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Some remarks on subspaces of Orlicz spaces of vector-valued finitely additive functions*

by

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In recent years, renewed efforts have been expanded in the study of spaces of finitely additive set functions (a good bibliography may be found in [10]). The present paper presents several results concerning subspaces of certain classes of finitely additive functions. These results have been obtained in somewhat different circumstances in [3] and are, of course, motivated by the analogous results of [5] and [9].

We shall assume throughout that Ω is any set, Σ a field of subsets of Ω and μ is a non-negative, real-valued, finitely additive function with domain Σ. Thus, in particular, $\mu(\Omega) < \infty$. We also assume familiarity with the work in [10] and shall use the notation and results of that paper throughout.

We start with the following observation the proof of which is completely contained in Lemma 13 of [10]: Suppose $F_n \in V^{\Phi}(\mathscr{X})$ for $n = 0, 1, 2, \ldots$ Then, if $F_n \to F_0$ (in N_{Φ}-norm) $F_n \to F_0$ in $V^1(\mathscr{X})$ norm and thus $F_n(E) \to F_0(E)$ uniformly for $E \in \Sigma$.

THEOREM 1. *Suppose Φ_1, Φ_2 are Young's functions and that $V^{\Phi_1}(\mathscr{X})$ is properly contained in $V^{\Phi_2}(\mathscr{X})$. Also suppose that Φ_1 is continuous. Then $V^{\Phi_1}(\mathscr{X})$ is meager in $V^{\Phi_2}(\mathscr{X})$.*

Proof. Consider the sets S_k for $k > 0$ defined by

$$S_k = \{v \in V^{\Phi_2}(\mathscr{X}) / N_{\Phi_1}(v) \leqslant k\}.$$

We will show that these sets S_k are closed in $V^{\Phi_2}(\mathscr{X})$ whence $V^{\Phi_1}(\mathscr{X})$ becomes an F_σ — subset of $V^{\Phi_2}(\mathscr{X})$ and we can apply Theorem 1, p. 36 of [1] to conclude to the desired result.

Let us prove S_k's closed. Suppose $v_n \in S_k$ for $n = 1, 2, \ldots$; and let

$$N_{\Phi_2}(v_n - v_0) \to 0$$

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where $v_0 \in V^{\Phi_2}(\mathscr{X})$. Then by the preceding remarks,

$$(*) \qquad v_n(E) \to v_0(E)$$

for each $E \in \Sigma$ the convergence being uniform for $E \in \Sigma$. That

$$N_{\Phi_1}(v_n) \leqslant k$$

for all n ($\geqslant 1$) is equivalent to

$$I_{\Phi_1}(v_n/k) \leqslant 1$$

for all $n \geqslant 1$; which in turn is the same as saying that for any partition $\Pi = \{F_1, \ldots, F_m\} \subseteq \Sigma$ of Ω we have for $n \geqslant 1$,

$$\sum_{i=1}^{m} \Phi_1\big(\|v_n(F_i)\|/k \cdot \mu(F_i)\big) \leqslant 1$$

which by continuity of Φ_1 and (*) yields

$$\sum_{i=1}^{m} \Phi_1\big(\|v_0(F_i)\|/k \cdot \mu(F_i)\big) \leqslant 1$$

and, of course, it follows immediately that

$$v_0 \in V^{\Phi_1}(\mathscr{X}) \quad \text{with} \quad N_{\Phi_1}(v_0) \leqslant k.$$

Thus, S_k is closed in $V^{\Phi_2}(\mathscr{X})$ and Theorem 1 is proved.

Our next result is a bit more delicate than the preceding one. We assume for convenience that all Young's functions are continuous. Recall from [8], that the space $V^{\infty}(\mathscr{X})$ denotes the collection of $\mathscr{X}$-valued, finitely additive functions v defined on Σ, vanishing on μ-null sets for which

$$\|v\|_{\infty} = \sup\{\|v(E)\|/\mu(E) \colon E \in \Sigma\} < \infty,$$

(where the usual limitations on the involved quotients are employed). Note that we have the relationship

$$V^{\infty}(\mathscr{X}) \subseteq V^{\Phi}(\mathscr{X}) \subseteq V^{1}(\mathscr{X})$$

for any (continuous) Φ and any Banach space $\mathscr{X}$. Moreover, if Φ is dominated by Φ' (i. e., if $\Phi(u) \leqslant M\Phi'(u)$ for $u \geqslant 0$) then $V^{\Phi'}(\mathscr{X}) \subseteq V^{\Phi}(\mathscr{X})$, and Φ' topology of $V^{\Phi'}(\mathscr{X})$ is stronger than the relative Φ-topology on $V^{\Phi'}(\mathscr{X})$.

