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Differential equations in a linear ring

by

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The purpose of this paper is to solve linear differential equations with coefficients in a linear ring, in particular ordinary linear differential equations with variable coefficients, using the definition of algebraic derivative given by D. Przeworska-Rolewicz [3].

Definition 1. Let X be a commutative linear ring (over a field F) with the unit e.

If for a linear operator D acting in X there is a linear operator R defined on X such that RX is contained in the domain $\mathscr{D}_D$ of D and

1° $DR = I$,

2° *the operator* $I+Rp$ *is invertible for any* $p \in X$ (¹),

3° $D(xy) = (Dx)y + x(Dy)$ for $x, y \in \mathscr{D}_D$,

then D will be called *algebraic derivative* and R — *algebraic integral* on X, the kernel $Z_D = \{z \in \mathscr{D}_D\colon Dz = 0\}$ is called the *space of constants*.

In particular, it follows from 2° that the operator $I-\lambda R$ is invertible for any $\lambda \in F$.

Indeed, $I-\lambda R = I+R(-\lambda e)$.

PROPOSITION 1.

(i) $D(\lambda e) = 0$ *for any* $\lambda \in F$.

(ii) $Dx^n = nx^{n-1}Dx$ *for* $n = 1, 2, \ldots$

(iii) $D^k R^k = I$ *for* $k = 1, 2, \ldots$

(iv) *There is a* q *such that* $Dq = e$, *namely* $q = Re$.

(v) $(D+pI)(I+Rp)^{-1} = D$ *for any* $p \in X$.

Proof. (i) For all $\lambda \in F$ and $x \in \mathscr{D}_D$

$$Dx = D(ex) = (De)x + e(Dx) = (De)x + Dx.$$

Hence $(De)x = 0$ and arbitrariness of x imply that $De = 0$. Therefore $D(\lambda e) = \lambda De = 0$.

(¹) By Rp we will mean always the superposition of the operator R and of the operator of multiplication by an element $p \in X$.

(ii), (iii) follows by an easy induction.

(iv) $Dq = DRe = e$ (compare with Theorem 3.4 of [3]).

(v) Since $D+pI = D+Ip = D+DRp = D(I+Rp)$, we find

$$(D+pI)(I+Rp)^{-1} = D(I+Rp)(I+Rp)^{-1} = D.$$

Now we will consider the following equation:

$$D^k x + p_1 D^{k-1}x + p_2 D^{k-2}x + \ldots + p_{k-1}Dx + p_k x = y \tag{1}$$

where $y \in X$ and $p_i \in X$ for $i = 1, 2, \ldots, k$. We denote $D^n p = p^{(n)}$ for $n = 0, 1, \ldots$

Lemma 1. *We have*

$$aD^k x = \sum_{i=0}^{k} (-1)^i \binom{k}{i} D^{k-i}(a^{(i)}x) \tag{2}$$

for all positive integers k and for an arbitrary $a \in \mathcal{D}_D$.

Proof, by induction. For $k = 1$ the formula (2) is true because $aDx = D(ax) - (Da)x$. We assume that (2) is true for $k = n$ and we shall prove that (2) is true for $k = n+1$, i. e. that

$$aD^{n+1}x = \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} D^{n+1-i}(a^{(i)}x).$$

Since

$$aD^{n+1}x = aD(D^n x) = D(aD^n x) - (Da)D^n x,$$

the induction assumption implies that

$$aD^{n+1}x = D\Big[\sum_{i=0}^{n} (-1)^i \binom{n}{i} D^{n-i}(a^{(i)}x)\Big] - \sum_{j=0}^{n} (-1)^j \binom{n}{j} D^{n-j}(a^{(j+1)}x)$$

$$= \sum_{i=0}^{n} (-1)^i \binom{n}{i} D^{n+1-i}(a^{(i)}x) + \sum_{j=0}^{n} (-1)^{j+1} \binom{n}{j} D^{n-j}(a^{(j+1)}x)$$

$$= \sum_{i=0}^{n} (-1)^i \binom{n}{i} D^{n+1-i}(a^{(i)}x) + \sum_{i=1}^{n+1} (-1)^i \binom{n}{i-1} D^{n+1-i}(a^{(i)}x)$$

$$= D^{n+1}(ax) + \sum_{i=1}^{n} (-1)^i \Big[\binom{n}{i} + \binom{n}{i-1}\Big] D^{n+1-i}(a^{(i)}x) +$$

$$+ (-1)^{n+1}a^{(n+1)}x = \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} D^{n+1-i}(a^{(i)}x)$$

what was to be proved.

Theorem 1. *The general form of a solution of the equation* (1) *is*

$$x = \Big(I + \sum_{j=1}^{k} R^j \tilde{p}_j\Big)^{-1} \Big(R^k y + \sum_{m=0}^{k-1} R^m z_m\Big)$$

where

1. $z_m \in Z_D$ *for* $m = 0, 1, \ldots, k-1$,
2. $\tilde{p}_j = \sum_{w=0}^{j-1} (-1)^w \binom{k+w-j}{w} p_{j-w}^{(w)}$,
3. *the operator* $(I + \sum_{j=1}^{k} R^j \tilde{p}_j)$ *is invertible* ([2]),
4. $p_j \in \mathcal{D}_{D^{k-j}}$ *for* $j = 1, 2, \ldots, k$.

