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On some countably modulared spaces

by

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1. The theory of countably modulared spaces was started by [1]. In that paper the notions of a countably modulared space and of a uniformly countably modulared space were defined, and investigated in case of atomless finite measures and purely atomic infinite measures. Also the problem of equality of these two spaces was solved in the above case.

Next, the theory of countably modulared spaces was developed in [3] and [4]. There are considered countably modulared spaces and uniformly countably modulared spaces defined by means of families of non-negative measures and by means of various sequences of pseudomodulars. The results of [3] generalize those of [1] concerning finite atomless measures.

In this paper we shall deal with countably modulared spaces and uniformly countably modulared spaces defined by means of families of infinite purely atomic measures. The problem of equality of the above two spaces is investigated.

1.1. In the sequel the following notations and terminology will be used:

Let a real linear space X be given and let ϱ be a functional defined on X with values $0 \leqslant \varrho(x) \leqslant \infty$. This functional will be called a *modular*, if it satisfies the following conditions:

A.1. $\varrho(x) = 0$ if and only if $x = 0$.

A.2. $\varrho(-x) = \varrho(x)$.

A.3. $\varrho(\alpha x + \beta y) \leqslant \varrho(x) + \varrho(y)$ for every $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$.

If in place of A.1, ϱ satisfies only the condition $\varrho(0) = 0$, then ϱ is called a *pseudomodular* (see [2]).

By a *φ-function* we understand a continuous, non-decreasing function $\varphi(u)$ defined for $u \geqslant 0$ and such that $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\varphi(u) \to \infty$ as $u \to \infty$.

Let φ_i be φ-functions. In the sequel we shall often make use of the following conditions:

(α) $\varphi_i(u)$ are equicontinuous at $u = 0$;

(β) for every index i there exist positive constants $\lambda_i, \beta_i, \vartheta_i$ such that for every $u \geqslant \vartheta_i$ and $k \geqslant i$ the inequality $\varphi_i(\lambda_i u) \leqslant \beta_i \varphi_k(u)$ holds;

(γ) there exist positive constants k, c, u_0 and an index i_0 such that $\varphi_i(cu) \leqslant k\,\varphi_{i_0}(u)$ for $u \geqslant u_0$ and $i \geqslant i_0$;

(δ) for every $\varepsilon > 0$ there exist numbers $u_\varepsilon > 0$ and $\alpha_\varepsilon > 0$, depending on i, such that $\varphi_i(\alpha u) < \varepsilon\varphi_i(u)$ for $0 \leqslant \alpha \leqslant \alpha_\varepsilon$, $u \geqslant u_\varepsilon$ (see [1] and [2]).

1.2. The *countably modulared space* X_ϱ and the *uniformly countably modulared space* X_{ϱ_0} are defined as follows. Let ϱ_i, $i = 1, 2, \ldots$ be a sequence of pseudomodulars in a real linear space X, and let $\varrho_i(x) = 0$ for all i imply $x = 0$. First, we define the modulars

$$\varrho(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\varrho_i(x)}{1+\varrho_i(x)}, \qquad \varrho_0(x) = \sup_i \varrho_i(x),$$

and then we define the spaces

$$X_\varrho = \{x: \varrho(\lambda x) \to 0 \text{ as } \lambda \to 0,\ x \in X\},$$
$$X_{\varrho_0} = \{x: \varrho_0(\lambda x) \to 0 \text{ as } \lambda \to 0,\ x \in X\}.$$

In [1] the problem under which conditions the identity $X_\varrho = X_{\varrho_0}$ holds was investigated. Two cases were considered.

In the first case, the pseudomodulars are defined as

(1) $$\varrho_i(x) = \int_E \varphi_i(|x(t)|)\,d\mu,$$

where μ is a finite atomless measure on a σ-algebra $\mathscr{E}$ of subsets of a set E, and X is the space of all μ-measurable functions defined on E.

