STUDIA MATHEMATICA, T. XXXVIII. (1970)

Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

II. SHORT SUMMARIES

Asterisk after the name of the author indicates
that the summary was delivered by title

p-ellipsoïdes de l^q
Mesures cylindriques gaussiennes

par

SIMONE CHEVET (Clermont Ferrand)

Dans ce qui suit, on considérera des espaces de type l^p ($p \in [1, \infty]$). La boule unité de l^p sera notée par U_p et sa norme par N_p.

1. p-ellipsoïdes de l^q. On se donne deux nombres quelconques p et q dans $[1, \infty]$ et une application diagonale T de l^p dans l^q soit T: $(x_n)_n \to (b_n x_n)_n$. On désignera par α le nombre tel que

$$\frac{1}{\alpha} = \frac{1}{p} - \frac{1}{q}.$$

Définition. Dans le cas où $b_n \downarrow 0$, TU_p sera dit *p-ellipsoïde de l^q d'axes b_n.*

Notations utilisées. $d_n(T)$ désignera la n-ième épaisseur $d_n(TU_p, U_q)$ de TU_p dans l'espace de Banach l^q, $r(T)$ l'exposant d'entropie $r(TU_p, U_q)$ de TU_p dans l'espace de Banach l^q (cf. par exemple définition de Mitjagin [4] ou de Lorentz [3]), $\alpha_n(T)$ le coefficient d'approximation de T par des opérateurs linéaires continus de rang au plus n (au sens de Pietsch [5]).

D'autre part, si α_n est une suite de réels positifs tels que $\alpha_n \downarrow 0$, $\lambda(\alpha_n)$ désignera l'indice de convergence de cette suite

On établit les résultats suivants:

Si $b_n \downarrow 0$, *on a*:

1. *Si* $p \leqslant q$, $(n+1)^{1/\alpha} b_{n+1} \leqslant d_n(T) \leqslant \alpha_n(T) \leqslant b_{n+1}$.
   Si $p \geqslant q$, $b_{n+1} \leqslant d_n(T) \leqslant \alpha_n(T) \leqslant N_\alpha[(b_i)_{i>n}]$.
2. *Sous l'une des deux hypothèses*

(i) $$1 \leqslant p \leqslant q \quad \textit{et} \quad \lambda(b_n) < +\infty,$$

(ii) $$1 \leqslant q \leqslant p$$

on a

$$\frac{1}{r(T)} = \frac{1}{\lambda(b_m)} - \frac{1}{\alpha}.$$

De plus, sous l'hypothèse (ii):

$$r(T) = \lambda\big(d_n(T)\big) = \lambda\big(\alpha_n(T)\big).$$

3. *Si $q = 2$, on a sous chacune des deux hypothèses* (i) *et* (ii)

$$r(T) = \frac{-2}{2EV(TU_p)+1}$$

($EV(TU_p)$ étant l'exposant de volume de TU_p au sens de Dudley [1]).

Donc, si $EV(TU_p)$ est strictement inférieur à $-\frac{1}{2}$, TU_p est volumétrique au sens de Dudley.

2. Application aux mesures cylindriques gaussiennes. Soit $(\Omega, F, P; L)$ une fonction aléatoire linéaire gaussienne normale sur l^2.

Comme d'après le résultat 3, sous chacune des hypothèses (i) et (ii) on a

$$EV(TU_p) \leqslant -1 \Leftrightarrow r(T) \leqslant 2 \Leftrightarrow \lambda(b_n) \leqslant p',$$

on en déduit que *si L est presque sûrement à trajectoires continues sur un p-ellipsoïde TU_p de l^2 tel que $\lambda(b_n) < +\infty$, on a $\lambda(b_n) \leqslant p'$* (cf. Dudley [1]).

On remarque que la conjecture suivante de Dudley [1] est ainsi vérifiée:

C étant un disque (compact) de l^2 tel que $EV(C) < -1$, L est presque sûrement à trajectoires continues sur C.

Plus précisément, on établit le résultat suivant:

Si p est dans $]1, \infty[$ et si C est un p-ellipsoïde de l^2 d'axes b_n, on a équivalence de:

(a) *L est presque sûrement à trajectoires continues sur C.*

(b) *L est presque sûrement à trajectoires bornées sur C.*

(c) *$\sum b_n^{p'} < +\infty$ (p' conjugué de p).*

(d) *$\sum b_n^{p'} |L(e_n)|^{p'}$ converge presque partout ($e_n = (\delta_{nj})_j$, où δ_{nj} est le symbole de Kronecker).*

Si de plus p est supérieur à 2, (a)-(d) *sont équivalentes à*

(e) *Il existe un 2-ellipsoïde de l^2 contenant C et sur lequel L est presque sûrement à trajectoires continues.*

Remarques. 1. Les cas $p = 1, 2, +\infty$ ont été étudiés par Dudley [1].

2. Si C est un p-ellipsoïde de l^2 sur lequel L est presque sûrement à trajectoires bornées, C est bien entendu volumétrique.

Conséquence. Si T: $(x_n) \to (b_n x_n)$ est une application diagonale de l^2 dans l^q avec $q \in [1, \infty[$, alors l'image par T de la mesure cylindrique gaussienne normale sur l^2 est une mesure de Radon sur l^q si et seulement si la série de terme général b_n^q converge. En particulier, si $q \in [1, 2]$ on retrouve le résultat (connu) suivant:

On a équivalence de:

T radonifiante,

T d'Hilbert-Schmidt,

$\sum_n b_n^q < +\infty$.

Travaux cités

[1] R. Dudley, Journal of Functional Analysis 1 (1967), p. 290-330.

[2] L. Groos, *Wiener measures*, 1967.

[3] G. Lorentz, *Approximation of functions.*

[4] B. S. Mitjagin, *The approximative dimension and bases in nuclear spaces.*

[5] Pietsch, *Nukleare lokalkonvexe Räume*, Berlin 1965.

On conditional bases in non-nuclear Fréchet spaces

by

W. WOJTYŃSKI (Warszawa)

Theorem 1. *In each non-nuclear Köthe space $l^p[a_{m,n}]$ there exists a basis which is not unconditional.*

Theorem 2. *If X is a Fréchet space with a basis and all bases of X are absolute, then X is nuclear.*

Theorem 3. *If X is a Fréchet-hilbertian space (i.e. a projective limit of Hilbert spaces) with a basis, and all bases of X are unconditional, then X is nuclear.*

The proofs are given in Studia Math. 35 (1970), p. 77-96.

A theorem on complemented subspaces of nuclear spaces

by

C. BESSAGA (Warszawa)

An infinite matrix $[a_{pn}]$ of positive numbers is said to have property (D), provided that (1) for each p, the sequence $(a_{pn}/a_{p+1,n})_{n=1}^{\infty}$ is monotone, and either (2) $a_{1n} = 1$ and for every p there is a q such that $a_{qn} \geqslant a_{pn}^2$

for all n, or (3) $\lim_{p} a_{pn} = 1$ for all n and for every p there is a q such that $\lim_{n} a_{pn}/a_{qn}^2 = 0$.

THEOREM. *Let $X = m(a_{pn})$ be a nuclear step space (in the sense of Köthe) such that* (a) *the matrix $[a_{pn}]$ has property* (D), *and* (b) *X is isomorphic to $X \times X$. Then every complemented in X subspace with a basis is isomorphic to a subspace of X spanned on a subsequence of the unit-vector basis. Moreover, every basis of this complemented subspace is quasi-similar to a subsequence of the unit-vector basis of X.*

The argument is similar to that used by the author in a special case (Studia Math. 31 (1968), p. 307-318) and uses the invariance of diametral dimension and "dual diametral dimension".

