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# Multiplicity theory *

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**1. Introduction.** The principal purpose of this paper is to develop a multiplicity theory for normal operators acting on a separable Hilbert space (i.e. a theory of unitary equivalence for normal operators) and to compare the theory we develop with some of the theory already in the literature.

The basic tool we use is what is called a canonical decomposition system for certain types of von Neumann algebras. The theory of such systems was introduced and developed in [3] and is a powerful tool in the study of operator algebras. In paragraph two of this paper we summarize parts of the decomposition theory needed for our multiplicity theory. In paragraph three we introduce the notion of spectral classes for a normal operator and a multiplicity function and show that such a function completely characterizes the operator. The domain of the function is the spectral classes and the range of the function is the cardinal numbers. A spectral class is a collection of Borel subsets of the complex numbers and membership in a given class is determined by an equivalence relation induced by the structure of the von Neumann algebra generated by the normal operator.

In paragraph four we present a second form of the theory using the weakly continuous positive linear functionals (on the von Neumann algebra generated by the operator) as the domain of the multiplicity function. This somewhat serves to emphasize the basic algebraic nature of the problem we are treating and, also, makes it easier to compare our theory with some of the theory in the literature. This comparison is made in paragraph five.

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**2. Notation and preliminaries.** We shall use the terminology employed in [2]. $H$ shall always denote a complex Hilbert space and $B(H)$ the ring of bounded linear operators on $H$ to $H$; if $K \subset B(H)$ and $\lambda \epsilon H$, then $K\lambda$ shall denote the closure of the linear span of $\{A\lambda | A \epsilon K\}$. If $K\lambda = H$, then $\lambda$ is called cyclic for $K$. $E$ shall always denote a weakly closed, commutative and symmetric subring of $B(H)$ and $M$ the maximal ideal space of $E$, for $A \epsilon E$, $A^{\wedge}(m)$ denotes the Gelfand transform of $A$. $E'$ shall denote the commutant of $E$; hence, if $H$ is separable, then $E'$ has a cyclic vector (cf. [5]). The techniques of measure and integration play an important role in multiplicity theory, so we shall adopt the usual notations of measure theory. If $X$ is a compact Hausdorff space and $\mu$ is a regular and positive Borel measure on $X$, then we may consider $L_\infty(X, \mu)$ as a ring of operators acting on the Hilbert space $L_2(X, \mu)$ as follows; for each $a \epsilon L_\infty(X, \mu)$ we define $A_a \epsilon B(L_2)$ by $A_a f = af$ for every $f$ in $L_2$. We shall use this convention throughout this paper and note that $L_\infty(X, \mu)$ is a symmetric, weakly closed, and a maximal commutative subring of $B(L_2)$ with a cyclic vector ($f \equiv 1$).

The following theorem summarizes certain properties of rings of operators that we shall need:

THEOREM 2. 1. *Suppose $E$ is a weakly closed, symmetric, and commutative subring of $B(H)$ and $M$ the maximal ideal space of $E$. Then*

1. *$E$ has a unit and, therefore, $M$ is compact;*

2. *$M$ is extremely disconnected i.e. the closure of an open set is open (the open-closed sets are called clopen and form a basis for the topology of $M$);*

3. *if $P \epsilon E$, then $P$ is a projection operator if and only if $P^{\wedge}(m)$ is the characteristic function of a clopen set;*

4. *if $\lambda \epsilon H$, then there exists uniquely a regular and positive Borel measure $\nu$ on $M$ such that*

$$(A\lambda, \lambda) = \int_M A^{\wedge}(m)\, d\nu(m) \quad \textit{for } A \epsilon E;$$

*moreover, if $S$ is a $\nu$-measurable subset of $M$, then there exists a clopen set $V \epsilon M$ such that $\nu(S \Delta V) = 0$, where $s\Delta V$ denotes the symmetric difference $(S \setminus V) \cup (V \setminus S)$.*

*We shall now assume the additional hypothesis that $E'$, the commutant of $E$, has a cyclic vector $\xi_0$ and that in accordance with (4) above $\mu$ denotes the Borel measure associated with $\xi_0$;*

5. *the support of $\mu$ is $M$;*

6. *each equivalent class of $\mu$ measurable sets contains exactly one clopen set (cf. (4) above);*