In the second case, the pseudomodulars are of the form

(2) $$\varrho_i(x) = \sum_{j=1}^{\infty} \omega_j \varphi_i(|t_j|),$$

where X is the space of all real sequences (or real bounded sequences), and $\{\omega_j\}$ is a sequence of positive numbers such that $\liminf_{j\to\infty} \omega_j > 0$ (or $\lim_{r\to\infty} \omega_{j_r} = 0$ and $\sum_{r=1}^{\infty} \omega_{j_r} = \infty$ for a sequence of indices $\{j_r\}$).

1.3. A pseudomodular more general than (1) was considered in [3] namely

(3) $$\varrho_i(x) = \sup_\tau \int_E \varphi_i(|x(t)|)\,d\mu_\tau,$$

where $\{\mu_\tau\}$ is a family of non-negative measures on $\mathscr{E}$, and $\tau \in \mathscr{T}$, where $\mathscr{T}$ is an abstract set. In this case the following two properties of the family $\{\mu_\tau\}$ are needed in order to investigate the problem of the identity $X_\varrho = X_{\varrho_0}$:

(u.b.) The family of measures $\{\mu_\tau\}$ is called *uniformly bounded*, if there exists a constant $K \geqslant 0$ such that $\mu_\tau E \leqslant K$ for all $\tau \in \mathscr{T}$.

(e) The family of measures is called *equisplittable*, if there exists $\eta > 0$ such that for any sequence of numbers $\varepsilon_k \downarrow 0$ for which $\varepsilon_k \leqslant \eta$ and $\varepsilon_{k+1}/\varepsilon_k \leqslant \frac{1}{2}$ for all k, there exist constants $M \geqslant \delta > 0$ and a sequence of pairwise disjoint sets $A_k \in \mathscr{E}$ satisfying the condition $\delta\varepsilon_k \leqslant \sup_{\tau\in\mathscr{T}} \mu_\tau A_k \leqslant M\varepsilon_k$.

The following theorem holds (see [3]):

If the family of measures $\{\mu_\tau\}$ possesses the properties (u.b.) *and* (e) *and if the φ-functions φ_i satisfy conditions* (α), (β), *and* (δ), *then the identity $X_\varrho = X_{\varrho_0}$ holds if and only if φ_i satisfy condition* (γ).

In case when $\mathscr{T}_0 = \mathscr{T} \cup \{\tau_0\}$, where $\tau_0 \notin \mathscr{T}$, is a topological space, a sequence of pseudomodulars $\{\varrho_i\}$ may be defined as follows (see [4]):

(4) $$\varrho_i(x) = \overline{\lim_{\tau\to\tau_0}} \int_E \varphi_i(|x(t)|)\,d\mu_\tau$$

or, equivalently,

(4′) $$\varrho_i(x) = \inf_U \sup_{\tau\in U} \int_E \varphi_i(|x(t)|)\,d\mu_\tau,$$

where U runs over the set of all neighbourhoods of τ_0 in $\mathscr{T}_0$.

In this case, in order to investigate the identity $X_\varrho = X_{\varrho_0}$, the family of measures $\{\mu_\tau\}$ must possess besides the property (u.b.), the following property (t.e.) called topological equisplittability:

(t.e.) The family of measures $\{\mu_\tau\}$ is called *topologically equisplittable* in $\mathscr{T}_0$, if there exists $\eta > 0$ such that for any sequence of numbers $\varepsilon_k \downarrow 0$ satisfying the inequalities $\varepsilon_k \leqslant \eta$, $\varepsilon_{k+1}/\varepsilon_k \leqslant \frac{1}{2}$ for all k, there exist constants $M \geqslant \delta > 0$ and a sequence of pairwise disjoint sets $A_k \in \mathscr{E}$ for which $\delta\varepsilon_k \leqslant \overline{\lim_{\tau\to\tau_0}} \mu_\tau A_k \leqslant M\varepsilon_k$.

The following theorem holds (see [4]):

If the family of measures $\{\mu_\tau\}$ possesses the properties (u.b.) *and* (t.e.) *and if the φ-functions φ_i satisfy conditions* (α), (β), *and* (δ), *then the identity $X_\varrho = X_{\varrho_0}$ holds if and only if φ_i satisfy condition* (γ).

