(a) *$\tilde{\mathscr{H}}$ is a $\tilde{\mathscr{A}}$-module.*

(b) *$\mathscr{A}$ is isomorphic to the algebra of all global sections in $\tilde{\mathscr{A}}$.*

(c) *$\mathscr{H}$ is isomorphic to the vector space of all global sections in $\tilde{\mathscr{H}}$.*

(d) *The two isomorphisms are consistent with the action of $\mathscr{A}$ on $\mathscr{H}$, and that of $\tilde{\mathscr{A}}$ on $\tilde{\mathscr{H}}$.*

The properties of the local algebras involved in the representation are then studied.

STUDIA MATHEMATICA, T. XXXVIII. (1970)

Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

III. UNSOLVED PROBLEMS

1. The operator ideals and measures in linear spaces

Problem 1. Let (i, H, B) be an abstract Wiener space, let g be a bounded continuous real-valued function on B with bounded support and suppose that g is either Lip a on B or vanishes in a neighborhood of the origin. Then its potential u has the second order Fréchet derivative $D^2u(0)$ at the origin and $D^2u(0)$ is of Hilbert-Schmidt type, but not necessarily nuclear (see Leonard Gross, *Potential theory on Hilbert space*, J. Funct. Analysis 1 (1967), p. 123, and, in particular, Theorem 2, p. 167).

Find a class $\mathfrak{A}$ of operators in H which includes the nuclear operators and define an extension $\mathcal{T}$ of the trace operation to $\mathfrak{A}$ such that $\mathfrak{A}$ contains all the operators $D^2u(0)$ arising as above and such that

$$\mathcal{T}(D^2u(0)) = -2g(0).$$

It would be nice if $\mathfrak{A}$ were an orthogonally invariant class, but, if necessary, it may depend on the choice of (i, H, B).

N. Aronszajn and L. Gross

Let L_0 be the space of all Lebesgue measurable functions on the unit interval with the topology of asymptotic convergence.

Problem 2. Let $A: L_\infty \to L_0$ be a continuous linear operator. Can A be factorized through a Hilbert space, i.e. can we write $A = C \circ B$, where $B: L_\infty \to L_2$ and $C: L_2 \to L_0$ are continuous linear operators?

Problem 3. Need every Banach subspace of L_0 be also a subspace of L_1?

S. Kwapień

Problem 4. Let X be a locally compact Hausdorff space, and $\mu \geqslant 0$ a Radon measure on X.

For $1 \leqslant p \leqslant \infty$ let $L^p_{\mathrm{loc}}(X, \mu)$ denote the space of all complex functions f on X for which $f|_K \in L^p(K, \mu)$ for all compact $K, K \subset X$. (We identify here functions which are equal locally almost everywhere with respect to μ.)

Equip $L^p_{\mathrm{loc}}(X, \mu)$ with the projective limit topology with respect to the spaces $L^p(K, \mu)$, $K \subset X$, K compact.

Give a necessary and sufficient condition for $L^p_{\mathrm{loc}}(X, \mu)$ to be barrelled (*tonnellé*).

Remarks. The following condition is necessary:

For all $A \subset X$ with the properties

1. A is measurable, $\mu(A) < \infty$;
2. for each $K \subset X$, K compact and $\mu(A \setminus K) > 0$, there is an element $f \in L^p_{\mathrm{loc}}(X, \mu)$ such that $f|_A \notin L^p(A, \mu)$.

A sufficient condition is that X is paracompact.

Example. Let N denote the set of natural numbers with the discrete topology, $\check{N}$ its Stone-Čech compactification. Take $\omega \in \check{N} \setminus N$ and define $X = \check{N} \setminus \{\omega\}$. We define a Radon measure μ on X by

$$\mu(A) = \sum_{n \in A \cap N} 2^{-n}, \qquad A \subset X.$$

It can be shown that $L^p_{\mathrm{loc}}(X, \mu)$ is *not* barrelled.

N. J. Nielsen

Problem 5. Let X be a non-nuclear Fréchet space. Does there exist a sequence (x_n) in X and $a \in [1, 2]$ such that $\sum |x^*(x_n)| < +\infty$ for every bounded linear functional x^* on X, but $\sum [p(x_n)]^a = +\infty$ for some (continuous) seminorm p on X?