7. *if $f \epsilon L_\infty(M, \mu)$, then there exists $A \epsilon E$ such that $f(m) = A^{\wedge}(m)$ a.e.;*



8. *if $f \epsilon L_p(M, \mu)$, $p = 1, 2$, then $f$ is equal a.e. to a continuous function from $M$ to the extended number system,*

9. *there is a one-to-one correspondence between the clopen subsets of $M$ and subspaces of $H$ that are invariant for $E'$ given by $V \leftrightarrow P_V H$, where $V$ denotes a clopen subset of $M$ and $P_V \epsilon E$ such that $P_V^{\wedge}(m)$ is the characteristic function of $V$; moreover, the sets $\{A' P_V | A' \epsilon E'\}$ are exactly the weakly closed two sided ideals of $E'$;*

10. *$E_*$, the set of ultrastrongly continuous linear functionals on $E$ is the same as the set of weakly, ultraweakly, and strongly continuous linear functions on $E$, and .*

11. *if $T \epsilon E_*$ and $T \geqslant 0$ (i.e. $T(A^* A) \geqslant 0$ for $A \epsilon E$), then there exists uniquely $\varphi \epsilon L_1(M, \mu)$ such that $\varphi$ is continuous (cf. (8) above) and $T(A) = \int_M A^{\wedge}(m)\varphi(m)\, d\mu(m)$ for $A \epsilon E$; also, there exists $\eta \epsilon H$ such that $T(A) = (A\eta, \eta)$ for $A \epsilon E$.*

Proof. With the exception of the last part of (9) the proofs all the above statements can be found in [3]. An indication of the proofs of 10 and 11 is also in [4]. We shall show now that if $I$ is a weakly closed two-sided ideal in $E'$, then $I = \{A' P_V | A' \epsilon E'\}$ for some clopen set $V \subset M$. Note that $I$ is a von Neumann algebra contained in $B(H)$ so that $Q \subset I \cap I'$, where $Q$ is the principal identity of $I$. Hence, $QH$ is an invariant subspace for $E'$ (in fact, $E'QH = (E'Q)QH \subset IQH = QIH \subset QH$ so that, by the first part of 9, $Q = P_V$ for some $V$. Since $A'x = 0$ for $A' \epsilon I$ and $x \epsilon (QH)^\perp$, we have $A' = A' P_V$.

It is useful to observe that 9 is valid even if $E'$ fails to have a cyclic vector. The importance of 9 is that it allows us to identify the weakly closed two-sided ideal of $E'$ with its invariant subspaces. Many useful facts follow from this; one, for example, is that $E'$ has a minimal weakly closed two sided ideal if and only if $M$ contains isolated points.

Returning to 11 above, suppose $\lambda \epsilon H$ and $T(A) = (A\lambda, \lambda)$ for $A \epsilon E$ (hence, by 10, $T \epsilon E_*$), then there is a continuous $L_1$-function $\varphi_2$ corresponding to $\lambda$. We shall denote the closure of $\{m|_{\varphi_\lambda}(m)\}$ by $S_\lambda$. Therefore, $S\lambda$ is a clopen set and $P_{S_\lambda} H = E'\lambda$. We shall use this notation throughout the remainder of this paper; also, we shall use $\pi(S_\lambda)$ to denote the characteristic function of $S_\lambda$.

Definition 2. 1. Suppose $E, E'$ and $\xi_0$ satisfy the hypothesis of Theorem 2.1. A *canonical decomposition system* is a collection $\{(K_a, \eta_a)\}_{a\epsilon\Gamma}$ such that $\eta_a \epsilon K_a \subset H$ and (1) $\Gamma$ is a well ordered set; (2) $K_a = E\eta_a$, $H = \Sigma_a \oplus K_a$, and $K_a \perp K_\beta$ for $a \neq \beta$; (3) $\varphi_{\eta_a} = \pi(S_{\eta_a})$ and (4) $a < \beta$ implies $S_{\eta_\beta} \subset S_{\eta_a}$.

These systems always exist and their essential properties can be expressed in terms of a dimension function.