Proof. Applying Lemma 1 to the left side of (1) we obtain:

$$D^k x + \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} D^{k-1-i}(p_1^{(i)}x) + \sum_{i=0}^{k-2} (-1)^i \binom{k-2}{i} D^{k-2-i}(p_2^{(i)}x) +$$

$$+ \ldots + \sum_{i=0}^{1} (-1)^i \binom{1}{i} D^{1-i}(p_{k-1}^{(i)}x) + p_k x = y.$$

If we order the left side of the equation with respect to the powers of D, we obtain

$$D^k x + D^{k-1}\Big[\binom{k-1}{0} p_1 x\Big] + D^{k-2}\Big[\binom{k-2}{0} p_2 - \binom{k-1}{1} p_1' x\Big] + \ldots +$$

$$+ [p_k x + (-1)^1 p_{k-1}' x + \ldots + (-1)^{k-1} p_1^{(k-1)} x] = y.$$

In virtue of Proposition 1 (iii) we have

$$D^k x + D^k R(p_1 x) + \ldots + D^k R^k [p_k x + \ldots + (-1)^{k-1} p_1^{(k-1)} x] = y.$$

Hence finally:

$$D^k\Big(I + \sum_{j=1}^{k} R^j \tilde{p}_j\Big) x = y, \quad \text{where} \quad \tilde{p}_j = \sum_{w=0}^{j-1} (-1)^w \binom{k-j+w}{w} p_{j-w}^{(w)}.$$

Basing on the assumption (3) we obtain

$$D^k x = \Big(I + \sum_{j=1}^{k} R^j \tilde{p}_j\Big)^{-1} y.$$

From the Corollary 2.5 in [3] we have

$$x = \Big(I + \sum_{j=1}^{k} R^j \tilde{p}_j\Big)^{-1} \Big(R^k y + \sum_{m=0}^{k-1} R^m z_m\Big)$$

where $z_m \in Z_D$ for $m = 0, 1, \ldots, k-1$ what was to be proved.

([2]) This assumption does not limit us essentially because it is equivalent to the assumption that the respective Volterra integral equation is solvable uniquely, what is satisfied.

EXAMPLE 1. Let us consider the space $C[0,1]$ and an ordinary differential equation:

$$x''+px'+qx = y,$$

where $p, q, p', y \in C[0,1]$.
Then, as follows from Theorem 1 we obtain:

$$[I+Rp+R^2(q-p')]x = R^2 y+Rz_1+z_2$$

what implies

$$x(t)+\int_0^t [p(s)+(t-s)(q(s)-p'(s))]x(s)\,ds = y_1(t), \tag{3}$$

where

$$y_1(t) = R^2 y+Rz_1+z_2.$$

Solving a corresponding Volterra integral equation (see [1]) we obtain:

$$x(t) = y_1(t)+\int_0^t \mathcal{N}(t,s)y_1(s)\,ds,$$

where the kernel

$$\mathcal{N}(t,s) = N(t,s)+\sum_{n=1}^{\infty} N_n(t,s),$$

$$N(t,s) = p(s)+(t-s)(q(s)-p'(s)),$$

$$N_n(t,s) = \int_s^t N(t,w)N_{n-1}(w,s)\,dw, \qquad N_0 = N.$$

It is easy to check that the solution $x(t)$ defined by the formula (3) is twice differentiable.

We denote now

$$v_p(z) = (I+Rp)^{-1}z \quad \text{for} \quad z\in Z_D; \quad p\in X.$$

As follows from Theorem 1, $v_p(z)$ is a solution of the equation $(D+pI)x = 0$. $v_p(z)$ is a linear family with respect to z, because for all $a, b\in F$

$$v_p(az_1+bz_2) = av_p(z_1)+bv_p(z_2).$$

From the definition we have

$$(D+pI)v_p(z) = (D+pI)v_p(z) = D(I+Rp)v_p(z) = Dz = 0.$$

For a fixed $p\in X$ let us denote

$$V_p = \{v_p(z):\ z\in Z_D\}.$$

The definition implies that the set $V_{-\lambda e}$ is the eigenspace corresponding to the eigenvalue $-\lambda$.

PROPOSITION 2. *For all $p, q\in X$*

$$V_pV_q\subset V_{p+q}.$$

Proof. By the assumption we have: $Dv_p(z) = pv_p(z)$, $Dv_q(z) = qv_q(z)$ and $Dv_{p+q}(z) = (p+q)v_{p+q}(z)$, for all $z\in Z_D$. Hence

$$D[v_p(z)v_q(z)] = (Dv_p(z))v_q(z)+v_p(z)(Dv_q(z)) = (p+q)v_p(z)v_q(z).$$

This implies that

$$V_pV_q\subset V_{p+q}.$$

We do not know if the opposite inclusion holds or not.

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References

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