It is easily seen that if we take in $\mathscr{T}_0$ the coarsest topology, then the pseudomodulars (4) are reduced to the pseudomodulars (3), and the property (t.e.) is identical with the property (e).

1.4. Now, let us consider the case of purely atomic finite measures defined in the following manner by means of a non-negative matrix $(a_{n\nu})$ containing no column consisting of zeros only:

(5) $$\mu_n A = \sum_i a_{n\nu_i} \quad \text{for } A = \{\nu_i\} \in \mathscr{E}, \quad \mu_n \emptyset = 0.$$

In [3], 5.1, there are given sufficient conditions in order that the family of measures $\{\mu_\tau\}$, defined by (5), be equisplittable.

Also in [3] and [4] there are given examples of matrices for which the respective families of measures possess property (e) or (t.e).

2.1. Now, we shall investigate the case of a family of infinite atomic measures $\{\mu_n\}$ defined by means of formula (5). Then the pseudomodulars (3) are of the form

$$\varrho_i(x) = \sup_n \sum_{\nu=1}^{\infty} a_{n\nu}\varphi_i(|t_\nu|). \tag{6}$$

Here, the following conditions will be of use in place of 1.1 (β) and (γ):

(β′) for every index i there exist positive constants $\lambda_i, \beta_i, \vartheta_i$ such that for every $u \leqslant \vartheta_i$ and $k \geqslant i$ the inequality $\varphi_i(\lambda_i u) \leqslant \beta_i \varphi_k(u)$ holds;

(γ′) there exist positive constants k, c, u_0 and an index i_0 such that $\varphi_i(cu) \leqslant k\varphi_{i_0}(u)$ for $0 \leqslant u \leqslant u_0$ and $i \geqslant i_0$.

2.2. The following conditions for the identity $X_\varrho = X_{\varrho_0}$ will be proved now in case of pseudomodulars (6):

THEOREM 1. *Let* $\liminf_{\nu\to\infty} a_{n_0\nu} > 0$ *for a fixed* n_0. *If the φ-functions* φ_i *satisfy conditions* (α) *and* (γ′), *then* $X_\varrho = X_{\varrho_0}$.

Proof. From (γ′) we conclude that there are positive constants k, c_0, u_0 and an index i_0 such that $\varphi_i(u) \leqslant k\varphi_{i_0}(c_0 u)$ for $0 \leqslant u \leqslant u_0$, $i \geqslant i_0$. Let $x \in X_\varrho$; then there are $\lambda_i > 0$ such that $\varrho_i(\lambda_i x) < \infty$ for all i. Hence

$$\lim_{\nu\to\infty} a_{n_0\nu}\varphi_i(\lambda_i|t_\nu|) = 0 \quad \text{for each } i,$$

and so

$$\lim_{\nu\to\infty} \varphi_i(\lambda_i|t_\nu|) = 0.$$

By continuity of φ_i, $t_\nu \to 0$ as $\nu \to \infty$, and so $\{t_\nu\}$ is bounded. Taking λ sufficiently small, we have $|\lambda t_\nu| \leqslant u_0$ for all ν. Hence $\varphi_i(\lambda|t_\nu|) \leqslant k\varphi_{i_0}(c_0\lambda|t_\nu|)$ for $i \geqslant i_0$ and all ν, i.e. $\varrho_i(\lambda x) \leqslant k\varrho_{i_0}(c_0\lambda x)$ for $i \geqslant i_0$. Thus $X_\varrho = X_{\varrho_0}$.

THEOREM 2. *Let us suppose that there exists a sequence of indices* $\{\nu_j\}$ *such that* $\sum_{j=1}^{\infty} a_{n_0\nu_j} = \infty$ *for a fixed index* n_0, *and* $\sup_n a_{n\nu_j} \leqslant M$ *for all* j, *where* $M > 0$ *is a constant. Let the φ-functions* φ_i *satisfy* (β′) *and* (δ). *If* $X_\varrho = X_{\varrho_0}$, *then* φ_i *satisfy* (γ′).