The positive answer to this problem will fill up the gap between the Dworetzky-Rogers theorem and Grothendieck's characterization of Fréchet nuclear spaces as those in which classes of absolutely convergent series and unconditionaly convergent ones coincide.

Definition. An operator $A: X \to Y$ (X and Y are Banach spaces) is $(a, 1)$-*absolutely summing* if there exists a constant $C > 0$ such that

$$\Big(\sum_n \|A(x_n)\|^a\Big)^{1/a} \leqslant C \sup_{\|x^*\| \leqslant 1} \sum_n |x^*(x_n)|$$

for each sequence (x_n) in X.

Problem 6. Find a composition formula for $(a, 1)$-absolutely summing operators.

Conjecture. If $A \in \Pi_{a,1}(X, Y)$, $B \in \Pi_{\beta,1}(Y, Z)$, $1 \leqslant a \leqslant 2$ and $1 \leqslant \beta \leqslant 2$, then

$$BA \in \Pi_{\gamma,1}(X, Z), \qquad \text{where } \left(\frac{1}{\gamma} = \min \frac{1}{a} + \frac{1}{\beta} - \frac{1}{2}, 1\right).$$

Problem 7. Show that the operator $\Sigma: l_1 \to c$ defined by

$$\Sigma((t_n)) = \Big(\sum_{i=1}^{n} t_i\Big) \qquad \text{for } (t_n) \in l_1$$

cannot be represented as $\Sigma = BA$, where $A \in \Pi_{a,1}(l_1, X)$, $B \in \Pi_{\beta,1}(X, C)$ for some Banach space X and for some $a, \beta \in [1, 2)$.

It was proved recently that $\Sigma \in \Pi_{(a,1)}(l_1, c)$ for every $a > 1$ (cf. S. Kwapień and A. Pełczyński, *The main triangle projection*, Studia Math. 34 (1969), p. 43-67).

Problem 8. Is every $(a, 1)$-absolutely summing operator for $1 \leqslant a < 2$ a Dunford-Pettis operator, i.e. one which carries weakly convergent sequences into norm convergent sequences?

Problem 9. Let X be a Banach space such that either there exists an infinite-dimensional Banach space Y with the property

(a) $\Pi_{1,1}(Y, X) = B(Y, X)$, where $B(Y, X)$ denotes the space of all bounded linear operators from Y into X, or

(b) the natural injection $l_1 \hat{\otimes} X \to l_2 \hat{\hat{\otimes}} X$ is an isomorphism.

Is X isomorphic to a Hilbert space?

The part (b) of the above problem is posed by A. Grothendieck in the paper *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Math. Sao Paulo 8 (1956).

If Y has an unconditional basis (more generally, sufficiently many Boolean algebras of projections in the sense of Lindenstrauss and Zippin, see J. Lindenstrauss and M. Zippin, *Banach spaces with sufficiently many Boolean algebras of projections*, J. Math. Analysis and Appl.), then the answer to Problem 9a is positive. If X has either an unconditional basis or sufficiently many Boolean algebras of projections, then the answer to 9b is positive.

Problem 10. Let $1 \leqslant p_1 < p_2 \leqslant +\infty$ and let $T: L_{p_2} \to L_{p_1}$ be a bounded linear operator. Does there exist, for every q with $p_2 \geqslant q \geqslant p_1$, a factorization $T = BA$ with A and B bounded linear operators $L_p \to L_q$ and $L_q \to L_{p_1}$, respectively?

It is proved (see J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in L_p-spaces and their applications*, Studia Math. 29 (1968), p. 275-326) that the answer to Problem 10 is "yes" if $p_2 \leqslant 2 \leqslant p_1$.

A. Pełczyński

Ein Operatorenideal $\mathfrak{A}$ heisst *klein*, wenn aus $\mathfrak{A}(E, F) = \mathscr{L}(E, F)$ stets $\dim E < \infty$ oder $\dim F < \infty$ folgt.

Problem 11 (Grothendieck). Ist $\mathfrak{N}$ (Ideal der nuklearen Operatoren) klein?