Definition 2.2. The *dimension of $H$ relative to $E$*, which we denote by $\dim_E H$, is defined to be the smallest cardinal number $c$ such that $H = \Sigma_{\alpha\in A} \oplus H_\alpha$, where each $H_\alpha$ is a non-trivial cyclic subspace for $E$ and $\operatorname{card} A = c$. For each clopen set $V \subset M$, we set $E_v = \{AP_v \mid A \in E\}$, and for $m \in M$, $d(m) = \inf_V \{\dim_{E_v} P_v H \mid m \in V \subset M\}$.

$d(m)$ is called the *local dimension function* and it has extensive applications (cf. [3]). One such application is:

THEOREM 2.2. *Suppose $E, E', \xi_0, \{S_{\eta_\alpha}\}_{\alpha\in\Gamma}$ satisfy the conditions of Definition 2.1, $H$ is separable, $n$ a positive integer and $m \in M$. Then $m \in S_{\eta_n}$ if and only if $d(m) \geqslant n$.*

For a proof of Theorem 2.2, see [3]. The importance of Theorem 2.2 is that it shows that the $\{S_{\eta_\alpha}\}_{\alpha\in\Gamma}$ depend only on $E$ and not on the choice of $\eta_\alpha$ or $\xi_0$. It is this uniqueness of the $S_{\eta_\alpha}$ that leads to a useful definition of spectral classes for normal operators.

**3. Multiplicity theory 1.** In order to motivate our development of multiplicity theory we shall present an example of two hermitian operators that are not unitarily equivalent. Let $L_2[0,1]$ denote the usual Lebesque space of square summable functions on $[0,1]$ and $L_\infty[0,1]$ the essentially bounded and measurable functions on $[0,1]$. Recall that if $f \in L_\infty[0,1]$ and is continuous on $[0,1]$, then the spectrum of $A_f$ is just the range of $f$; if, in addition, $f$ is strictly increasing on $[0,1]$, $E_\lambda = \{x \mid f(x) \leqslant \lambda\}$, $g_\lambda(x) = \pi(E_\lambda)(x)$ (the characteristic function of $E_\lambda$), and $P(\lambda)$ the spectral resolution of $A_f$, then $P(\lambda) = A_{g_\lambda}$.

Example 3.1. Let $c(x)$ be the Cantor function on $[0,1]$, i.e. $c(x)$ is a continuous non-decreasing function on $[0,1]$ such that $c(1)-c(0) = c(1) = 1$ and $c'(x_0) = 0$ for $x_0$ not in the Cantor set where by Cantor set we mean the one obtained from the usual middle third construction on $[0,1]$. Let $f(x) = x + c(x)$ and $g(x) = 2x$. Then, again considering $L_\infty[0,1] \subset B(L_2[0,1])$, $A_f$ and $A_g$ belong to $L_\infty[0,1]$, have the same spectrum, and generate the same subring of $B(L_2[0,1])$ in both the norm and weak operator topologies ($C_0[0,1]$ and $L_\infty[0,1]$ respectively and, in fact, it is important to note that the latter ring is its own commutant). However, $A_f$ and $A_g$ are not unitarily equivalent. To see this, let $S$ denote the Cantor set, $D_0 = f(S)$, and $P(\lambda)$ and $Q(\lambda)$ the spectral resolutions of $A_f$ and $A_g$ respectively. Then $P(D_0)$ is multiplication by zero and $Q(D_0)$ multiplication by the characteristic function of a set which has Lebesque measure one half.

The basic difference between $A_f$ and $A_g$ is that $f$ takes a zero set onto a set of positive measure whereas $g$ does not. The following theorem gives further insight into situation. In what follows $N_i$ $(i = 1, 2)$ denotes a normal operator in $B(H)$, $E^i$ the weakly closed symmetric ring generated in $B(H)$ by $N_i$ and $N_i^*$, and $M_i$ the maximal ideal space of $E^i$. If $D$ is



a Borel subset of the complex numbers, then $N_i^{\wedge-1}(D) = \{m \mid m \in M_i$ and $N_i^\wedge(m) \in D\}$

THEOREM 3.1. *Suppose each of $N_i$, $E^i$ and $M_i$ $(i = 1, 2)$ has the meaning cited in the previous paragraph and $U$ is a unitary operator in $B(H)$ such that $UN_1 = N_2U$. If $D$ is a Borel subset of the complex plane, then $N_1^{\wedge-1}(D)$ has void interior if and only if $N_2^{\wedge-1}(D)$ has void interior.*

Proof. $A \to UAU^{-1}$ defines a homeomorphism $\varphi\colon M_1 \to M_2$ such that is $z$ is a complex number, then $N_1^\wedge(m) = z$ if and only if $N_2^\wedge(\varphi(m)) = z$.