The second statement of the theorem generalizes Dragilev's result (Mat. Sbornik 68 (1965), p. 153-173). Observe also that "most" of the classical nuclear Fréchet spaces do satisfy conditions (a) and (b). The following is open:

PROBLEM. Is the statement of the theorem true, when (a) or (b) are not assumed?

Optimal conditioning of operators

by

C. McCARTHY (Minneapolis)

In many problems of numerical analysis, the problem arises of estimating the *minimal condition number* $c(S)$ of an invertible matrix S:

$$c(S) = \inf_{\mathcal{D}} \|(DS)^{-1}\| \ \|DS\|,$$

where $\mathcal{D} = \{D\}$ is the family of all diagonal matrices. Even more important for applications is to find an explicit $D \in \mathcal{D}$ for which $\|(DS)^{-1}\| \ \|DS\|$ is of the same order of magnitude as $c(S)$. We can estimate $c(S)$ in terms of

$$\Lambda(S) = \sup_{\mathcal{U}} \|S^{-1} US\|$$

(which is trivially dominated by $c(S)$), where $\mathcal{U} = \{U\}$ is the set of all diagonal unitary matrices, and we can rather explicitly determine a D for which $\|(DS)^{-1}\| \|DS\| \leqslant 2\Lambda(S) \leqslant 2\sqrt{3}\ \Lambda_0(S)$, $\Lambda_0(S) = \sup \|S^{-1} VS\|$, V diagonal with entries $+1$ or -1.

Generalization to the case where S is an invertible operator on Hilbert space, $\mathcal{D}$ is a von Neumann algebra of operators and $\mathcal{U}$ is the group of unitary operators in $\mathcal{D}'$, is possible. We can show that the infimum defining $c(S)$ is always attained, and if $S > 0$ is best conditioned in the sense that $c(S) = \|S^{-1}\| \ \|S\|$, then S^{a} is best conditioned for all real a. If $\mathcal{D}$ is maximal abelian, the preceding estimates persist. If $\mathcal{U}$ is commutative, the estimate $c(S) \leqslant \Lambda(S)^2$ is known and is essentially Wermer's theorem on Boolean algebras of projections in Hilbert space.

The aesthetically pleasing conjecture $c(S) = \Lambda(S)$ is true for all 3×3 matrices S, but fails for a certain 8×8 matrix.

On algebraic derivative

by

D. PRZEWORSKA-ROLEWICZ (Warszawa)

Let X be a linear space over an algebraically closed field $\mathfrak{F}$ of characteristic zero. For any operator $T\colon X \to X$ we write

$$a_T = \dim Z_T = \dim\{x \in D_T\colon Tx = 0\}.$$

Mikusiński [1], [2] characterized the algebraic derivative as an endomorphism of X satisfying the following conditions:

Let $P(t), Q(t)$ be arbitrary polynomials with coefficients in $\mathfrak{F}$. Then (1) $a_{P(D)} =$ degree of $P(t)$, (2) $a_{P(D)Q(D)} = a_{P(D)} + a_{Q(D)}$, (3) for every $x \in X$, there is a $P(t)$ such that $P(D)x = 0$.

We propose another, more general definition: If for an endomorphism D of X there is an endomorphism R of X such that $DR = I_X$ and $I_X - \lambda R$ is invertible for any $\lambda \in \mathfrak{F}$, then D and R are called an *algebraic derivative* and *algebraic integral*, respectively.

Let

$$P(T) = \prod_{k=1}^{n} (t - t_k)^{r_k},$$

$r_1 + \ldots + r_n =$ degree of $P(t)$. Write

$$P(t,s) = \prod_{k=1}^{n} (t - t_k s)^{r_k}.$$

THEOREM 1. $a_{P(D)} = a_D \cdot$*(degree of P).*

THEOREM 2. *$z \in Z_{P(D)}$ if and only if*

$$z = \sum_{k=1}^{n} (I - t_k R)^{-r_k} \Big[\sum_{m=0}^{r_k - 1} R^m z_{m,k}\Big], \quad \textit{where } z_{m,k} \in Z_D.$$

THEOREM 3. *Any solution of the equation* $P(D)x = y$ *is of the form*

$$x = [P(I, R)]^{-1}R^{N}y+z,$$

where $z \in Z_{P(D)}$ *and* N = *degree of* P.

THEOREM 4. *If* $X = Z_{P(D)}$ *for a polynomial* $P(t)$, *then there is no endomorphism* T *of* X *satisfying the condition* $DT = TD+I$.

THEOREM 5. *If* X *is a complete linear metric space, then every endomorphism* D *satisfying conditions* (1)-(3) *of Mikusiński is non-continuous.*

The results have been anounced in [4]. Detailed proofs will appear in [5].

References

[1] J. Mikusiński, *Sur les espaces linéaires avec dérivation*, Studia Math. 16 (1958), p. 113-123.

[2] — *Extensions de l'espace linéaire avec dérivation*, ibidem 16 (1958), p. 156-172.

[3] R. Sikorski, *On Mikusiński's algebraic theory of differential equations*, ibidem 16 (1958), p. 230-236.

[4] D. Przeworska-Rolewicz, *A characterization of algebraic derivative*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., 7 (1969), p. 11-13.

[5] — *Algebraic derivative and abstract differential equations*, Anais Acad. Brazil. de Ciencias (in print).

A remark on (p, q)-absolutely summing operators in L_p-spaces

by

N. TOMCZAK (Warszawa)

X, Y and Z denote Banach spaces, and T and S linear operators.

Definition 1. For every $T: X \to Y$ and every p and q with $1 \leqslant q \leqslant p < \infty$ we let $a_{pq}(T)$ be the supremum of

$$\Big(\sum_{i=1}^{n} \|Tx_i\|^p\Big)^{1/p}$$

over all finite sequences $x_1, \ldots, x_n$ with the following property:

$$\sup_{\|x^*\|\leqslant 1}\Big(\sum_{i=1}^{n} |x^*(x_i)|^q\Big) < 1.$$

Let $A_{pq}(X, Y) = \{T: a_{pq}(T) < \infty\}$. Operators in $A_{pq}(X, Y)$ are said to be (p, q)-*absolutely summing.*

It is well known that $A_{pq}(X, Y)$ is a Banach space under the norm $a_{pq}(\cdot)$.

PROPOSITION. *If* $T \in A_{pp}(X, Y)$ *and* $S \in A_{st}(Y, Z)$, *then* $ST \in A_{rq}(X, Z)$, *where* $r^{-1} = p^{-1}+s^{-1} \leqslant 1$ *and* $q^{-1} = p^{-1}+t^{-1} \leqslant 1$.

THEOREM. *Let* X *be isomorphic to a subspace of* $L_1(\mu)$ *for some measure* μ and *let* Y *be an arbitrary Banach space. Then for each* $1 \leqslant r \leqslant 2$ *we have* $A_{r1}(X, Y) = A_{s2}(X, Y)$, *where* $s^{-1} = r^{-1}-2^{-1}$.

COROLLARY. *Let* $1 \leqslant r \leqslant 2$ *and* $1 \leqslant p \leqslant 2$. *Then, for every* Y, *we have* $A_{r1}(l_p, Y) = A_{s2}(l_p, Y)$, $A_{r1}(L_p(0, 1)Y) = A_{s2}(L_p(0, 1), Y)$, *where* $s^{-1} = r^{-1}-2^{-1}$.

The proofs are published in Studia Math. 35 (1970), p. 97-100.