Sei $\mathfrak{S}_p^{\mathrm{app}}$ das Ideal aller Operatoren T mit

$$\sum [a_n(T)]^p < +\infty,$$
$$a_n(T) = \inf\{\|T-A\|:\ \dim R(A) < n\}.$$

SATZ. *Das Ideal $\mathfrak{S}_p^{\mathrm{app}}$ ist klein für $p \leqslant 1/2$.*

Problem 12. Ist $\mathfrak{S}_1^{\mathrm{app}}$ klein? $(\mathfrak{S}_1^{\mathrm{app}} \subset \mathfrak{N})$

Problem 13. Welche anderen Operatorenideale sind klein?

Wir betrachten die folgende Verallgemeinerung der streng singularen und cosingularen Operatoren:

$T \in \mathfrak{S}(E, F) \Leftrightarrow$ wenn $BTA: E_0 \xrightarrow{A} E \xrightarrow{T} F \xrightarrow{B} F_0$ für $A \in \mathscr{L}(E_0, E)$ und $B \in \mathscr{L}(F, F_0)$ ein Isomorphismus ist, so gilt $\dim E_0 = \dim F_0 < \infty$.

Problem 14. Ist $\mathfrak{S}$ ein Ideal? $\big(S, T \in \mathfrak{S}(E, F) \rightarrow S+T \in \mathfrak{S}(E, F)\big)$

A. Pietsch

Problem 15. La composée d'un opérateur p-radonifiant et d'un opérateur q-radonifiant, est-elle r-radonifiante, $1/r = 1/p+1/q$, si $r < 1$?

L. Schwartz

Problem 16. Si E et F sont espaces de Banach, un opérateur p-radonifiant de E dans F, avec $p < 1$ est-il 0-radonifiant?

S. Kwapień and L. Schwartz

Problem 17. Soit (λ_n) une suite de nombres complexes telle que $\sum |\lambda_n|^2 < +\infty$. Existe t-il un opérateur nucléaire T, tel que le spectre de T soit la suite (λ_n)?

Problem 18. Si la réponse au Problem 16 est *non*, caracteriser les suites (λ_n) qui sont spectre d'un opérateur nucléaire.

P. Saphar

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2. Schauder bases and linear topological invariants

Etant donnée une application diagonale $T: (x_n)_n \rightarrow (b_n x_n)_n$ de l_p dans l_q $(p, q \in [1, \infty])$.

Problem 19. Quelle est la valeur de $r(TU_p, U_q)$, quand $p \leqslant q$ et $\lambda(b_n) = +\infty$?

Problem 20. Si $1 \leqslant p \leqslant q$ et $\lambda(b_n) < +\infty$, a-t-on $r(TU_p, U_q)$ égal à l'indice de convergence de $d_n(TU_p, U_q)$ (resp. à l'indice de convergence de $a_n(T)$-définition de Pietsch des coefficients d'approximation de T par des opérateurs de rang au plus n)?

Problem 21. Si $\sum\limits_n d_n^r(TU_p, U_q) < +\infty$ a-t-on aussi $\sum\limits_n a_n^r(T) < +\infty$ (tout ou moins si $1 \leqslant q \leqslant p$)?

Problem 22. Peut on exprimer simplement $d_n(TU_p, U_q)$ et $a_n(T)$ en fonction de n et des b_k?

Problem 23. Déterminer une condition nécessaire et suffisante sur les b_n pour que T soit une applications de type k au sens de Pietsch (c.-à.-d. $\sum\limits_n a_n^k(T) < \infty$ si $k < \infty$).

(U_p est la boule unité de l_p, U_q celle de l_q.)

Problem 24. Sous l'hypothèse (i) (cf. sommaire, p. 439) a-t-on $r(T) = \lambda d_n(T)$ ou même $r(T) = \lambda a_n(T)$?

S. Chevet

Problem 25. Let B be a compact absolutely convex set in Hilbert space H. Is

$$r(B) = \frac{-2}{1+2EV(B)},$$

whenever $r(B) < 2$ or $EV(B) < -1$?

Here $r(B)$ is the exponent of entropy and $EV(B)$ the exponent of volume, as defined in R. Dudley, *The sizes of compact subsets of Hilbert space and continuity of Gaussian processes*, J. Funct. Analysis 1 (1967), p. 290-330.

R. Dudley

Problem 26. Give an example of a separable barrelled locally convex space which is sequentially complete but not complete. Such a space cannot have a basis.