Remark 3.1. In Example 3.1, $M_i (i = 1, 2)$ is the maximal ideal space of $L_\infty[0,1]$ and $A_f^{\wedge-1}(D_0)$ has void interior while $A_g^{\wedge-1}(D_0)$ contains an open set. Note that the maximal ideal space of the normed closed ring generated each of $A_f$ and $A_g$ is $[0,1]$.

Before proceeding to a converse form of Theorem 3.1, we shall develop a lemma which is technical in nature but quite important.

LEMMA 3.1. *Suppose $N$ is a normal operator in $B(H)$, $E$ the weakly closed symmetric ring generated in $B(H)$ by $N$ and $N^*$, $M$ the maximal ideal space of $E$, $\xi_0$ a unit cyclic vector for $E$, and $V$ a clopen subset of $M$. Then there exists a Borel subset $D$ of complex numbers such that the symmetric difference $V \Delta N^{\wedge-1}(D)$ has void interior.*

Proof. In accordance with 4 and 5 of Theorem 2.1, let $\mu$ be the measure on $M$ corresponding to the cyclic vector $\xi_0$. We shall now introduce an important equivalence relation among the $\mu$-measurable subsets of $M$. If each of $F$ and $G$ is such a set, we write $F \simeq G$ to mean that the symmetric difference $F \Delta G$ has void interior.

If $D_1 = N^\wedge(V)$, then $D_1$ is a compact subset of the complex numbers. If $\{V_\alpha\}_\alpha$ is a tower of clopen sets such that $V \subset V_\alpha \subset N^{\wedge-1}(D_1)$ and $W_0 = \bigcup_\alpha V_\alpha$, then $V \subset \overline{W}_0 \subset N^{\wedge-1}(D_1)$ and we can assume $\overline{W}_0$ is a maximal clopen set with respect to the inclusion relations. Hence, $\overline{W}_0 \simeq N^{\wedge-1}(D_1)$. We shall denote $\overline{W}_0$ by $W$. Since $N$ and $N^*$ generate $E$ in the weak operator topology, there exist polynomials $p_n$, $n = 1, 2, \ldots$, in $N$ and $N^*$ such that

$$\int_M |p_n^\wedge(m) - P_V^\wedge(m)|\, d\mu(m) < \frac{1}{2n}. \tag{3.1a}$$

For each $n$ let

$$S_n = \{m \in W \mid |p_n^\wedge(m)| \leqslant \tfrac{1}{2}\} \quad \text{and} \quad T_n = \{m \in W \mid |p_n^\wedge(m)| > \tfrac{1}{2}\}.$$

Hence, $S_n \cup T_n = W$ and $S_n \cap T_n = \emptyset$. We set $\varepsilon_2 = \mu(W)$, $\varepsilon_1 = \mu(V)$ and we assume that $\varepsilon_2 > \varepsilon_1$ (if $\varepsilon_2 = \varepsilon_1$, then $W \simeq V$). For $m \in S_n \cup V$, $|p_n^\wedge(m) - P_V^\wedge(m)| \geqslant \frac{1}{2}$ and (see (3.1a))

$$\frac{1}{2n} > \int_{S_n \cap V} |p_n^\wedge(m) - P_V^\wedge(m)|\, d\mu(m) \geqslant \frac{1}{2}\mu(S_n \cap V).$$

Hence,

$$\mu(S_n \cap V) < \frac{1}{2^{n-1}} \quad \text{and} \quad \mu(T_n \cap V) > \varepsilon_1 - \frac{1}{2^{n-1}}.$$