Proof. Let us suppose $X_\varrho = X_{\varrho_0}$, but for every $k, c, u_0 > 0$ and every i_0 there exist $0 \leqslant u \leqslant u_0$ and $i \geqslant i_0$ such that $\varphi_i(cu) \geqslant k\varphi_{i_0}(u)$. Given $k > 0$, we choose $c = 2^{-k}$. Then there exist $i_{n,m,k} \geqslant n$ and $u_{n,m,k} \leqslant 1/m$ such that

$$\varphi_{i_{n,m,k}}(2^{-k}u_{n,m,k}) > 2^k\varphi_n(u_{n,m,k}) \tag{7}$$

for $n, m, k = 1, 2, \ldots$ Now, we define an increasing sequence of indices m_k as follows. We choose m_1 so large that $m_1 \geqslant 1/\vartheta_1$ and $\varphi_1(1/m_1) \leqslant \min(1, 1/2M)$, and we put $u_1 = u_{1,m_1,1}$. Let us suppose the numbers $m_1, m_2, \ldots, m_{k-1}$ are defined in such a manner that $\varphi_i(1/m_i) \leqslant 1$, $m_i \geqslant 1/\vartheta_i$ and $\varphi_i(1/m_i) \leqslant \varphi_{i-1}(u_{i-1})$, where $u_i = u_{i,m_i,i}$, $i = 2, 3, \ldots, k-1$. Then we take m_k so large that $\varphi_k(1/m_k) \leqslant \min(1, 1/2^k M)$, $m_k \geqslant 1/\vartheta_k$ and $\varphi_k(1/m_k) \leqslant \varphi_{k-1}(u_{k-1})$, and we put $u_k = u_{k,m_k,k}$. Then $\varphi_k(u_k) \leqslant \varphi_k(1/m_k) \leqslant \varphi_{k-1}(u_{k-1})$ for $k = 2, 3, \ldots$

We show now that there exists a sequence $A_1, A_2, \ldots$ of pairwise disjoint sets of indices such that

$$\frac{1}{2^k} < \sup_n \mu_n A_k \varphi_k(u_k) \leqslant \frac{1}{2^{k-1}} \tag{8}$$

for all k. It is sufficient to construct one set $A_k = \{\nu_{j_1}, \nu_{j_2}, \ldots\}$ in such a manner that ν_{j_1} is arbitrarily large. Let ν_{j_1} be fixed. We shall prove indirectly that such a set A_k exists. In other case, there are three possibilities:

1° the inequality

$$\sup_n \sum_{r=1}^{m} a_{n\nu_{j_r}}\varphi_k(u_k) \leqslant \frac{1}{2^k} \tag{9}$$

holds for all sequences $\{j_r\}$ and all $m \geqslant 1$;

2° there is a sequence $\{j_r\}$ and $m \geqslant 1$ such that (9) holds, but

$$\sup_n \sum_{r=1}^{m+1} a_{n\nu_{j_r}}\varphi_k(u_k) > \frac{1}{2^{k-1}}; \tag{10}$$

3° (10) holds always.

In the case 1°, we take in (9) $j_r = j_1 + (r-1)$ and $m \to \infty$, and the assumption $\sum_{j=1}^{\infty} a_{n_0\nu_j} = \infty$ yields a contradiction.

In the case 2°, there exists an index $n = n_k$ such that

$$\sum_{r=1}^{m} a_{n_k\nu_{j_r}}\varphi_k(u_k) \leqslant \frac{1}{2^k} \quad \text{and} \quad \sum_{r=1}^{m+1} a_{n_k\nu_{j_r}}\varphi_k(u_k) > \frac{1}{2^{k-1}}.$$

Hence $a_{n_k\nu_{j_{m+1}}}\varphi_k(u_k) > 1/2^k$, and so

$$\frac{1}{2^k} \geqslant M\varphi_k\left(\frac{1}{m_k}\right) \geqslant M\varphi_k(u_k) \geqslant a_{n_k\nu_{j_{m+1}}}\varphi_k(u_k) > \frac{1}{2^k},$$

a contradiction.