Decompositions of operator representations of function algebras

by

W. MLAK (Kraków)

Let $A \in C(D)$ (D compact, Hausdorff) be a uniformly closed, separating and containing constants sup norm algebra. The norm continuous linear map $T: A \to L(H)$ ($L(H)$ = the algebra of all linear bounded operators in the Hilbert space H) is a *representation* if it is multiplicative and $T(1) = I$.

Assume that the dual $M = C(D)^* = E_1+E_2$ (direct sum of closed subspaces) and that for $f \in M$ the relation $f \perp A$ implies $f_i \perp A$ $(i = 1, 2)$, where $f = f_1+f_2, f_i \in E_i$. This is called the *Riesz property of decomposition* $M = E_1+E_2$. Write now $T \in Z(E_i)$ if for each $x, y \in H$ $(T(u)x, y) = f_i(u: x, y)$ (all $u \in A$) for $f_i \in E_i$. Now the main point is that the Riesz property implies that $T = T_1+T_2$ (direct sum), where T_1 are representations of class $Z(E_i)$. If T is contractive, then the direct sum becomes an orthogonal one. Moreover, if (G_a) is the totality of all Gleason parts of A, then $T = (\oplus T_a) \oplus T_s$, where $T_a \in Z(E_a)$, E_a consisting of $f \ll G_a$ and $T_s \in Z(E_s)$, E_s consisting of completely singular f. The part T_a admits an extension to a representation of H^∞-type algebra corresponding to G_a, which is dilatable if points in G_a have unique representing probability measures. If $z \subset D$ is an intersection of peak sets of A, then $M = E_z+E_{D-z}$, E_z = functionals with support in z, has the Riesz property.

It follows that $T = (\oplus T_b) \oplus T'$, where $T_b \in Z(E_b)$, $D = \bigcup b$ is the Bishop decomposition of D into maximal sets of antisymmetry for A, T' suitably singular.

All investigations are dilation free and a variety of modifications of developed methods and related theorems is avaible. Several applications may be given to the theory of von Neumann spectral sets, representations of concrete algebras, etc.

References

[1] W. Mlak, *Absolutely continuous operator valued representations of function algebras*, Bull. Acad. Polon. Sci., Ser. sci. math. astr. et phys., 17 (1969), p. 547-549.

[2] — *Decompositions and extensions of operator-valued representations of function algebras* (to appear in Acta Math. Sci. (Szeged)).

[3] — *Decompositions of operator-valued representations of function algebras* (to appear in Studia Math.).

---

p-integral operators

by

A. PERSSON (Lund)

In his thesis *Produits tensoriels topologiques et espaces nucléaires* A. Grothendieck introduced the so-called *integral operators* between Banach spaces E and F. These are operators that have a factorization of the form $E \to C(K) \overset{\text{id}}{\to} L^1_\mu(K) \to F$, where μ is a positive measure on the compact space K.

Grothendieck proved that such an operator is nuclear provided that one of the following conditions is satisfied:

(i) E is reflexive;

(ii) E has a separable dual;

(iii) F is reflexive;

(iv) F is separable and isomorphic to the dual of a Banach space.

In a recent paper (Studia Math. 33 (1969)) A. Pietsch and A. Persson studied operators with a representation of the form $\sum \langle x, a_n \rangle y_n$ with $\sum \|a_n\|^p < \infty$ and $\sup\limits_{\|y'\| \leqslant 1} \sum |\langle y_n, y' \rangle|^{p'}$ (called *p-nuclear*) and operators with a factorization $E \to C(K) \overset{\text{id}}{\to} L^p(K) \to F$ (called *p-integral*). Conditions (iii) and (iv) are not sufficient in order that a p-integral operator be p-nuclear when $p > 1$. This is shown by the injection $C(K) \overset{\text{id}}{\to} L^p(K)$. However, assuming (i) or (ii), every p-integral mapping $E \to F$ is p-nuclear. The first case is obtained as a corollary of the more general statement that the composition of a weakly compact mapping and a p-integral mapping (in that order!) is p-nuclear. This generalizes a theorem of Grothendieck in the case $p = 1$.

Elementary characterization of operators with absolutely summing adjoint

by

U. SCHLOTTERBECK (Tübingen)

A null sequence $\{x_n\}$ in a locally convex space E is said to be *hypermajorized* if for any continuous linear operator S from E in any Banach lattice G the sequence $\{Sx_n\}$ is majorized in G. The following theorem holds:

THEOREM. *Let E, F be Banach spaces, and let T be a linear operator from E into F. T has an absolutely summing adjoint if and only if T maps null sequences in E into hypermajorized null sequences in F.*

Connected with this are the following results:

1. *An (F)-space E is nuclear if and only if every null sequence in E is hypermajorized.*

2. *A locally convex space E is nuclear if and only if the space of null sequences in E, topologized as usual, coincides with the space of hypermajorized null sequences in E, the latter endowed with a certain natural locally convex topology.*

Modulus of prenuclear maps in ordered spaces

by

J. SCHMETS (Liège)

We shall prove that in quite general cases every prenuclear map T from a function space E to another F admits a modulus, i.e. a linear positive operator $|T|$ from E to F such that

$$|Tf| \leqslant |T|\,|f|, \ \forall f \in E,$$

$$|cT| = |c|\,|T|, \ \forall c \in C,$$

and if T_1 and T_2 are such prenuclear maps

$$|T_1+T_2| \leqslant |T_1|+|T_2|.$$

This operator $|T|$ is continuous from E to F and verifies relations of continuity similar to those of T. We shall examine prenuclearity of $|T|$.

Finally, for a prenuclear map from $\mu-L_2$ to $\nu-L_2$

$$T. = \int K(x, y).d\mu,$$

we prove that

$$|T|. = \int |K(x, y)|.d\mu.$$

References

[1] R. V. Chacon-U. Krengel, *Linear modulus of a linear operator*, Proc. Am. Math. Soc. 154 (1964), p. 553-559.

[2] M. de Wilde et J. Schmets, *Module des opérateurs prénucléaires dans les espaces de fonctions*, Bull. Soc. Roy. Sc. Liège 9-10 (1966), p. 558-565.

[3] N. Dunford and J. T. Schwartz, *Convergence almost everywhere of operator averages*, J. Math. Mech. 5 (1956), p. 129-178.

On strictly cosingular operators in locally convex spaces

by

YU. N. VLADIMIRSKIĬ (Kostrome, USSR)

Let X and Y be locally convex linear topological spaces. $L(X, Y)$ denotes the space of all continuous linear transformations acting from X to Y. An operator $\varphi \in L(X, Y)$ is called *Φ-operator* [1], if φ is open, the range R_φ is closed and codim $R_\varphi < \infty$. An operator $\alpha \in L(X, Y)$ is called *strictly cosingular* [2], if there exists no infinite-dimensional locally convex space Z admitting surjective open mappings $h_1 \in L(X, Z)$, $h_2 \in L(Y, Z)$ such that $h_2\alpha = h_1$.

THEOREM. *Let X be fully complete and let Y be a Banach space. Then*

(1) *$\alpha \in L(X, Y)$ is strictly cosingular iff for every closed subspace $Y_1 \subset Y$ with* codim $Y_1 = \infty$ *there is a closed subspace $Y_2 \subset Y$ with* codim $Y_2 = \infty$ *such that $Y_1 \subset Y_2$ and $\pi_2\alpha$ is compact;*

(2) *if $\varphi \in L(X, Y)$ is a Φ-operator, $\alpha \in L(X, Y)$ is strictly cosingular, then $\varphi+\alpha$ is a Φ-operator;*

(3) *the set of all strictly cosingular operators acting from X to Y is a linear subspace of $L(X, Y)$. This subspace is a two-sided ideal, if $X = Y$.*

COROLLARY. *If X and Y are Banach space, then every strictly cosingular operator $\alpha \in L(X, Y)$ is a Φ-admissible perturbation.*

References

[1] I. C. Gohberg, A. S. Markus and I. A. Feldman, Izv. Moldavsk. Fil. Akad Nauk SSSR 1960, No 10 (76), p. 51.