For $m \in T_n \cap (W \setminus V)$, $|p_n^\wedge(m) - P_V^\wedge(m)| > \frac{1}{2}$ and

$$\frac{1}{2^n} > \int_{T_n \cap (W \setminus V)} |p_n^\wedge(m) - P_V^\wedge(m)|\, d\mu(m) \geqslant \frac{1}{2}\, \mu\big(T_n \cap (W \setminus V)\big).$$

Hence

$$\mu\big(T_n \cap (W \setminus V)\big) < \frac{1}{2^{n-1}} \quad \text{and} \quad \mu\big(S_n \cap (W \setminus V)\big) > \varepsilon_2 - \varepsilon_1 - \frac{1}{2^{n-1}}.$$

Let

$$K = \bigcup_{n=1} \Big[\bigcap_{i=n} S_i\Big] \quad \text{and} \quad L = \bigcup_{n=1} \Big[\bigcap_{i=n} T_i\Big].$$

$K$ and $L$ are measurable subsets of $W$ and it can be easily verified that they are disjoint. We shall show that the symmetric difference $V \Delta L$ has $\mu$-measure zero. To this end,

$$\mu\Big(V \setminus \bigcap_{n=N}^{\infty} T_n\Big) = \mu\Big(\bigcup_{n=N}^{\infty} [V \setminus T_n]\Big) = \mu\Big(\bigcup_{n=N}^{\infty} [V \cap S_n]\Big)$$

$$\leqslant \sum_{n=N}^{\infty} \mu(V \cap S_n) < \sum_{n=N}^{\infty} \frac{1}{2^{n-1}} < \frac{1}{2^{N-2}}.$$

Hence, $\mu(V \setminus L) = 0$. Also,

$$L \setminus V \subset \bigcap_{n \geqslant N} [T_n \setminus V] \subset T_n \setminus V \subset T_n \cap [W \setminus V].$$

But $\mu(T_n \cap [W \setminus V]) < 1/2^{n-1}$, therefore, $\mu(L \setminus V) = 0$. Consequently, $\mu(L \Delta V) = 0$ and $L \simeq V$. Similarly, $\mu([W \setminus V]\, |K) = \mu(K \setminus [W \setminus V]) = 0$ so that $W \setminus V \simeq K$. We set $D = N(L)$. If $m \in K$, then $|p_n^\wedge(m)| \leqslant \frac{1}{2}$ for large $n$ and if $m \in L$, then $|p_n^\wedge(m)| > \frac{1}{2}$ for large $n$. Hence, if $m_1 \in K$ and $m_2 \in L$, then $N^\wedge(m_1) \neq N^\wedge(m_2)$. Therefore $N^{\wedge -1}(D) \cap K = \emptyset$ and $V \simeq N^{\wedge -1}(D)$. In order to see the last assertion note that $V \cup [W \setminus V] = W \supset N^{\wedge -1}(D)$ so that either $V \setminus N^{\wedge -1}(D)$ or $N^{\wedge -1}(D) \setminus V$ must contain an open set if $V \Delta N^{\wedge -1}(D)$ does not have void interior. The first case contradicts $L \simeq V$ since $L \subset N^{\wedge -1}(D)$. In the second case, $N^{\wedge -1}(D) \cap W \setminus V \simeq N^{\wedge -1}(D) \cap K$ (since $W \setminus V \simeq K$) contains an open set which contradicts $N^{\wedge -1}(D) \cap K = \emptyset$.

We shall now present a converse form of Theorem 3.1. As we shall see later, the following theorem corresponds to case where the multiplicity function has the constant value of one. Also, later we shall reduce the general case to special case of multiplicity one and it is in this reduction that the nature of the multiplicity function becomes more transparant.



THEOREM 3.2. *Suppose $H$ is a separable Hilbert space, $N_i$ $(i = 1, 2)$ a normal operator in $B(H)$, $E^i$ the weakly closed symmetric ring generated by $N_i$ and $N_i^*$, $\xi_i$ a unit cyclic vector for $E^i$, $M_i$ the maximal ideal space of $E^i$, and $\mu_i$ the regular Borel measure corresponding to $\xi_i$. Suppose further that for each Borel subset $D$ of the complex numbers, $N_1^{\wedge -1}(D)$ has void interior if and only if $N_2^{\wedge -1}(D)$ has void interior. Then $N_1$ is unitarily equivalent to $N_2$.*

