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Colloquium on
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Diagonals of operators

by

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1. Introduction. Let $\{x_j, x_j'\}$ and $\{y_j, y_j'\}$ be complete biorthogonal sequences in the respective Banach spaces X and Y. For F a continuous linear operator from X into Y the sequence $\Delta(F) = \{y_j'(Fx_j)\}$ will be called the *diagonal* of F. The operator F is said to be a *diagonal operator* if $y_j'(Fx_k) = 0$ for $j \neq k$. In this note the properties and characteristics of two spaces of sequences are discussed: N consisting of all sequences $\Delta(F)$ as F ranges over the space $\mathcal{N}$ of all nuclear operators from X into Y and M consisting of all sequences $\Delta(F)$ as F ranges over the space $\mathcal{D}$ of all diagonal operators from Y into X. Specialized work in this direction appears in [7] where the two complete biorthogonal sequences coincide, in [8] where the biorthogonal sequences are both $\{e_j, E_j\}$ in perfect coordinate spaces and in [9] where the biorthogonal sequences are $\{e_j, E_j\}$ in the spaces of type l^p. Some of the content of my communication is included in [7] and is not repeated here. It is there shown that the nature and relationship of the two sequence spaces M and N are reflected in the summability properties of the series $\sum_j x_j(x)x_j$ as x ranges over X.

A Banach space X in which there is a complete biorthogonal sequence $\{x_j, x_j'\}$ can be identified with space S of sequences through the correspondence of x in X to the sequence $s_x = \{x_j'(x)\}$. Under this correspondence x_j becomes the sequence $\{\delta_{jk}: k = 1, 2, \ldots\}$ which is denoted by e_j and x_j' becomes the j-th coordinate functional which is given by $E_j(s) = s(j)$. Here $s = \{s(1), s(2), \ldots\}$. With the norm

$$\|s_x\|_S = \|x\|$$

S is a Banach space isometric to X. The conjugate space X^* can be identified with the space S' consisting of all sequences $s_{x'} = \{x'(x_j)\}$ as x' ranges over X^*. With the norm

$$\|s_{x'}\|^S = \|x'\|$$

S' is isometric to X^*.

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It thus suffices to consider the case of two Banach spaces of sequences S and T in both of which $\{e_j, E_j\}$ forms a complete biorthogonal sequence. A Banach space of sequences on which each functional E_j is continuous is called a *BK-space*. The space of all sequences is called ω, and the subspace of sequences in ω which are eventually 0 will be called φ. For $s \in \varphi$ and $t \in \omega$, (s, t) is the finite sum $\sum_j s(j)t(j)$. For A a subset of φ, $A^{(\omega)}$ consists of all t in ω such that $|(s, t)| \leqslant 1$ for each s in A. For B a subset of ω, $B^{(\varphi)}$ consists of all s in φ such that $|(s, t)| \leqslant 1$ for each t in B. The respective norms on S, T, S^f and T^f are $\| \|_S$, $\| \|_T$, $\| \|^S$, $\| \|^T$. For s in S and $s' = s'_{x'}$ in $S^f s'(s)$ is the value of x' at s, i.e., $x'(s)$. It is easy to see that if either s or s' is in φ, $s'(s) = (s', s)$. The product of two sequences is defined coordinatewise; thus if $v = st$, $v(j) = s(j)t(j)$ for each j.

The proof of the following statement is essentially the same as the proof of 4.1 of [5] and is omitted:

1.1. *If $\{e_j, E_j\}$ is a complete biorthogonal sequence in the BK-space S, then*

$$S^f = \bigcup_{n=1}^{\infty} nA^{(\omega)}$$

and $A^{(\omega)}$ is the closed unit ball of S^f, where A consists of all s in φ such that $\|s\|_S \leqslant 1$.

2. Diagonals of nuclear operators. In this section and the following S and T denote Banach spaces of sequences in which $\{e_j, E_j\}$ forms a complete biorthogonal sequence. The space N then consists of all sequences of the form

$$v_F = \Delta(F) = E_j(Fe_j)$$

as F ranges over the Banach space $\mathcal{N}$ of all nuclear operators from S into T.