D. J. H. Garling and N. J. Kalton

A sequence (x_n) is said to be *regular* if there exists a neighborhood V of 0 such that $x_n \notin V$ for all n. A bounded regular sequence is called *normalized*. If there exist scalars α_n such that $(\alpha_n x_n)$ is normalized, then (x_n) is said to be *normal*.

Problem 27. Let E be a locally convex space in which every Schauder basis sequence is normal. Is E normed?

Problem 28. Let E be a non-normed linear locally convex space with a Schauder basis. Does E possess an abnormal Schauder basis?

Problem 29. Classify locally convex spaces E which have Schauder bases and are such that every Schauder basis in E is abnormal.

N. J. Kalton

Let H be a Hilbert space, let A be strictly positive operator, $A \geqslant 1$, and let

$$E = \bigcap_{n>0} D(A^n)$$

be the intersection of the domains of the powers A^n with the topology induced by the system of norms

$$\|x\|_n = (A^n x, A^n x)^{1/2}, \quad n \geqslant 0.$$

Problem 30. Is it true that a complemented subspace X in E has the same structure, i.e. $X \simeq \bigcap_{n>0} D(B^n)$, where B is a strictly positive operator in a Hilbert space G?

Let S_1, S_0 be convex symmetric sets in a linear space E. Put:

$$b_k(S_1, S_0) = \sup_{\dim E = k} \sup \{\beta : \beta S_0 \cap E \subset S_1\},$$

$$r_k(S_1, S_0) = \sup_{\dim E = k} \frac{\mathrm{Vol}_k(S_1 \cap E)}{\mathrm{Vol}_k(S_0 \cap E)},$$

$$\beta(S_1, S_0) = \overline{\lim} \left(\frac{\log b_k}{\log k}\right), \qquad \varrho(S_1, S_0) = \overline{\lim} \left(\frac{\log r_k}{k \log k}\right).$$

It is obvious that $b_k \leqslant r_k$ and $\beta(S_1, S_0) \leqslant \varrho(S_1, S_0)$.

Problem 31. Is it true that $\varrho(S_1, S_0) \leqslant \beta(S_1, S_0)$ or $\varrho \leqslant A\beta$, where A is an absolute constant?

It is easy to prove that $\varrho \leqslant \beta + 1$ (see B. Mitjagin, Mathem. zam. 5:1(1969), p. 99-106).

B. Mitjagin

Since every nuclear Fréchet space with a basis is isomorphic to a Köthe space, we shall often formulate our problems for a nuclear Köthe space instead of nuclear Fréchet space with a basis.

We do not know whether every nuclear Fréchet space has a Schauder basis. We do not know even whether every nuclear Fréchet space has the approximation property (Grothendieck). Those problems seem to be very difficult. The following one seems to be less difficult:

Problem 32 (conjecture — M. I. Kadec and A. Pełczyński, see Studia Math. 25 (1965), p. 315, Remark 1). A Fréchet space X is nuclear if (and only if) every closed linear subspace of X with a basis is nuclear.

The substantioning this conjecture would allow us to formulate Wojtyński's characterizations of nuclearity of a Fréchet space for arbitrary Fréchet spaces, not only for Fréchet spaces with a basis (cf. W. Wojtyński, *On conditional bases in Fréchet spaces*, Studia Math. 35 (1970)). In particular, the following is unknown:

Problem 33. Is a Fréchet space X nuclear if (and only if) every basis for each closed linear subspace of X is unconditional?

An analysis of Wojtyński's proof indicates that an essential step toward an answer to Problem 33 will be the positive answer to the following:

Problem 34. Let (x_n) be a normalized basis in a Banach space X. Does there exist a conditional normalized basis (y_n) in X which satisfies the following conditions:

1° $y_n = \sum_{i \leqslant n} c_i x_i$ for $n = 1, 2, \ldots$;

2° there exist constants $C > 0$, $\alpha > 0$ and a sequence $\eta = (\varepsilon_n)$ with $\varepsilon_n = \pm 1$ such that

$$\|P_\eta^n\| \geqslant C n^\alpha,$$

where

$$P_\eta^n(x) = \sum_{i=1}^{n} \varepsilon_i f_i(x) y_i \quad \text{for } x = \sum_{i=1}^{\infty} f_i(x) y_i ?$$

The next few problems are related to the Dragilev-Mitjagin theorem which roughly speaking, asserts that in "nice" nuclear Köthe spaces all bases are quasi-equivalent.