[2] A. Pełczyński, Bull. Acad. Polon. Sci. Sér. sci. math. astr. et phys., 13 (1965), p. 31-36.

On weakly convergent sequences in Banach spaces

by

W. SZLENK (Warszawa)

Let X be a separable Banach space, let (x_n), $x_n \in X (n = 1, 2, \ldots)$, be a weakly convergent sequence to an element $x_0 \in X$. Let $\sigma_1, \ldots, \sigma_r$ be increasing sequences of integers and $\sigma_{m,n}$ be the set of the n first elements of σ_m. We define the sequences $(x_n^{(1)}), \ldots, (x_n^{(r)})$ as follows:

$$x_n^{(1)} = \frac{1}{n} \sum_{k \in \sigma_{r,n}} x_k, \quad \ldots, \quad x_n^{(r)} = \frac{1}{n} \sum_{k \in \sigma_{r,n}} x_k^{(r-1)}.$$

We say that the space X has BS_r-property if for each weakly convergent sequence (x_n) there exists sequences $\sigma_1, \ldots, \sigma_r$ such that

$$\lim_{n=\infty} \|x_n^{(r)} - x_0\| = 0.$$

The BS_1-property is called *Banach-Saks property* [1]. We present the following results:

THEOREM 1. *Every uniformly smooth Banach space* (for definition, see [2]) *has BS_1-property.*

This is the dual result to the Kakutani's theorem [3], concerning uniformly convex Banach spaces.

Let β be an ordinal number and let $C(\beta)$ be the space of all continuous real functions on the set of all ordinal numbers which are $\leqslant \beta$.

THEOREM 2. *If $\beta < \omega^{\omega^r}$, then the space $C(\beta)$ has BS_r-property.*

Remark. The space $C(\omega^{\omega^\omega})$ has no BS_r-property for $r = 1, 2, \ldots$ (see [4]).

The proofs will be published in Bulletin Acad. Polon. Sci.

References

[1] S. Banach et S. Saks, *Sur la convergence forte dans les champs L^p*, Studia Math. 2 (1930), p. 51-57.

[2] M. M. Day, *Normed linear spaces*, Berlin—Göttingen—Heidelberg 1958.

[3] S. Kakutani, *Weak convergence in uniformly convex spaces*, Tohoko Math. J. 45 (1938), p. 188-193.

[4] J. Schreier, *Ein Gegenbeispiel zur Theorie der schwachen Konvergenz*, Studia Math. 2 (1930), p. 58-62.

Nuclearity and local convexity of sequence spaces

by

CHRISTIAN FENSKE and EBERHARD SCHOCK (Bonn)

Let $\varphi\colon [0,\infty)\to[0,\infty)$ be a continuous, subadditive, increasing function satisfying $\varphi(0)=0$, and P a set of non-negative sequences, such that for any n there is $\varrho\in P$ with $\varrho_n>0$, and for any two $\varrho,\sigma\in P$ there is $\tau\in P$ with $\tau_n\geqslant\max(\varrho_n,\sigma_n)$. Then $\Lambda_\varphi P$ denotes the space of all sequences ξ, such that

$$\sum_{n=0}^{\infty}\varphi(|\xi_n|\varrho_n)<\infty$$

for any $\varrho\in P$, equipped with the obvious topology. If $\varphi(t)=t$, $\Lambda P\colon=\Lambda_\varphi P$ is a "gestufter Raum" of Köthe; if, on the other hand, all ϱ_n equal 1, this concept is due to Nakano and Gramsch. For $\varrho\in P$ let

$$U_\varrho\colon=\Big\{\xi\in\Lambda_\varphi P\Big|\sum_{n=0}^{\infty}\varphi(|\xi_n|\varrho_n)\leqslant 1\Big\}.$$

If $U_\sigma\subset U_\varrho$, then the n-th diameter $\delta_n(U_\sigma, U_\varrho)$ is computed as follows: Rearrange the sequence $(\varrho_n\sigma_n^{-1})_{n\geqslant 0}$ into a decreasing sequence, after deleting all terms with $\sigma_n=0$. Then the term with number n will equal $\delta_n(U_\sigma, U_\varrho)$. Now, calling a topological linear space E with a basis $\mathcal{U}$ of neighbourhoods of zero *nuclear*, if for every $U\in\mathcal{U}$ there is $V\in\mathcal{U}$ such that

$$\sum_{n=0}^{\infty}\delta_n(V, U)<\infty,$$

we immediately obtain the Grothendieck-Pietsch criterion: $\Lambda_\varphi P$ is nuclear if and only if for every $\varrho\in P$ there are $\sigma\in P$ and $\lambda\in l^1$ with $\varrho_n\leqslant\lambda_n\sigma_n$. Similarly, by considering diametral dimension, we obtain Köthe's criterion:

Let $P=(\varrho^{(k)})$ be an increasing system of sequences, $\Lambda_\varphi P$ a sequence space. $\Lambda_\varphi P$ possesses the Montel-property if and only if there does not exist an infinite set $I\subset N$, a $k_0\in N$, and a positive sequence (M_k), such that for every $k\geqslant k_0$ and $n\in I$ $0<\varrho_n^{(k)}\leqslant M_k\varrho_n^{(k_0)}$.

Finally, a sequence space $\Lambda_\varphi P$ is locally convex if and only if it is isomorphic to ΛP. Details and further results may be found in [1] and [2].

References

[1] C. Fenske und E. Schock, *Über die diametrale Dimension von lokalkonvexen Räumen*, Berichte „Gesellschaft für Mathematik und Datenverarbeitung", Bonn, 10 (b) (1969).

[2] — *Nuklearität und lokale Konvexität von Folgenräumen*, Mathematische Nachrichten (to appear).

The normalization of Schauder bases of locally convex spaces

by

N. J. KALTON (Cambridge)

The report is covered by two papers in this volume, p. 243-266, and the paper *The normalization properties of Schauder bases* submitted to Proc. London Math. Soc.

Nuclear operators and potential theory on Hilbert space

by

LEONARD GROSS (Ithaca, N. J.)

As is well known the potential of a function f on R^n can be written in the form

(1) $$u=\int_0^\infty p_t*f\,dt$$

where $*$ denotes convolution and $p_t(dx)$ is the measure

(2) $$p_t(dx)=(2\pi t)^{-n/2}e^{-|x|^2/2t}dx$$

on R^n. Moreover, u satisfies Poisson equation

(3) $$\Delta u = -2f,$$

where Δu denotes the Laplacian of u, i.e., Δu = trace of second Fréchet derivative of u.

When R^n is replaced by an infinite-dimensional real separable Hilbert space H, equations (1)-(3) possess an immediate generalization as follows. It is possible to complete H with respect to a suitably weak norm (a so-called *measurable norm*) to obtain a Banach space B (known as an abstract Wiener space) which has the property that equation (2) defines in a natural way a countably additive probability measure on the Borel field of B. If f is a real-valued function on B which, let us say, is bounded and vanishes off a bounded set, its potential may be defined by (1). It can be proved that under various smoothness conditions on f, u will satisfy equation (3) in the sense that the second Fréchet derivative of u (taken in H directions only) is actually a nuclear operator on H and its trace equals $-2f$. In general, it is possible for the second Fréchet derivative of u to exist without being nuclear. Thus a new type of regularity theorem appears in infinite-dimensional potential theory which is not present in finite-dimensions since the existence of the $(n\times n)$-matrix of second derivatives automatically implies its nuclearity in finite dimensions.