Proof. Suppose that each of $D_1$ and $D_2$ is a Borel subset of the complex numbers such that $N_2^{\wedge -1}(D_1) \simeq N_2^{\wedge -1}(D_1)$. Then $N_1^{\wedge -1}(D_1) \simeq N_1^{\wedge -1}(D_2)$ and $\mu_1(N_1^{\wedge -1}(D_1)) = \mu_1(N_1^{\wedge -1}(D_2))$. Therefore, we define a measure $\alpha$ on $M_2$ in the following manner; for each clopen subset $W$ of $M_2$ there exists a Borel subset $D$ of complex numbers such that $N_2^{\wedge -1}(D) \simeq W$ (cf. Lemma 3.1). We set $\alpha(W) = \mu_1(N_1^{\wedge -1}(D))$. If $K$ is a $\mu_2$-measurable subset of $M_2$, then there exists a unique clopen set $V$ (cf. 6 of Theorem 2.1) such that $K \simeq V$. We set $\alpha(K) = \alpha(V)$. It is easy to show that $\alpha$ is a positive measure on $M_2$, $\alpha(M_2) = 1$, and $\alpha$ is absolutely continuous with respect to $\mu_2$. Consequently (cf. 11 and 8 of Theorem 2.1), there exists a continuous $L_1$ function $\varphi$ such that $\varphi(m) \geqslant 0$ and

$$\int_{M_2} A^\wedge(m)\, d\alpha(m) = \int_{M_2} A^\wedge(m)\varphi(m)\, d\mu_2(m).$$

for $A \in E^2$. Again, from 11 of Theorem 2.1, there exists a vector $\xi_2' \in H$ such that

$$\int_{M_2} A^\wedge(m)\varphi(m)\, d\mu_2(m) = (A\xi_2', \xi_2')$$

for $A \in E^2$. Since $\alpha(M_2) = 1$, $(I\xi_2', \xi_2') = \|\xi_2'\|^2 = 1$. Also, $\xi_2'$ is cyclic for $E^2$ since, in the contrary case, it would follow from 11 and 8 of Theorem 2.1 that $M_2 \setminus S_{\xi_2'}$ would contain a clopen set $W$ and $\alpha(W) = 0$. Hence, if $D$ is a subset of the complex plane such that $N_2^{\wedge -1}(D) \simeq W$, then $N_1^{\wedge -1}(D)$ has non-void interior and $0 < \mu_1(N_1^{\wedge -1}(D)) = \alpha(W) = 0$ which is absurd.

We now define a mapping $\theta\colon C(M_1) \to C(M_2)$ as follows: if $V_i$, $i = 1, \ldots, n$, is a pairwise disjoint covering of $M_1$ with clopen sets, then there exist $n$ Borel subsets of the complex numbers $D_i$, $i = 1, \ldots, n$, and $n$ clopen subsets of $M_2$, $W_i$, $i = 1, \ldots, n$, such that $N_1^{\wedge -1}(D_i) \simeq V_i$, $N_2^{\wedge -1}(D_i) = W_i$, $i = 1, \ldots, n$. Our hypothesis ($N_1^{\wedge -1}(D)$ has void interior if and only if $N_2^{\wedge -1}(D)$ has void interior) guarantees $W_i$, $i = 1, \ldots, n$, is a covering of $M_2$ with pairwise disjoint clopen sets. If

$$f(m) = \sum_{i=1}^{n} a_i \pi\big(V_i(m)\big),$$

then we set

$$\theta(f)(m) = \sum_{i=1}^{n} a_i \pi\big(W_i(m)\big).$$

Again, our hypothesis guarantees that $\theta$ is well defined; moreover, $\theta$ is an isometric mapping from a dense subset of $L_\infty(M_1, \mu_1)$ to a dense subset of $L_\infty(M_2, \alpha)$. Hence, $\theta$ can be extended to an isometry from $L_\infty(M_1, \mu_1)$ onto $L_\infty(M_2, \alpha)$ or, equivalenty, $C(M_1)$ onto $C(M_2)$ and, hence, from $E^1$ onto $E^2$. If $P_i(D)$ denotes the spectral function for $N_i$ $(i = 1, 2)$, then it easily follows that $\theta\big(P_1(D)\big) = P_2(D)$. Hence, $\theta(N_1) = N_2$ and $\theta(N_1^*) = N_2^*$ so that $\theta$ is a symmetric isomorphism. Also, $\theta(AB) = \theta(A)\theta(B)$ whenever $A^\wedge(m)$ and $B^\wedge(m)$ are clopen step functions on $M_1$ or when $A$ and $B$ are polynomials in $N_1$ and $N_1^*$. Either fact together with the continuity of $\theta$ in the operator norm implies $\theta(AB) = \theta(A)\theta(B)$ for all $A, B \in E^1$. Also