For s' in S^f and t in T,

$$|s'(j)|\,|t(j)| \leqslant \|s'\|^S \|e_j\|_S \|e_j\|^T \|t\|_T.$$

Thus if $\{s'_n\}$ is a sequence in S^f and $\{t_n\}$ is a sequence in T such that

$$\sum_n \|s'_n\|^S \|t_n\|_T < \infty$$

it follows that

$$\sum_n |s'_n(j)|\,|t_n(j)| < \infty \quad \text{for each } j.$$

2.1. Theorem. (a) *The set N consists of all sequences v of the form*

$$v = \sum_{n=1}^{\infty} s'_n t_n, \tag{2.1}$$

where $\{s'_n\}$ is a sequence in S^f and $\{t_n\}$ is a sequence in T such that

$$\sum_{n=1}^{\infty} \|s'_n\|^S \|t_n\|_T < \infty.$$

(The convergence in (2.1) is coordinatewise).

(b) *With the norm*

$$\|v\|_N = \Big\{\sum_{n=1}^{\infty} \|s'_n\|^S \|t_n\|_T \colon \sum_{n=1}^{\infty} s'_n t_n = v;\ \{s'_n\} \subset S^f, \{t_n\} \subset T\Big\}; \tag{2.2}$$

N is a BK-space.

(c) *The biorthogonal sequence $\{e_j, E_j\}$ is complete in N.*

(d) *The mapping from $\mathcal{N}$ onto N defined by $F \to \Delta(F)$ is a continuous linear operator.*

Proof. (a) If v has the form (2.1), define F from S into T by

$$F(s) = \sum_{n=1}^{\infty} s'_n(s) t_n.$$

Then $F \in \mathcal{N}$ and $\Delta(F) = v$.

If $v \in N$, let $v = \Delta(F)$ for F in $\mathcal{N}$. There are sequences $\{s'_n\}$ in S^f and $\{t_n\}$ in T such that $Fs = \sum_n s'_n(s)t_n$ for s in S and $\sum_n \|s'_n\|^S \|t_n\|_T < \infty$. For each j

$$\sum_n s'_n(j)t_n(j) = E_j\Big(\sum_n s'_n(e_j)t_n\Big) = E_j(Te_j) = v(j),$$

so that $\sum_n s'_n t_n = v$. Thus v has the form (2.1).

(b) It follows from (a) that N is a linear space. The proof that $\| \|_N$ is a norm is omitted. To see N is complete with the norm $\| \|_N$ assume $\sum_n \|v_n\|_N < \infty$. Let $\{s'_{nk}\}$ and $\{t_{nk}\}$ be sequences in S^f and T respectively such that $\sum_k s'_{nk}t_{nk} = v_n$ and $\sum_k \|s'_{nk}\|^S \|t_{nk}\|_T < \|v_n\|_N + 2^{-n}$ for $n = 1, 2, \ldots$ Then

$$\sum_n \sum_k \|s'_{nk}\|^S \|t_{nk}\|_T < \infty,$$

so that if v is the coordinatewise sum of $\sum_n \sum_k s_{nk}t_{nk}$, it follows that v is in N. Since

$$\Big\|\sum_{n=1}^{m} v_n - v\Big\|_N \leqslant \sum_{n=m+1}^{\infty} \sum_k \|s'_{nk}\|^S \|t_{nk}\|_T \leqslant \sum_{n=m+1}^{\infty} \|v_n\|_N + 2^{-m};$$

$\sum_n v_n$ converges to v in N.

Each coordinate functional on N is continuous because if $v = \sum_n s_n' t_n$,

$$|v(j)| \leqslant \sum_n |s_n'(j)|\, |t_n(j)| \leqslant \Big(\sum_n \|s_n'\|^S \|t_n\|_T\Big) \|e_j\|_S \|e_j\|^T.$$

(c) Let v in N have the form (2.1). Since $\{e_j, E_j\}$ is a complete biorthogonal sequence in T, φ is dense in T. Let $\{t_{nk}\}$ be sequences in φ such that $\sum_k t_{nk} = t_n$ and $\sum_k \|t_{nk}\|_T < \|t_n\|_T + 2^{-n}$ for each n. Then

$$\sum_n \sum_k \|s_n' t_{nk}\|_N \leqslant \sum_n \sum_k \|s_n'\|^S \|t_{nk}\|_T < \infty$$

and $\sum_n \sum_k s_n' t_{nk} = v$ coordinatewise and thus in N. Therefore, φ is dense in N so that $\{e_j, E_j\}$ is a complete biorthogonal sequence in N.