In the case 3°, the contradiction is obtained as in 2°, taking $m+1 = j_1$. Now, we define $x = \{t_\nu\}$, where

$$t_\nu = \begin{cases} u_k & \text{for } \nu \in A_k,\ k = 1, 2, \ldots, \\ 0 & \text{elsewhere.} \end{cases}$$

Then

$$\varrho_i(\lambda_i x) \leqslant \sup_n \sum_{k=1}^{i-1} \mu_n A_k \varphi_i(\lambda_i u_k) + \sup_n \sum_{k=i}^{\infty} \mu_n A_k \varphi_i(\lambda_i u_k).$$

But, by (8), the first term at the right-hand side of the above inequality is less than or equal to

$$\frac{(i-1)\varphi_i(\lambda_i u_k)}{2^{k-1}\varphi_k(u_k)},$$

and the second one is less than or equal to $\beta_i \sum_{k=i}^{\infty} 2^{-k+1} < \infty$. Hence $\varrho_i(\lambda_i x) < \infty$.

Let us take in (δ), $\varepsilon = c/2\varrho_i(\lambda_i x)$, and let $\lambda_i' > 0$ be so small that $\lambda_i' \leqslant \lambda_i$ and $\lambda_i' u_k \leqslant u_\varepsilon$ for all k; this is possible because $u_k \leqslant 1/m_k$. Let $\varepsilon' = c/2\varrho_i(\lambda_i' x) \geqslant \varepsilon$; then $u_{\varepsilon'} \geqslant u_\varepsilon$, and so $\lambda_i' u_k \leqslant u_{\varepsilon'}$ for all k. Hence, if $0 < \lambda < \alpha_{\varepsilon'} \lambda_i'$, then

$$\varrho_i(\lambda x) \leqslant \varepsilon' \sup_n \sum_{k=1}^{\infty} \mu_n A_k \varphi_i(\lambda_i' |u_k|) = \varepsilon' \varrho_i(\lambda_i' x) = \tfrac{1}{2}c.$$

Hence $x \in X_\varrho$. Now, by (7) and (8),

$$\varrho_{i_k, m_k, k}(2^{-k}x) \geqslant \sup_n \mu_n A_k \varphi_{i_k, m_k, k}(2^{-k}u_k) \geqslant 2^k \varphi_k(u_k) \sup_n \mu_n A_k > 1,$$

and so $x \notin X_{\varrho_0}$. Hence $X_\varrho \neq X_{\varrho_0}$, a contradiction.

Let us remark, that the results of 3.2 and 3.3 in [1] follow from our theorems 1 and 2 if we put $a_{n\nu} = \omega_\nu$ for $n, \nu = 1, 2, \ldots$; in case of 3.3, the assumption $\sum_{k=1}^{\infty} \omega_k = \infty$ must be replaced by $\sum_{r=1}^{\infty} \omega_{j_r} = \infty$, where $\lim_{r\to\infty} \omega_{j_r} = 0$, which was not mentioned in [1].

Theorems 1 and 2 imply the following

THEOREM 3. *Let us suppose that there exists a sequence of indices* $\{\nu_j\}$ *such that* $\lim_{j\to\infty} a_{n_0\nu_j} > 0$ *for a fixed index* n_0, *and* $\sup_n a_{n\nu_j} \leqslant M$ *for all* j, *where* $M > 0$ *is a constant. Let the* φ*-functions* φ_i *satisfy conditions* (α), (β') *and* (δ). *Then* $X_\varrho = X_{\varrho_0}$ *if and only if* φ_i *satisfy* (γ').

2.3. The results of 2.2 may be proved also if we replace the pseudomodulars (6) by means of (4), where the measures are given by (5).

References

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