$$\int_{M_1} A^\wedge(m)\, d\mu_1(m) = \int_{M_2} \theta(A)^\wedge(m)\, d\alpha(m)$$

whenever $A^\wedge(m)$ is the characteristic function of a clopen set and, therefore, because of the dominance of the $L_\infty$-norm for all $A \in E^1$.

We now define an operator $U$ by $UA\xi_1 = \theta(A)\xi_2'$. Hence,

$$\|UA\xi_1\|^2 = \big(\theta(A)\xi_2', \theta(A)\xi_2'\big) = \int_{M_2} \theta(A^*A)^\wedge(m)\, d\alpha(m)$$
$$= \int (A^*A)(m)\, d\mu_1(m) = \|A\xi_1\|^2.$$

Since $\xi_1$ and $\xi_2'$ are cyclic for $E^1$ and $E^2$ respectively, $U$ can be extended to a unitary operator.

For $A \in E'$, $UN_1A\xi_1 = \theta(N_1A)\xi_2' = \theta(N_1)\theta(A)\xi_2' = N_2UA\xi_1$. Hence $UN_1 = N_2U$ on a dense subset of $H$ so that $UN_1 = N_2U$ on all of $H$. This completes the proof of Theorem 3.2.

Before extending Theorem 3.2 to the case where cyclic vectors do not exist, we shall look at an example. Let $f(x) = x^p$, $g(x) = x^q$ $(p, q = 1, 2, \ldots)$ and $A_f$ and $A_g$ the corresponding multiplication operators acting on $L_2[0,1]$. If the hypothesis of Theorem 3.2 are not satisfy, $P(\lambda)$ the spectral function for $A_f$, and $Q(\lambda)$ the spectral function for $A_g$, then there exists a Borel subset $D$ of the complex numbers such that either $P(D) = 0$ and $Q(D) \neq 0$ or $P(D) \neq 0$ and $Q(D) = 0$. However, it is easy to see that no such set $D$ exists.

In the first paragraph of the proof of Lemma 3.1 we introduced an equivalence relations among subsets of a maximal ideal space $M$. We shall now, in a similar manner, introduce an equivalence relation among Borel subsets of complex numbers with respect to a given normal operator acting on a separable Hilbert space and use this relation to define a multiplicity function. We shall do this in two steps.

Definition 3.1. Suppose $N$ is a normal operator acting on a separable Hilbert space $H$, $E$ the weakly closed symmetric ring generated by $N$ and $N^*$, $M$ the maximal ideal space of $E$. We define an equivalence relation among the Borel subsets of the complex numbers in the following manner: if each of $D_1$ and $D_2$ is a Borel subset of the complex numbers, then $D_1$ is equivalent to $D_2$ if and only if the symmetric difference $N^{\wedge -1}(D_1) \Delta N^{\wedge -1}(D_2)$ has void interior in $M$, in this case we write $D_1 \simeq D_2$; the induced equivalence classes will be called the *spectral classes of* $N$.

Let us recall that if $E$ is a symmetric and commutative ring acting on a separable Hilbert space $H$, then $E'$, the commutant of $E$, always has a cyclic vector.

Definition 3.2. Suppose that $N, H, E$, and $M$ are as in Definition 3.1 and, in accordance with Definition 2.1, $\{(K_\alpha, \eta_\alpha)\}_{\alpha \in \Gamma}$ is a canonical decomposition system for $E$ and that, for $\alpha \in \Gamma$, $S_{\eta_\alpha}$ is the support of $\varphi_{\eta_\alpha}$. Suppose $\Omega$ is a spectral class of $N$ and $D \in \Omega$. If $n$ is the least positive integer for which $N^{\wedge -1}(D) \cap S_{\eta_{n+1}}$ has void interior, then we say that $\Omega$ has *multiplicity* $n$; if $N^{\wedge -1}(D) \cap S_{\eta_{n+1}}$ has non-void interior for all $n$, then we say $\Omega$ has *infinite multiplicity*.