References

[1] L. Gross, *Abstract Wiener spaces*, Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability, 1965, p. 31-42.

[2] — *Potential theory on Hilbert space*, Journal of Functional Analysis 1 (1967), p. 123-181.

On the structure of nuclear spaces

by

G. KÖTHE (Frankfurt)

T. and Y. Kōmura proved that every nuclear space is isomorphic to a subspace of a product s^{A}, s the space of rapidly decreasing sequences. Using their ideas it is possible to prove that, roughly speaking, every nuclear sequence space can be represented as an intersection of sequence spaces which arise from s using a permutation of coordinates and a diagonal transformation (cf. Studia Mathematica 31 (1968), p. 267-271).

Développement en série de polynômes orthogonaux des fonctions indéfiniment différentiables

par

M. ZERNER (Nice)

Sur un ouvert borné lipschitzien, le développement en série de polynômes orthonormés établit un isomorphisme vectoriel topologique entre l'espace des fonctions indéfiniment différentiables sur l'adhérence de l'ouvert et l'espace des suites à d'ecroissance rapide.

Le contenu de ce rapport a été résumé dans une note chez Comptes Rendus de l'Académie des Sciences de Paris portant le même titre (tome 268, p. 218-220, 27 janvier 1969).

A nuclear space of functions on a locally compact group

by

T. PYTLIK (Wrocław)

A nuclear space $D(G)$ of functions on a large class of locally compact groups G, analogous to the class D of Schwartz on a Lie group was constructed by F. Bruhat (Bull. Soc. Math. France 89 (1961), p. 43-75) and K. Maurin (Bull. Acad. Polon. Sci, Sér. sci. math. astr. et phys., 9 (1961) p. 845-850). These constructions are based on Yamabe's approximation theorem.

A. Hulanicki suggested a construction based on an infinite process of regularizing of functions in $L_2(G)$ by convolutions, and asked whether this leads to a non-trivial nuclear space Φ of functions with compact supports on an arbitrary locally compact group G with the property that Φ is translation invariant and translations of functions of Φ give a strongly continuous representation of G.

The aim of this paper is to give a positive answer to this question. The idea of the construction is the following. Select a sequence of functions $g_n \in C_0(G)$ and, for arbitrary compact set V, we build a space Φ_V with the property that $f \in \Phi_V$ if, for any n, $f = h_n * g_n * \ldots * g_1$ with $h_n \in L_2(G)$ and $\operatorname{supp} f \subset VU$, where U is a fixed open neighbourhood of unity. Φ_V is a nuclear countably normed space, and for $V \subset W$ the embedding of Φ_V into Φ_W is bicontinuous. The required space is then defined as inductive limit of spaces Φ_V.

For details see Bull. Acad. Polon. Sci., Sér. sci. math. astr. et phys., 18 (1969), p. 161-166.

Montel spaces of Hölder continuous functions

by

EBERHARD SCHOCK (Bonn)

A continuous function $f: [0,1] \to \mathbf{R}$ is *α-Hölder-continuous* $(0 < \alpha \leqslant 1)$ if

$$\|f\|_\alpha := \sup\left\{\frac{|f(s)-f(t)|}{|s-t|^\alpha},\ s,t \in [0,1]\right\} < \infty.$$

The space $C_\alpha[0,1]$ of all functions f with $f(0) = 0$ and $\|f\|_\alpha < \infty$ is a Banach space with the norm $\|\ \|_\alpha$. Pełczyński and Ciesielski have shown, that the space $C_\alpha[0,1]$ is isomorphic to l^∞. Let $\alpha \in (0,1]$ and

$$H_{\alpha-}[0,1] = \operatorname*{proj}_{\beta<\alpha} C_\beta[0,1] = \bigcap_{\beta<\alpha} C_\beta[0,1].$$

THEOREM 1. *The space $H_{\alpha-}[0,1]$ is a Montel space.*

Let (s_α) be the following sequence space:

$$(s_\alpha) = \{\xi\colon p_\beta(\xi) = \sup|\xi_n| n^\beta < \infty,\ \beta < \alpha\}.$$

(s_α) is a non-nuclear Montel space.

THEOREM 2. *The spaces $H_{\alpha-}[0,1]$ and (s_α) are isomorphic.*

THEOREM 3. *The functions φ_n,*

$$\varphi_n(t) = \int_0^t \chi_n(\tau)\, d\tau,$$

$\{\chi_n\}$ the Haar system, form an unconditional basis in $H_{\alpha-}[0,1]$.

Let $\alpha \in [0,1]$ and

$$H_{\alpha+}[0,1] = \operatorname*{ind}_{\beta>\alpha} C_\beta[0,1] = \bigcup_{\beta>\alpha} C_\beta[0,1].$$

THEOREM 4. *The space $H_{\alpha+}[0,1]$ is a Montel space.*

Let $l_n^\infty = \{\xi\colon \sup|\xi_n| n^\beta < \infty\}$ and $(\sigma_\alpha) = \bigcup_{\beta>\alpha} l_\beta^\infty$.

(σ_α) is a non-nuclear Montel space.

THEOREM 5. *The spaces $H_{\alpha+}[0,1]$ and (σ_α) are isomorphic.*

THEOREM 6. *The functions φ_n form an unconditional basis in $H_{\alpha+}[0,1]$.*

To appear in J. reine angew. Math.

Linear operators with sufficiently many a priori given eigenvectors

by

DIMITER SKORDEV (Sofia)

Let X be a real Banach space and let T be a Hausdorff locally convex topology in X. The unit ball S of X is assumed to be T-bounded and T-closed.

Let M be a set of non-zero elements of X such that the T-closed convex hull of $M \cup (-M)$ is equal to S. We denote by $\mathscr{A}(X, T, M)$ the set of all linear mappings A of X into X which satisfy the following two conditions:

(a) All the elements of M are eigenvectors of A.

(b) The restriction of A on S is T-continuous.

The set $\mathscr{A}(X, T, M)$ is not empty: at least the operators A which have the form $Ax = \lambda x$ (with constant real λ) belong to it. More interesting examples can be found in [2] and in other papers of Tagamlitzki.

Let $\mathscr{L}(X)$ be the Banach algebra of all bounded linear mappings of the Banach space X into itself.

THEOREM I. *$\mathscr{A}(X, T, M)$ is a closed subalgebra of $\mathscr{L}(X)$.*

To each A belonging to $\mathscr{A}(X, T, M)$ we make to correspond the real-valued function $A^\wedge(x)$ which is defined for every $x \in M$ by means of the equation

$$Ax = A^\wedge(x)x.$$

Let $\mathscr{A}^\wedge(X, T, M)$ be the set of all such functions and let $C(T, M)$ be the Banach algebra of all bounded T-continuous real-valued functions defined on M.