With these comments in mind recall that if $\mathscr{X}$ is any reflexive Banach space and Φ and its conjugate Young's function Ψ both satisfy the Δ_2-condition, then $V^{\Phi}(\mathscr{X})$ is similarly a reflexive Banach space with dual space identifiable as $V^{\Psi}(\mathscr{X}^*)$. This is particularly true in the case of $\mathscr{X}$ being any finite-dimensional Banach space. If, in particular, $\mathscr{X} = K^n$ (K = scalars), then Leader's proof — with only trivial modifications — shown that $V^1(K^n)^*$ is none other than $V^{\infty}(K^n)$. This allows us to prove the following analog to Theorem 1 of [5]:

THEOREM 2. *Suppose H is a linear subspace of $V^{\infty}(K^n)$ which is closed in $V^{\Phi}(K^n)$ for some (continuous) Young's function Φ which is dominated by a Φ_1 satisfying $\Phi_1, \Psi_1 \in \Delta_2$. Then H is a finite dimensional linear space.*

Proof. We start with the following result which might be of independendet interest:

THEOREM 3. *Let H be any linear subspace of $V^{\infty}(\mathscr{X})$ ($\mathscr{X}$—any Banach space) and suppose H is closed in $V^{\Phi}(\mathscr{X})$ for some (continuous) Young's function Φ. Then H is closed in $V^{\infty}(\mathscr{X})$ and, in fact, is closed in all intermediate Orlicz spaces $V^{\Phi'}(\mathscr{X})$.*

Proof. To show that H is closed in $V^{\infty}(\mathscr{X})$ we will show that the identity surjection i_H of H onto itself is, in fact, a uniform isomorphism of H in its relative $V^{\infty}(\mathscr{X})$ topology to H in its relative $V^{\Phi}(\mathscr{X})$ topology; this being done, we can conclude by the completeness of H in the latter structure the completeness—hence closedness — of H in the former. Note that we have immediately that the map $i_H\colon H_{\infty} \to H_{\Phi}$ is continuous. To see that i_H is a uniform isomorphism, we need only apply the comment preceeding Theorem 1, and the Closed Graph Theorem and we can conclude H to be as claimed, i. e., H is closed in $V^{\infty}(\mathscr{X})$. That H is thereby closed in intermediate Orlicz spaces $V^{\Phi'}(\mathscr{X})$ is immediate.

Returning to the proof of Theorem 2; we note that $V^1(\mathscr{X})$ (in the case, of $\mathscr{X} = K^n$ being a finite dimensional space) can readily be seen to be an abstract (L)-space (norm $\mathscr{X}$ with $\|(x_1, \ldots, x_n)\| = \sum_1^m |x_i|$) whence $V^{\infty}(\mathscr{X}) = V^1(\mathscr{X})^*$ is equivalent to an abstract M-space (in fact, is such a space with correct norming of $\mathscr{X}$) with unit (obviously, $\mu^n = (\mu, \ldots, \mu)$ is such a unit). Thus using the comments preceding Theorem 2, we have that H is a closed linear subspace of the Banach space $V^{\infty}(\mathscr{X})$ (a space which satisfies the strict Dunford Pettis property of [4] and of the reflexive space $V^{\Phi_1}(\mathscr{X})$. The situation is now identical to that of Theorem 1 of [5] and that proof can now be mimicked to complete the present proof.

Of particular note in all the above theorems, especially Theorem 1, is the verification of Leader's claim that "by consistently avoiding integrals, we reduce to a minimum the limiting processes necessary for the theory of L^p". One need only refer to the proofs in [3] of the analogous results for point functions to be convinced of the verity of the above quote.

Addendum. In a paper soon to appear in Math. Annalen, Professor S. Saxon has introduced the notion of a Baire-like locally convex space; a locally convex space E is said to be Baire-like whenever E cannot be written as the union of an increasing sequence of nowhere dense, closed, balanced, convex sets. Theorem 2.10 of the paper (entitled "Product spaces, Baire-like spaces and the strongest locally convex topology") states that every countably co-dimensional subspace of a Baire-like space is Baire-like. Our Theorem 1 therefore can be strengthened to: *$V^{\Phi_1}(\mathfrak{X})$ is a meager, uncountably codimensional linear subspace of $V^{\Phi_2}(\mathfrak{X})$.* Indeed, the S_k's are an increasing sequence of closed, convex, balanced, nowhere dense sets, $\bigcup_k S_k = V^{\Phi_1}(\mathfrak{X})$.

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The chi function in generalized summability

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1. INTRODUCTION

In 1949 Wilansky, [11], introduced the conull and coregular classification of scalar summability matrices by the use of the chi functional. Yurimyae in [12] and Snyder in [8] and [9] showed that these properties can be characterized without the use of matrices.

Other authors, see [1], [4], [5], [6], and [7], have considered the topic of generalized summability and, in particular, have obtained analogues of the Silverman–Toeplitz and the Kojima–Schur conditions.

In this paper, we extend the concept of the chi function to the generalized situation, in a certain setting, and obtain an analogue of Snyder's result, Theorem 1, p. 378 of [9]. We also show that some of the usual summability methods utilizing the chi function carry over to this new setting.

2. *FK*-SPACES

Let F be a Fréchet space, i. e., a locally convex complete linear metric space. Recall, p. 217 of [10], the topology of E may be generated by a sequence of continuous seminorms, $\{p_j\}$. We shall use the following notation:

$E(s)$ is the space of all sequences in E with pointwise addition and scalar multiplication;

$E(m)$ is that subspace of $E(s)$ consisting of bounded sequences, i. e., $\{x_n\}$ is in $E(s)$ and $\{x_n|\ n \in \omega\}$ is a bounded subset of E;

$E(c)$ is that subspace of $E(m)$ consisting of convergent sequences;

$E(c_0)$ is that subspace of $E(c)$ consisting of sequences convergent to zero.

If X is any one of the above spaces, let $C_n\colon X \to E$ be defined by $C_n(x) = x_n$.