\lim_B \dot{x}) + \sum_k A_k(B\dot{x})_k - \sum_p C_p(x) \\ &= \chi(A)\Big(\chi(B)(x) + \sum_p B_p(x)\Big) + \sum_k A_k\Big(\sum_p B_{kp}x\Big) - \sum_p C_p(x) \\ &= \chi(A)\circ\chi(B)(x) + \chi(A)\Big(\sum_p B_p(x)\Big) + \sum_k\sum_p A_k B_{kp}(x) \\ &\quad - \chi(A)\Big(\sum_p B_p(x)\Big) - \sum_p\sum_k A_k B_{kp}(x). \end{aligned}$$

But a now familiar argument shows that the iterated sums appearing in the above are equal (in the weak topology). Thus

$$\chi(AB) = \chi(A)\circ\chi(B).$$

COROLLARY. 5.2. *The set of triangular conull matrices is a closed ideal in $T(E)$.*

Proof. This follows from 5.1 and 3.10.

Note added in proof. In 4.2, 4.4 and 4.10 the hypothesis that $\chi(A)$ is invertible may be replaced by $\chi(A)$ is onto.

References

[1] A. Alexiewicz and W. Orlicz, *Consistency theorems for Banach space analogues of Toeplitzian methods of summability*, Studia Math. 18 (1959), pp. 199–210.

[2] N. Dunford and J. T. Schwartz, *Linear Operators*, Part I: *General Theory*, New York 1964.

[3] J. L. Kelley, I. Namioka, et al, *Linear Topological Spaces*, Princeton 1963.
[4] G. G. Lorentz and M. S. Macphail, *Unbounded operators and a theorem of A. Robinson*, Trans. Roy. Soc. Canada (Series III) 44 (1952), pp. 33-37.
[5] H. Melvin-Melvin, *Generalized k-transformations in Banach spaces*, Proc. London Math. Soc. (second series) 53 (1951), pp. 83-108.
[6] M. S. Ramanujan, *Generalized Kojima–Toeplitz matrices in certain linear topological spaces*, Math. Ann. 159 (1965), pp. 365-373.
[7] A. Robinson, *On functional transformations and summability*, Proc. London Math. Soc. (second series) 52 (1947), pp. 132-160.
[8] A. K. Snyder, *On a definition for conull and coregular FK spaces*, Notices Amer. Math. Soc. 10 (1963), p. 183.
[9] — *Conull and coregular FK spaces*, Math. Zeitschr. 90 (1965), pp. 376-381.
[10] A. Wilansky, *Functional Analysis*, New York 1964.
[11] — *An application of Banach linear functionals to summability*, Trans. Amer. Math. Soc. 67 (1949), pp. 59-68.
[12] E. I. Yurimyae, *Einige Fragen über verallgemeinerte Matrixverfahren*, Eesti NSV Tead. Akad. Toim. Tehn. Füüs-Mat. 8 (1959), pp. 115-121.

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On a class of absolutely p-summing operators

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§ 1. Introduction. In this note we use the conventions and notation of A. Persson [4]. A normed linear space is denoted by E, and E' is its topological dual with the strong dual topology. We use $\langle f, f'\rangle$ to indicate the action of a vector f in E and a functional f' in E'. For $1 \leqslant p < \infty$, p and p' are the usual conjugate numbers. $L^p(X, \mu; E')$ is the Banach space of equivalent classes of strongly μ-measurable E'-valued functions K such that $\int_X \|K(x)\|^p d\mu < \infty$. In case E' is C — the complex numbers, we simply write $L^p(X, \mu)$ instead of $L^p(X, \mu; C)$. All the measures in this note are countably additive and positive, and all the operators are bounded.

We aim to investigate a class of linear operators $T\colon E \to L^p(X, \mu)$ and their adjoint $T^*\colon L^{p'}(X, \mu) \to E'$ such that

$$|Tf(x)| \leqslant \gamma(x)\|f\|$$

for some γ in $L^p(X, \mu)$. When E is a reflexive Banach space or E' is separable, it turns out that each of them can be represented by a unique K in $L^p(X, \mu; E')$ in the following way:

$$Tf(x) = \langle f, K(x)\rangle \text{ a. e.} \quad \text{and} \quad T^* g = \int_X g(x) K(x)\, d\mu$$

for f in E, g in $L^{p'}(X, \mu)$ where the integral is taken to be the Bochner integral. In this case, T is an operator of type $-N_p$ and T^* is of type $-N^p$ ([4], Theorem 1 and 2). They are all completely continuous operators. Our result is quite similar to Dunford–Pettis Theorem in which they obtained a representation for any operators $T\colon E \to L^\infty(X, \mu)$ and $S\colon L^1(X, \mu) \to E'$ by a bounded weakly measurable E'-valued function under the hypotheses that E is a separable Banach space (see [6], p. 469). As a consequence of our result, we obtain an alternative proof for a theorem of Shields–Wallen–Williams which asserts that if operator $T\colon E \to L^p(X, \mu)$ satisfies $|Tf(x)| \leqslant \gamma(x)\|f\|$ for some γ in $L^p(X, \mu)$, then T is absolutely p-summing. In case $E = L^{p'}(Y, \nu)$, our result also includes