THEOREM II. *The correspondence $A \to A^\wedge$ is an isometric and isomorphic mapping of $\mathscr{A}(X, T, M)$ into $C(T, M)$.*

COROLLARY 1. *The algebra $\mathscr{A}(X, T, M)$ is commutative.*

COROLLARY 2. *$\mathscr{A}(X, T, M)$ is a closed subalgebra of $C(T, M)$.*

THEOREM III. *If R is a closed ideal in $\mathscr{A}(X, T, M)$, A is an operator belonging to $\mathscr{A}(X, T, M)$ and $A^2 \in R$, then $A \in R$.*

COROLLARY 3. *If A is an operator belonging to $\mathscr{A}(X, T, M)$, x is an element of X and $A^2x = 0$, then $Ax = 0$.*

THEOREM IV. *If A is an operator belonging to $\mathscr{A}(X, T, M)$, then the following 3 conditions are equivalent:*

(A) *The range of A is equal to X.*

(B) *The inverse operator* A^{-1} *exists and belongs to* $\mathscr{A}(X, T, M)$.

(C) *There exists a real number* $\varrho > 0$ *such that* $|A^{\wedge}(x)| \geqslant \varrho$ *for every* x *in* M.

COROLLARY 4. *If* M *is* T*-compact, then for every operator belonging to* $\mathscr{A}(X, T, M)$ *the Fredholm alternative holds.*

The proofs of all these results are essentially the same as in [1], where these statements are proved under the restriction that S is T-complete.

References

[1] D. Skordev, Acad. Bulgare des Sci., Bull. de l'Institut de math., 10 (1969) (to appear).

[2] Y. Tagamlitzki, Ann. de l'Univers. de Sofia, Fac. des sci. phys. et math., 48 (1953/54), livre 1, I partie, p. 69-85.

Some questions concerning support points

by

N. T. PECK (Urbana, Ill.)

We begin with an example of a closed bounded convex set in a Fréchet space which can not be supported at any of its points by a non-trivial continuous linear functional; this answers a question of Klee and Phelps. We then show that if C is a closed bounded convex set in a Fréchet space and F is the set of all linear functionals on the Fréchet space which are continuous on C, then some support-point theorems involving functionals in F can be obtained which parallel the support-point theorems of Bishop and Phelps for Banach spaces. As an application we extend a recent theorem of Krein-Milman type due to Lindenstrauss and Bessaga-Pełczyński.

When the containing linear space is assumed only to be complete and locally convex, not necessarily Fréchet, we have some weaker support theorems, which we mention briefly.

The detailed discussion will be presented in the paper "Support points in locally convex spaces", to appear in Duke Math. Journal.

On the equivalence of weak and Schauder basis

by

M. DE WILDE (Liège)

Let E denote a locally convex topological vector space (LCTVS). A *weak* (or *strong*) basis in E is a sequence f_k such that every element f of E can be expressed uniquely in the form

$$f = \sum_{k=1}^{\infty} c_k(f) f_k$$

with convergence in $\sigma(E, E')$ (or in the original topology of E). The sequence is said to be a (weak or strong) *Schauder basis* if the $c_k(f)$ are continuous linear forms on E.

Moreover, let us call a *strictly netted* space an LCTVS E in which there exists a family of absolutely convex sets $e_{n_1, \ldots, n_k}$, $n_1, \ldots, n_k$, $k = 1, 2, \ldots$, which satisfy the following conditions:

(a) $$E = \bigcup_{n_1=1}^{\infty} e_{n_1}, \ldots, = \bigcup_{n_{k+1}=1}^{\infty} e_{n_1, \ldots, n_{k+1}}, \ldots,$$

(b) for any sequence n_k, there exists a sequence $\lambda_k > 0$ such that, if $\mu_k \in [0, \lambda_k]$ and $g_k \in e_{n_1, \ldots, n_k}$, the series $\sum \mu_k g_k$ converges in E and

$$\sum_{k=k_0}^{\infty} \mu_k g_k \in e_{n_1, \ldots, n_{k_0}}, \forall k_0.$$

Such spaces verify a general form of the closed graph theorem and have numerous examples and wide permanence properties (see [3]).

We prove the following theorem:

If E is bornological, sequentially complete and strictly netted, then any weak basis in E is a strong Schauder basis.

This theorem was proved for Fréchet spaces by C. Bessaga and A. Pełczyński (see [1]).

As examples of convenient E which are not Fréchet spaces, let us point the classical spaces $\mathscr{D}$, $\mathscr{D}'$, $\mathscr{E}'$, $\mathscr{S}'$, ... (see [2]).

References

[1] C. Bessaga i A. Pełczyński, *Własności baz w przestrzeniach typu* B_0, Prace Matem. 3 (1959), p. 123-142.

[2] L. Schwartz, *Sur le théorème du graphe fermé*, C. R. Paris 263 (1966), p. 602-605.

[3] De Wilde, *Réseaux dans les espaces linéaires à semi-normes*, Thèse.

On differentiability in some class of locally convex spaces

by

W. SZCZYRBA (Warszawa)

In this paper we present a theory of differentiation in locally convex linear topological spaces which are either metrizable quasi-normable or are of type DF-S (in sense of Grothendieck, Summa Bras. Math. 3 (1954), 81-127).

By E, F, G we denote locally convex spaces; T is a map from a subset of E into F; e is an interior point of the domain of T; $L_b(E, F)$ denotes the space of continuous linear operators from F to E with the topology of bounded convergence.

Definition. T is *differentiable* at e if there is a linear operator $T'(e) \in L_b(E, F)$ such that $r_e(h) = T(e+h) - T(e) - T'(e)h$ has the following property:

for every $V \in \mathfrak{N}(F)$ there is a $U \in \mathfrak{N}(E)$ such that for every Moore-Smith sequence $(h_\lambda)_{\lambda \in \Lambda}$ convergent to zero in E, we have

$$\lim_{\lambda \leftarrow \Lambda} \|r(h_\lambda)\|_V / \|h_\lambda\|_U = 0$$

$\mathfrak{N}(E)$ and $\mathfrak{N}(F)$ denote the sets of all absolutely convex neighbourhoods of zero in E and in F, respectively).

We say that T is *continuously differentiable* if the map $e \to T'(e) \in L_b(E, F)$ is continuous.

THEOREM. *Suppose that the space E is metrizable quasi-normable and that the map T is continuously differentiable in a neighbourhood of e. Then for every $V \in \mathfrak{N}(F)$ there are $U, W \in \mathfrak{N}(E)$ such that for every $x \in e + U$ and for every neighbourhood $Y \in \mathfrak{N}(E)$, we have*

$$\|r_x(h)\|_V \leqslant C_x(Y) \cdot \|h\|_W \quad \textit{for } h \in Y,$$

$$C_x(Y) = \sup\{\|T'(x+k)s - T'(x)s\| : s \in W,\ k \in Y\} \quad \textit{and} \quad \lim_{Y \in \mathfrak{N}(E)} C_x(Y) = 0.$$

COROLLARY 1. *If E, G are metrizable quasi-normable, and $T: E \times G \to F$ is continuously partially differentiable at a point $(e, g) \in E \times G$, then T is differentiable at (e, g).*

The map $T: E \to F$ is said to be *twice differentiable* at a point $e_0 \in E$ if the map $e \to T'(e) \in L_b(E, F)$ is differentiable at e_0.

COROLLARY 2. *If E is metrizable quasi-normable, and T is twice differentiable at e, then $T''(e) \in L_b(E, E; F)$ and is a symmetric bilinear form.*

Similar results are valid for DF-S spaces.

The detailed proofs will appear in Studia Mathematica.

Fredholm links and perturbation theorems

by

G. NEUBAUER (Constanz)

Let E be a Banach space and $\mathscr{G}(E)$ its Grassmann space (space of closed linear subspaces of E with the gap-topology). Let us call the configuration (M, N, M', N') a *link* if $M + N \subset M' \cap N'$, and a *Fredholm link* if, in addition, $M' + N'$ is closed and $\dim M' \cap N'/M + N < \infty$.