Problem 35. Give an example of a nuclear Köthe space in which there exist two non-quasi-equivalent bases.

(Bases (x_n) and (y_n) are said to be *quasi-equivalent* if there exist a sequence of scalars (a_n) and a permutation $n \to \sigma(n)$ of the set of integers, such that the mapping $x_n \to a_n y_{\sigma(n)}$ defines a linear homeomorphism of the space.

Problem 36. Suppose that X and Y are nuclear Köthe spaces in which all bases are quasi-equivalent. Have the spaces $X \times Y$ and $X \hat{\otimes} Y$ the same property?

In particular, if $\mathscr{H}$ is the space of holomorphic functions in the open unit disc $\{z: |z| < 1\}$, and $\mathscr{E}$ is the space of entire functions, is it true that in the space $\mathscr{E} \hat{\otimes} \mathscr{H}$ all bases are quasi-equivalent?

As I have learned from prof B. Mitjagin, Zacharjuta proved very recently that in the space $\mathscr{E} \times \mathscr{H}$ all bases are quasiequivalent.

The next problem is related to the recent result of C. Bessaga and B. Mitjagin.

Problem 37. Let Y be a complemented subspace of a nuclear Fréchet space with a basis. Does Y have a basis?

In particular, suppose that Y is a complemented subspace of a Köthe space in which all bases are quasi equivalent. Does there exist a basis in Y?

Problem 38. Does there exist a nuclear Köthe space F such that every nuclear Köthe space is isomorphic to a complemented subspace of F?

Problem 39. Does there exist a nuclear Fréchet space F such that every nuclear Fréchet space is isomorphic to a complemented subspace of F?

Problem 40. Does there exist a nuclear Fréchet space G such that every nuclear Fréchet space is isomorphic to a quotient space of G?

In particular, has this property the Komura's universal space of differentiable scalar-valued functions on the real line?

In connection with Problems 38 and 39 let me make the following digression: there exists a separable Banach space (with a basis), say U, such that every separable Banach space with a basis is isomorphic to a complemented subspace of U (cf. A. Pełczyński, *Universal bases*, Studia Math. 32 (1969), p. 247-268). It is thus natural to ask the following question:

Problem 41. Does there exist a separable Banach space P such that every separable Banach space (resp. every separable Banach space with a metric approximation property) is isomorphic to a complemented subspace of P?

I believe that the answer to Problem 41 is negative and this, together with the result mentioned above, would lead to the negative solution of the basis problem. It would be a non-constructive solution. It seems to be interesting to investigate the following family of spaces:

Let M be a subset of the integers. Let $[e^{int}]_M$ denote the closed linear span in $C[0, 2\pi]$ of the functions $[e^{int}]_{n \in M}$.

Problem 42. Does there exist a separable Banach space P_{spec} such that for every set M of the integers the space $[e^{int}]_M$ is isomorphic to a complemented subspace of P_{spec}?

A. Pełczyński

Let X be a linear metric complete space (F-space in the Banach terminology). We say that the space X is *locally pseudoconvex* if there are a basis of neighborhoods of zero $\{U_n\}$ and a sequence of positive constants $\{c_n\}$ such that

$$U_n + U_n \subset c_n U_n.$$

If, additionally, $c_n \leqslant 2^{1/p}$, then the space X is called *locally p-convex*. Spaces locally 1-convex are called briefly *locally convex*.

Let A and B be two sets in X and let L be a subspace of X. Write

$$d(A, B, L) = \inf\{a > 0 : A \supset L + aB\}$$

and

$$d_n(A, B) = \inf\{d(A, B, L) : \dim L = n\}.$$

We say that the space X is *nuclear* if for any balanced neighborhood of zero B there is a balanced neighborhood of zero A such that

$$\lim_n n d_n(A, B) = 0.$$

If the space X is locally convex, then the definition given above is equivalent to the usual one (see Б. Митягин, *Аппроксимативная размерность и базисы в ядерных пространствах*, УМН 16 (1961), p. (63-132).