(d) The proof that Δ is linear is omitted. The norm on $\mathscr{N}$ is given by the formula

$$\|F\| = \inf\Big\{\sum_n \|s_n'\|^S \|t_n\|_T \colon \sum_n s_n'(s) t_n = Fs \text{ for } s \text{ in } S\Big\}. \tag{2.3}$$

If $\sum_n s_n'(s) t_n = F(s)$, then

$$E_j(Fe_j) = \sum_n s_n'(e_j) E_j(t_n) = \sum_n s_n'(j) t_n(j)$$

for each j so that the set in (2.3) is smaller than the set in (2.2). Thus $\|F\| \geqslant \|\Delta(F)\|_N$ for F in $\mathscr{N}$.

For a sequence space S, $M(S)$ called the *multiplier algebra* of S, consists of all u in ω such that $us \in S$ whenever $s \in S$. Multiplier algebras are studied in [5].

2.2. $M(S) \cup M(T) \subset M(N)$.

Proof. Suppose $u \in M(T)$ and $v = \sum_n s_n t_n \in N$, where $\sum_n \|s_n'\|^S \|t_n'\|_T < \infty$. Then $uv = \sum_n u s_n t_n$ and $\sum_n \|s_n'\|^S \|ut_n\|_T < \infty$ because there is $a > 0$ such that $\|ut\|_T \leqslant a\|t\|_T$ for each t in T. See 3.2 of [5]. Therefore $M(T) \subset M(N)$.

By 3.5 of [5], $M(S) \subset M(S^f)$, and by an argument similar to that in the preceeding paragraph $M(S^f) \subset M(N)$.

The following theorem is an immediate consequence of 2.2 above and of Theorem 3 and Corollary 2 of [3]:

2.3. Theorem. *If $\{e_j\}$ is a Schauder (an unconditional) basis for S or T, then $\{e_j\}$ is a Schauder (an unconditional) basis for N.*

3. Diagonal operators. Examples. With S and T as in Section 2, let A consist of all $s \in \varphi$ such that $\|s\|_S \leqslant 1$ and B consist of all $t \in \varphi$ such that $\|t\|_T \leqslant 1$. The space M consists of all sequences of the form

$$u_F = \Delta(F) = E_j(Fe_j)$$

as F ranges over the set of all diagonal operators from T into S. It can be shown that M then consists of all sequences u such that $ut \in S$ whenever $t \in T$, i.e. M is the space called $(T \to S)$ by Goes in [4] and elsewhere.

3.1. *With the norm*

$$\|u\|_M = \sup\{\|ut\|_S \colon \|t\|_T \leqslant 1\} \tag{3.1}$$

M is a BK-space in which the closed unit ball is $(A^{(\omega)} B)^{(\omega)}$.

Proof. Let $\mathscr{L}(T, S)$ denote the Banach space of continuous linear operators from T into S with the uniform norm. It can be verified directly that for each pair i, j the functions defined by $\psi_{ij}(F) = E_i(Fe_j)$ is continuous and linear on $\mathscr{L}(T, S)$. The space $\mathscr{D}$ consists of all F such that $\psi_{ij}(F) = 0$ for $i \neq j$. Thus $\mathscr{D}$ is a closed subspace of $\mathscr{L}(T, S)$. Under the correspondence of F in $\mathscr{D}$ to $\Delta(F)$ in M, M is isomorphic to $\mathscr{D}$. The norm given by (3.1) coincides with the uniform norm on $\mathscr{D}$. Hence M is a Banach space. Each E_j is continuous on M since

$$E_j(u_F) = E_j(Fe_j) = \psi_{ij}(F).$$

By 1.1, $A^{(\omega)}$ is the closed unit ball of S^f. Thus if $\|u\|_M \leqslant 1$,

$$\|ut\|_S = \sup\{|(ut, s')| \colon s' \in A^{(\omega)}\} \leqslant 1$$

so that $|(u, v)| \leqslant 1$ for v in $A^{(\omega)} B$.