Let $\mathfrak{M}$ be a set of links (with the topology induced by $\mathscr{G}(E)$). Then the set $\mathfrak{M}_\varphi$ of Fredholm links in $\mathfrak{M}$ is open in $\mathfrak{M}$ and $\dim M' \cap N'/M + N$ has a local maximum in every Fredholm link. $M' + N'$ and $M \cap N$ are continuous in any subset of $\mathfrak{M}_\varphi$ where $\dim M' \cap N'/M + N$ is locally constant.

These results permit various applications to perturbation theory, in particular to invariance theorems for Fredholm complexes.

On finite-dimensional perturbations

by

N. ARONSZAJN (Lawrence, Kansas)

The lecture is covered by the paper of N. Aronszajn and R. D. Brown, *Finite-dimensional perturbations*, Studia Math. 36 (1969), p. 1-76.

Unbeschränkt teilbare zufällige Elemente in L^p-Räumen

von

G. DENNLER (Jena)

Für unbeschränkte teilbare zufällige Elemente in L^p-Räumen, $p > 2$, wird eine der bekannten Levy-Chintschin Darstellung in R^n analoge Darstellung bewiesen. Insbesondere wird dabei auf die Gaußschen Maße in L^p-Räumen eingegangen.

Streng unabhängige Distributionen

von

G. DENNLER (Jena)

Es ist bislang wohl noch nicht gelungen eine vollständige Übersicht über die zufälligen Distributionen mit unabhängigen Werten in jedem Punkt zu gewinnen. Für die Klasse der zufälligen Distributionen mit streng unabhängigen Werten in jedem Punkt ist das möglich. Jede derartige zufällige Distribution läßt sich als Faltungsprodukt einer unbeschränkt teilbaren zufälligen Distribution und höchstens abzählbar vielen zufälligen Distributionen darstellen, die auf einen Punkt konzentriert sind. Die unbeschränkt teilbaren zufälligen Distributionen lassen sich leicht mittels der Levy-Chintschin Darstellung charakterisieren.

A representation theorem for unconditionally converging linear operators on $C_0(T, X)$

by

IVAN DOBRAKOV (Bratislava)

Let X and Y be (real, complex) Banach spaces. According to Pełczyński [3] a bounded linear operator $U: X \to Y$ is called *unconditionally converging* if it transforms weakly unconditionally convergent series into (strongly) unconditionally convergent ones. By Orlicz's theorem every weakly compact linear operator is unconditionally converging. If Y is a weakly complete Banach space, then every bounded linear operator $U: X \to Y$ is unconditionally converging. Pełczyński in [3] proved that if X is a reflexive Banach space, then a bounded linear operator $U: C_0(T, X) \to Y$ is unconditionally converging if and only if it is weakly compact. Here T is a locally compact Hausdorff topological space and $C_0(T, X)$ denotes the Banach space of all X-valued continuous functions on T tending to zero at infinity with the usual supremum norm. It may be shown that the next representation theorem extends this results.

THEOREM 1. *Every unconditionally converging bounded linear operator $U: C_0(T, X) \to Y$ can be uniquely expressed in the form*

$$Uf = \int_T f\,dm, \quad f \in C_0(T, X),$$

where $m: B_0 \to L(X, Y)$ is an operator-valued Baire measure with the following properties:

1. *the values of m are unconditionally converging bounded linear operators from X to Y,*

2. *there is a finite non-negative countably additive measure λ on B_0 such that*

$$\lim_{\lambda(E) \to 0} \tilde{m}(E) = 0, \quad E \in B_0.$$

In that case

$$|U| = \tilde{m}(T) = \sup_{|y| \leqslant 1} v(y^* m, T).$$

Here $\tilde{m}$ denotes the semivariation of the measure m.

It remains an open problem if these conditions are sufficient in general for U to be unconditionally converging (for X reflexive they are).

We mention an application of this theorem. Using property 2 and some results of Foiaș and Singer [2] we have

THEOREM 2. *Let T contain no isolated points (and only such T). Then for every unconditionally converging bounded linear operator $U: C_0(T, X) \to C_0(T, X)$ the norm equality*

$$|1 + U| = 1 + |U|$$

holds.

In [2] this is proved for compact and majorable operators. A detailed version of these and related results will appear in [1].

References

[1] I. Dobrakov, *On representation of linear operators on $C_0(T, X)$*, Czech. Math. J. (to appear).

[2] C. Foiaș and I. Singer, *Points of diffusion of linear operators and almost diffuse operators in spaces of continuous functions*, Math. Zeitschrift 87 (1965), p. 434-450.

[3] A. Pełczyński, *Banach spaces on which every unconditionally converging operator is weakly compact*, Bull. Acad. Pol. 10 (1962), p. 641-648.

Sur les espaces complementés de $C(S)$

par

M. L. JONAC et C. SAMUEL (Marseille)

Soient S un espace compact, $C(S)$ l'espace de Banach réel des fonctions définies et continues sur S, à valeurs réelles, muni de la norme uniforme.

Si $\sigma: S \to S$ est un homéomorphisme involutif de S,

$$C_0(S) = \{f \in C(S) | \forall s \in S, f(\sigma s) = -f(s)\}.$$

Le résultat démontré ici est le suivant:

Si B est un sous-espace de $C(S)$, $\pi: C(S) \to B$, une projection de norme 1, alors il existe un espace compact K, un homeomorphisme involutif σ de K, tels que B soit isométriquement isomorphe à $C_\sigma(K)$.

COROLLAIRE. *Soit S un espace métrique compact, B un sous-espace de $C(S)$, $\pi: C(S) \to B$ une projection de norme 1, alors B possède une base monotone.*

Ce résultat est à paraître dans le Bulletin des Sciences Mathématiques.

Die Nuklearität der Lösungsräume der partiellen Differentialgleichungen *

von

YUKIO KŌMURA (Tokyo)

Sei Ω der Raum aller lokalsummierbarer Funktionen auf R^n. Für einen Teilraum Λ des Ω wird der Raum $\Lambda^* = \{f \in \Omega: \int |fg|\, dx < \infty$ für alle $g \in \Lambda\}$ der *Köthe-Dual* von Λ genannt. Λ heisst ein *vollkommener Raum*, wenn $\Lambda = \Lambda^{**}$.

Definition. Die starke Topologie $\tau_b(\Lambda, \Lambda^*)$ des Λ bezüglich des Köthe-Dualen Λ^* heisst die *natürliche Topologie* von Λ.

Für den vollkommenen Raum Λ hat der lineare Differentialoperator P mit konstanten Koeffizienten: $\Lambda \to \Lambda$ eine kleinste abgeschlossene Erweiterung, die wir wieder mit P bezeichnen wollen. Damit ist der Lösungsraum $E_\Lambda = \{u \in \Lambda: Pu = 0\}$ in Λ abgeschlossen.

Sei Λ ein vollkommener Raum, dessen Köthe-Dual nur Funktionen mit kompakten Träger enthält. Sei der Lösungsraum E_Λ des P nuklear bezüglich der natürlichen Topologie $\tau_b(E_\Lambda, E_\Lambda^*)$. Dann ist der Lösungsraum E_Λ in $C(R^n)$ enthalten und seine natürliche Topologie stimmt mit der gleichmässigen Konvergenz auf den kompakten Teilmengen von R^n überein.