The space $l^{(p_n)}$ is nuclear provided (p_n) is decreasing and $p_n \log n \to 0$ (see S. Rolewicz, *On the characterization of Schwartz spaces by properties of the norm*, Studia Math. 20 (1961), p. 87-92). Another example of a non-locally pseudoconvex nuclear space is given by Ch. Fenske and E. Schock in *Nuklearität und lokale Konvexität von Folgenräumen*, Math. Nachr. (in print).

Problem 43. Let X be a nuclear F-space. Is each bounded set in X compact?

This is the case for locally pseudoconvex spaces.

Problem 44. Do there exist non-trivial linear continuous functionals in each nuclear space?

Problem 45. Is each locally pseudoconvex nuclear space locally convex?

It is so for locally p-convex spaces with bases.

Let U be a balanced neigborhood of zero. We write

$$\|x\|_U = \inf\left\{t>0: \frac{x}{t}\in U\right\}.$$

Problem 46. Suppose that for any unconditionally convergent series $\sum x_n$ and for any open balanced set U, the series $\sum \|x_n\|_U$ is convergent.

Is this equivalent to the fact that the space X is nuclear?

It is so for locally convex spaces (see A. Grothendieck, *Produit tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1965)).

Let

$$d(A, B) = \left\{(t_n):\ \lim \frac{t_n}{d_n(A,B)} = 0\right\}$$

and

$$\delta(X) = \bigcup_A \bigcup_B d(A, B),$$

where the union is taken over all compact sets A and open sets B.

Problem 47. Let Y be a subspace of a nuclear space X. Does the inclusion $\delta(Y)\subset\delta(X)$ hold?.

It is so for locally convex spaces.

S. Rolewicz

Problem 48. Let E be a locally convex space, $U(E)$ a basis of neighborhoods of zero. Then the diametral dimension $\Delta(E)$ is the set of all positive sequences $\delta = (\delta_n)$ with the following property:

For each $u\in U(E)$ there exists a $v\in U(E)$ with $v<u$ and

$$\lim_n \delta_n(v, u)\,\delta_n^{-1} = 0$$

($\delta_n(v, u)$ is the n-th diameter of v with respect to u).

Characterize the spaces E with the following property:

For each $\delta\in\Delta(E)$, $\gamma\in\Delta(E)$, there exists $\beta\in\Delta(E)$ with

$$\beta_n \leqslant \delta_n\gamma_n$$

All spaces with regular bases (Dragilev) have this property.

E. Schock

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3. Various problems

Problem 49. Let H be a Hilbert space, A a continuous self-adjoint operator in H, and let $\mathscr{B}$ denote the Banach space of all continuous operators in H. Operators from $\mathscr{B}$ into $\mathscr{B}$ are called *transformations*. We shall discuss the transformation τ defined by

$$\tau(X) = AX - XA \qquad (X\in\mathscr{B}).$$

Let

$$\mathfrak{N}_\tau = \{X\in\mathscr{B}:\quad \tau(X) = 0\},$$
$$\mathfrak{R}_\tau = \{Y\in\mathscr{B}:\ Y = \tau(X),\quad X\in\mathscr{B}\}.$$

If H is finite-dimensional, then $\mathscr{B} = \mathfrak{N}_\tau \oplus \mathfrak{R}_\tau$.

Indeed, $\mathscr{B}$ equipped with the scalar product $\{X, Y\} = \operatorname{Trace} X\circ Y^*$ is a Hilbert space, τ is then selfadjoint, and thus each operator from $\mathscr{B}$ can be represented uniquely as a sum of two operators; one of them commutes with A and the other is of the form $AX-XA$, $X\in\mathscr{B}$. This decomposition may be written explicitly: if

$$A = \sum_{k=1}^{n} \lambda_k P_k,$$

the spectral decomposition of A, then

$$Y = Y_1 + (AY_2 - Y_2A),$$

where

$$Y_1 = \sum_{k=1}^{n} P_k Y P_k, \qquad Y_2 = \sum_k \sum_{j\neq 1} \frac{P_k Y P_j}{\lambda_k - \lambda_j}.$$

It is not difficult to generalize this result to the Hilbert-Schmidt operators.

What can be said in general case?