If $u \in (A^{(\omega)} B)^{(\omega)}$, then for $s' \in A^{(\omega)}$ and $t \in \varphi$, $|(s', ut)| = |(u, s't)| \leqslant \|t\|_T$ so that $\|ut\|_S \leqslant \|t\|_T$. Since φ is dense in both S and T, there is a continuous extension F_u of the linear operator $t \to ut$ from T into S such that $\|F_u(t)\|_S \leqslant \|t\|_t$. But $F_u t = ut$ for t in T. Thus $\|u\|_M \leqslant 1$.

3.2. $M(S) \cup M(T) \subset M(M)$.

Proof. Let $w \in M(S)$ and $u \in M$. If $t \in T$, $us \in S$, so $wut \in S$, so $wu \in M$. Hence $w \in M(M)$. A similar argument shows $M(T) \subset M(M)$.

From 3.2, Theorem 3 and Corollary 2 of [3] it follows that if $\{e_j\}$ is a Schauder (an unconditional) basis for S or T, then $\{e_j\}$ is a Schauder (an unconditional) basis for M^0, the closed linear span of $\{e_j\}$ in M.

3.3. Theorem. *If $\{e_j\}$ is a Schauder basis for S or T, then $N^f = M$ and $N = M^{f0}$.*

Proof. It is always true that $N^f \subset M$. Let C denote the set of all v in φ such that $\|v\| \leqslant 1$. Then, by 1.1,

$$N^f = \bigcup_{n=1}^{\infty} nC^{(\omega)}.$$

But $A^{(\omega)} B \subset C$, so that $C^{(\omega)} \subset (A^{(\omega)} B)^{(\omega)}$, which implies by 3.9 that $N^f \subset M$.

If $\{e_j\}$ is a basis for S, let $u \in M$ and $v \in N$, and let $v = \sum_n s'_n t_n$, where $\sum_n \|s'_n\|^S \|t_n\| < \infty$. For $s' \in S^f$ and each k let

$$P_k s' = \sum_{j=1}^{k} s'(j) e_j.$$

Then $\sup_k \|P_k\| < \infty$ since $\{e_j\}$ is a basis for S (p. 67 of [1]). Thus

$$\Big|\sum_{j=1}^{k} u(j) v(j)\Big|_k = \Big|\sum_{j=1}^{k} \sum_n s'_n(j) t_n(j) u(j)\Big|$$
$$= \Big|\sum_n \sum_{j=1}^{k} s_n(j) t_n(j) u(j)\Big|$$
$$= \Big|\sum_n P_k s'_n (u t_n)\Big|$$
$$\leqslant (\sup_k \|P_k\|) \|u\|_M \sum_n \|s'_n\|^S \|t_n\|_T.$$

The refore $M = N^f$ by 3.3 of [6].

A similar argument applies if $\{e_j\}$ is a basis for T.

The following table shows M and N for certain familiar spaces of sequences. See [2], p. 239, for a discussion of these spaces. The space M was calculated directly by means of the relation

$$M = \bigcup_{n=1}^{\infty} n (A^{(\omega)} B)^{(\omega)}$$

and N was calculated by 3.3. In particular, observe that cs is not contained in bv so that there are nuclear operators in cs whose diagonals do not map cs into itself. Hence theorems of type 3.1, 3.2 and 3.3 of [8] do not hold for the diagonals of continuous operations from a sequence space into itself even when the space is associated with a basis and the operators are nuclear.

S	c_0	l^1	l^p	l^q	cs	bv_0	cs
T	l^1	c_0	l^q	l^p	cs	bv_0	l^p
$M: T \to S$	m	l^1	$l^{pq/(p+q)}$	m	bv	bv	l^q
$N: S \to T$	l^1	c_0	$l^{pq/(pq-p-q)}$	l^1	cs	cs	l^p
			$(1 \leqslant p < q < \infty)$	$(1 \leqslant p \leqslant q < \infty)$			$\left(\frac{1}{p} + \frac{1}{q} = 1\right)$

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