SATZ. *Die folgenden Bedingungen für einen linearen partiellen Differentialoperator P mit konstanten Koeffizienten sind äquivalent:*

*) Hier ist der zweite Teil des Vortrages zusammengefasst. Der erste wird in der Arbeit *Duality of linear spaces of functions and nuclearity of solution spaces* in demselben Bande referiert.

1. *P ist hypoelliptisch.*
2. *Der Lösungsraum $\{u \in C(R^n): Pu = 0\}$ ist nuklear.*
3. *Die funktionale Dimension des Lösungsraumes $\{u \in C(R^n): Pu = 0\}$ ist endlich.*

P ist elliptisch dann und nur dann, wenn die funktionale Dimension des Lösungsraumes mit der Anzahl der Veränderlichen übereinstimmt.

Literatur

[1] Y. Kōmura, *Die Nuklearität der Lösungsräume der hypoelliptischen Gleichungen* Funkcialaj Ekvacioj 9 (1966), p 313-324.

[2] — *A note on solution spaces of elliptic equations*, to appear.

On certain topological invariants of Köthe spaces

by

M. M. DRAGILEV (Rostov on Don)

Let N and M denote the class of nuclear spaces and the class of Montel spaces, respectively, defined by matrices with monotonic quotients of the successive rows. In the classes N and M as well as in the class E of spaces which, in a sense, do not differ much from finite-dimensional spaces ($E \subset N \cap M$), the invariants of an algebraic character which dominate the diametral dimension are constructed. Specific problems concerning isomorphisms in the classes E and $N \times E$ are dealt with. A maximal class $M_0 \subset M$ of Köthe spaces which are differentiated by their diametral dimension is distinguished ($M_0 \neq M$; in particular, $E \setminus M_0 \neq \emptyset$). It is shown that in the nuclear spaces of the class $M_0 \cup E$ all bases are quasi-equivalent.

Interpolation of linear operators and its applications

by

E. M. SEMENOV (Voronezh)

Let Banach spaces E_0, E_1 and F_0, F_1 be continuously embedded into locally convex Hausdorff spaces $\mathscr{E}$ and $\mathscr{F}$, respectively. A pair of space (E, F) will be called an *interpolation pair* if

$$\|A\|_{E\to F} \leqslant C \max_{i=0,1} \|A\|_{E_i\to F_i}, \quad \forall A \in \bigcap_{i=0,1} Z(E_i, F_i).$$

A Banach space E of measurable functions on $[0,1]$ is called *symmetric* if

(1) $|x(t)| \leqslant |y(t)|$ and $y(t)\in E$ implies $x(t)\in E$ and $\|x\|_E \leqslant \|y\|_E$;

(2) $x(t)$, $y(t)$ equimeasurable $y(t)\in E$ implies $x(t)\in E$ and $\|x\|_E \leqslant \|y\|_E$;

The function $\varphi_E(\tau) = \|\varkappa_{[0,\tau]}\|_E$ is an important characteristic of the space E. We assume that the space E is separable or is the dual of a separable space. A method leading to interpolation theorems in symmetric spaces is developed. The problem of a description of the class of interpolation pairs is, in a sense, fully solved in the case where E_i and F_i are L_{p_j}-spaces ($j = 1, 2, 3, 4$).

Several concrete problems are then solved with use of the interpolation theorems.

1. The conditions

$$(*) \qquad 1 < \lim_{\tau\to 0} \frac{\varphi_E(2\tau)}{\varphi_E(\tau)}, \qquad \lim_{\tau\to 0} \frac{\varphi_E(2\tau)}{\varphi_E(\tau)} < 2$$

are necessary and sufficient for the boundedness of the Hilbert operator in a symmetric space E.

2. Condition (*) is also necessary and sufficient for the existence of an unconditional basis (i.e. in order that the trigonometric system be a basis) in a symmetric space E.

3. Let σ_Φ denote the symmetrically normed ideal of compact operators in a separable Hilbert space H with the symmetric norming function $\Phi(\xi)$. Let $A = A_R + iA_J$ be a Volterra operator. In order that

$$\sup_{A_J\in\sigma_\Phi} \frac{\|A_R\|\sigma_\Phi}{\|A_J\|\sigma_\Phi} < \infty$$

holds, it is necessary and sufficient that there exists an $\varepsilon > 0$ such that the inequalities

$$(1+\varepsilon)\Phi(\underbrace{1,1,\ldots,1}_{n},0,0,\ldots) \leqslant \Phi(\underbrace{1,1,\ldots,1}_{2n},0,0,\ldots) \leqslant (2-\varepsilon)\Phi(\underbrace{1,1,\ldots,1}_{n},0,0,\ldots)$$

are satisfied for sufficiently large n.

4. Applications to the theory of Fourrier series are considered.

Sur les produits tensoriels ordonnés

par

NICOLAE POPA (Bucharest)

En utilisant la notion de ffl-espace réticulé de type dénombrable (Voir N. Popa, *Les duals des espaces de Fréchet ordonnés*, Rév. Roum. Math. Pures et Appl. 14 (1969), N° 2) on démontre deux théorèmes.

THÉORÈME 1. *Si E est un treillis de Fréchet nucléaire et F est un treillis de Fréchet qui admet un système fondamental d'ensembles relativement compacts solides, alors $B_{c\times c}(E, F)$, muni de l'ordre naturel, est un treillis localement-convexe, complet au sens de l'ordre.*

Soit E, F deux espaces localement convexes séparés ordonnés, ayant des cônes normaux K_1, K_2. Alors sur $E\otimes F$ on considère le cône $C(K_1\otimes K_2)$, engendré par $K_1\otimes K_2$.

Sur $E\hat{\otimes} F$ nous considérons la fermature C de ce cône et nous disons que $E\hat{\otimes} F$ est *ordonné projectif*.

THÉORÈME 2. *Si E est un treillis de Fréchet nucléaire, F est treillis de Fréchet ayant un système fondamental d'ensembles relativement compacts solides, alors $E\hat{\otimes} F$ avec l'ordre projectif est un treillis de Fréchet.*

Representation of finite von Neumann algebras by sheaves

by

SIVLIU TELEMAN (Bucharest)

Let $\mathscr{A}$ be a finite von Neumann algebra on the Hilbert space $\mathscr{H}$, $\mathscr{Z}$ the center of $\mathscr{A}$. The mapping $m \mapsto m \cap \mathscr{Z}$ is a bijection between the set $\mathscr{M}(\mathscr{A})$ of all two-sided maximal ideals in $\mathscr{A}$ and the corresponding set $\mathscr{M}(\mathscr{Z})$. It follows that $\mathscr{A}$ is regular in the sense of Shilov-Willcox. By applying a representation theorem for regular (strongly) semi-simple rings, proved by the author elsewhere, the following theorem is obtained:

There exists a soft sheaf $\tilde{\mathscr{A}}$ of local algebras over C, and a soft sheaf $\tilde{\mathscr{H}}$ of complex vector spaces, having a common compact space as a basis, such that

(a) $\tilde{\mathscr{H}}$ *is a* $\tilde{\mathscr{A}}$*-module.*

(b) $\mathscr{A}$ *is isomorphic to the algebra of all global sections in* $\tilde{\mathscr{A}}$.

(c) $\mathscr{H}$ *is isomorphic to the vector space of all global sections in* $\tilde{\mathscr{H}}$.

(d) *The two isomorphisms are consistent with the action of* $\mathscr{A}$ *on* $\mathscr{H}$, *and that of* $\tilde{\mathscr{A}}$ *on* $\tilde{\mathscr{H}}$.

The properties of the local algebras involved in the representation are then studied.

STUDIA MATHEMATICA, T. XXXVIII. (1970)

Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

III. UNSOLVED PROBLEMS