J. Daleckiĭ

Problem 50. Let X and Y be Banach spaces with the Dunford-Pettis property. Does this imply that $X\otimes Y$ and $X\hat{\hat{\otimes}} Y$ also have the Dunford-Pettis property?

J. Dobrakov

Problem 51. If X is Hausdorff and not necessarily compact, is $C_b(X, Q)$ Q-uniform?

Problem 52. Characterize operators $T \in L(H, H)$ such that either

(a) $R_T \equiv R\ (I, T, T)$ is Q uniform

or

(b) $R_T \equiv C\ (X, Q)$ for some simple Q.

B. Gelbaum

S. Havinson, 1951, and V. Pták, 1958, proved the existence of a metric projection $Q: L^1(S) \to H^1$, i.e. for a function $f \in L^1(S^1)$,

$$f \sim \sum_{-\infty}^{+\infty} f_k e^{ikt},$$

here exists a unique function $Qf = g_0 \in H^1$,

$$g_0 \sim \sum_{0}^{\infty} g_k e^{ikt},$$

such that

$$\|f - g_0\|_{L_1} = \inf_{g \in H^1} \|f - g\|.$$

Problem 53. Estimate the modulus of continuity of this non-linear operator Q. In particular, is it true that

$$\|Qf_1 - Qf_2\| \leqslant M \log \frac{1}{\|f_1 - f_2\|}, \qquad \|f_1\|,\ \|f_2\| \leqslant 1,$$

where M is an absolute constant?

Problem 54. Let $C^p(S^n)$ denote the Banach space of all p times continuously differentiable functions on the n-dimensional sphere.

Can the spaces $C^p(S^n)$ (for different pairs (p, n), $p \geqslant 1$, $n \geqslant 2$) be isomorphic?

It has been proved that a space of this type is not isomorphic to the $C(S^n)$.

Let D be a domain of holomorphity in C^n, and

$$D^m = \{z \in C^m:\ |z_j| \leqslant 1,\ j = 1, 2, \ldots, m\},$$

$$B^n = \left\{z \in C^m:\ \sum_{j=1}^{n} |z_j|^2 \leqslant 1\right\}.$$

Let $A(D)$ denote the Banach space of all functions holomorphic in the interior of D and continuous in closure $\bar{D}$, with the norm

$$\|f\| = \sup\{|f(z)|:\ z \in D\}.$$

Problem 55. Can the spaces $A(D^m)$ (for different $m \geqslant 2$) be isomorphic? Or prove that these spaces are pairwise non-isomorphic.

Problem 56. Can the spaces $A(B^n)$ (for different $n \geqslant 1$) be isomorphic? Or prove that these spaces are pairwise non-isomorphic.

It has been proved that a space of the first type is not isomorphic to a space of the second type (cf. authors abstract p. 267).

G. Henkin

Problem 57. The terminology will be that of Nachbin's *Topology on spaces of holomorphic mappings*, Berlin 1969.

Let $\mathscr{H}(E)$ be the vector space of all entire complex-valued functions on the complex normed space. Let $\mathscr{H}_b(E)$ be the vector subspace of those f in $\mathscr{H}(E)$ bounded on bounded subsets. The natural topology $\mathscr{T}_b$ on $\mathscr{H}_b E$ is defined by the family of seminorms

$$f \to \sup_{\|z\| \leqslant r} |f(z)| \quad \text{for every } r \geqslant 0.$$

Three natural topologies are considered on $\mathscr{H}(E)$. Firstly, $\mathscr{T}_0$ is defined by the family of seminorms

$$f \to \sup_{z \in K} |f(z)|$$

for every $K \subset E$ compact. Secondly, $\mathscr{T}_\infty$ is defined by the family of seminorms

$$f \to \sup_{z \in K} \|d^m f(z)\|$$

for every $K \subset E$ compact and $m = 0, 1, 2, \ldots$ Thirdly, $\mathscr{T}_\omega$ is defined by the family of seminorms p, each of which is ported by some compact subset $K \subset E$ in the sense that, for any neighborhood V of K, there is $C(V) > 0$ such that

$$p(f) \leqslant C(V) \sup_{z \in V} |f(z)| \quad \text{for any } f \in \mathscr{H}(E).$$

It is known that $\mathscr{T}_0 = \mathscr{T}_\infty = \mathscr{T}_\omega$ if $\dim E < \infty$, and $\mathscr{T}_0 < \mathscr{T}_\infty < \mathscr{T}_\omega$ if $\dim E = \infty$.

An exponential-polynomial on E is a function pe^φ, where p is a continuous complex-valued polynomial on E and $\varphi \in E'$.

A convolution operator θ on $\mathscr{H}_b(E)$, resp. $\mathscr{H}(E)$, is a linear mapping on that space which commutes with translations on it, and is continuous for $\mathscr{T}_b$ resp. $\mathscr{T}_\omega$, or $\mathscr{T}_\infty$, or $\mathscr{T}_0$.

Prove that $\theta^{-1}(0)$ is the closure of its vector subspace of all finite sums of exponential-polynomials lying in $\theta^{-1}(0)$.

If $\dim E < \infty$, see B. Malgrange, Ann. Inst. Fourier 6 (1955-6). For arbitrary E and $\mathscr{H}_b(E)$, resp. $\mathscr{H}(E)$ replaced by the nuclear cases $\mathscr{H}_{Nb}(E)$, resp. $\mathscr{H}_N(E)$, see C. P. Gupta, Univ. of Rochester thesis, 1966, published in Notas de Matematica # 37, Rio de Janeiro (1968), and L. Nachbin, Proc. Conference on Functional Analysis and Related Fields, Univ. of Chicago, 1968, Springer Verlag (1969). Since the Borel transform on $\mathscr{H}'_b(E)$, resp. $\mathscr{H}'(E)$, is not one-to-one if $\dim E = \infty$, it may be convenient to replace $\mathscr{H}_b(E)$, resp. $\mathscr{H}(E)$, by $\mathscr{H}_{bc}(E)$, resp. $\mathscr{H}_c(E)$, by imposing that differentials be compact homogeneous polynomials at any point.

L. Nachbin

Problem 58. Let X be a compact absolutely convex subset of a topological vector space. Is the induced topology of X locally convex? Is this topology locally convex when the topological vector space is locally pseudo-convex (for the definition of a pseudo-convex space see Problem 43).

N. T. Peck and L. Waelbroeck

Problem 59. H^∞ is the space of bounded analytic functions on the open disc. In the sense of isomorphic isometry does H^∞ have a unique predual?

P. Porcelli

Problem 60. Let L^ϱ be a normed function space on the unit interval with Lebesgue measure (for definitions see author's paper p. 1131-185).

Does every closed subspace of this space have a quasi-complement? ($A \subset L^\varrho$, a closed subspace is said to have a *quasi-complement* if there exists a closed subspace $B \subset L^\varrho$ such that $A \cap B = \{0\}$ and $A+B$ is dense in L^ϱ)?

M. M. Rao

Problem 61. Let X, Y be Banach spaces. It is known that every strictly cosingular operator is a Φ-admissible perturbation. But it is not known whether there exists a Φ-admissible perturbation which is not strictly consingular.

Problem 62. Compare the following properties of an open bounded set Ω in R^m:

(a) $b^\infty(\bar{\Omega})$ is nuclear;

(b) $b^\infty(\bar{\Omega})$ is isomorphic to s;

(c) there are constants p, n, C such that

$$\|f\|_{L^\infty(\bar{\Omega})} \leqslant C\|f\|_{W^{n,p}(\Omega)}$$

(the norm in Sobolev's space);

(d) there are constants C, q such that for any k, any polynomial P of degree k, and any j $(1 \leqslant j \leqslant m)$, we have

$$\left\|\frac{\partial^j}{x_j}P\right\| \leqslant Ck^q\|P\|$$

(various forms of this property, according to the norm $\|\cdot\|$);

(e) extension properties.

M. Zerner

Problem 63. Let $\mathscr{K}$ denote the collection of all subsets of the complex plane C^1 which are countable unions of pairwise disjoint disks

$$D_n = \{z \in C^1\colon\ |z-a_n| < \varrho_n\}$$

with $a_n \to \infty$.

Is it true that among the spaces $H(C^1 \setminus K)$, $K \in \mathscr{K}$, there is a continuum of different isomorphic types?

V. Zakharjuta