Remark 3.2. We recall, in view of Theorem 2.2, that the $S_{\eta_j}$ are uniquely determined by the ring $E$. Hence, there is no ambiquity in our definition of multiplicity. Also, we could have defined multiplicity in terms of the local demension function $d(m)$. That is, the multiplicity of $\Omega$ is the least positive integer $n$ such that $N^{\wedge -1}(D) \cap \{m \mid d(m) > n\}$ has void interior. Finally, we should mention that the role of the multiplicity is to reduce to general case to the special case considered in Theorem 3.1. This is apparent in the proof of the sufficiency part of next theorem.

THEOREM 3.3. *Suppose $H$ is a separable Hilbert space, $N_i$ $(i = 1, 2)$ a normal operator on $H$, $E^i$ the weakly closed and symmetric ring generated by $N_i$ and $N_i^*$ $(i = 1, 2)$, and $I \in E^i$ $(i = 1, 2)$, where $I$ denotes the identity of $B(H)$. Then $N_1$ is unitarily equivalent to $N_2$ if and only if $N_1$ and $N_2$ have the same spectral classes with the same respective multiplicities.*

Proof. We shall prove the sufficiency first. Let $\{(K_i, \eta_i)\}$ and $\{(L_i, \xi_i)\}$ be canonical decomposition systems for $E^1$ and $E^2$ respectively. $E^1|_{K_i}$ and $E^2|_{L_i}$ are maximal commutative subrings of $B(K_i)$ and $B(L_i)$ respectively. Our hypothesis (same spectral classes with same respective multiplicities) tells us that if $D$ is a Borel subset of the complex numbers, then $N_1^{\wedge -1}(D) \cap S_{\eta_i}$ has void interior if and only if $N_2^{\wedge -1}(D) \cap S_{\xi_i}$ has void interior. Inasmuch as $S_{\eta_i}(S_{\xi_i})$ is the maximal ideal space $E^1|_{K_i}(E^2|_{L_i})$ we have, by Theorem 3.2, that there exists $U_i$ such that $U_iK_i = L$

and $U_iN_i = N_2U_i$. If $U = \sum_i U_i$, then $U$ is a unitary operator and $UN_1 = N_2U$.

We shall now prove the necessity. Suppose $UN_1 = N_2U$, where $U$ is unitary. We first show that the mapping $A \to UAU^{-1}$ takes $(E^1)'$ onto $(E^2)'$. For $A \in (E^1)'$ and $B \in E^2$, $(U^{-1}BU)A = A(U^{-1}BU)$, so that upon multiplying the last equation on the left by $U$ and the right by $U^{-1}$, we get $B(BAU^{-1}) = (UAU^{-1})B$, so that $U(E^1)'U^{-1} \subset (E^2)'$. Similarly, $U^{-1}(E^2)'U \subset (E^1)'$; hence $U(E^1)'U^{-1} = (E^2)'$. Also, if $\xi_0$ is cyclic for $(E^1)'$, then $H = (E^1)'\xi_0 = U(E^1)'\xi_0 = U(E^1)'U^{-1}U\xi_0 = (E^2)'U\xi_0$, so that $U\xi_0$ is cyclic for $(E^2)'$.

Suppose now that $\{(K_i, U\eta_i)\}$ is a canonical decomposition system for $E^1$ and $L_i = E^2U\eta_i$. We shall show that $\{(L_i, U\eta_i)\}$ is a canonical decomposition system for $E^2$. Inasmuch as $H = \sum_i \oplus UE^1U^{-1}U\eta_i = \sum_i \oplus E^2U\eta_i$, we need to show that $\varphi_{U\eta_i} = \pi(S_{U\eta_i})$ (cf. Definition 2.1). To this end, if $A \in E^2$, then

$$(AU\eta_i, U\eta_i) = (U^{-1}AU\eta_i, \eta_i) = \int_{S_{\eta_i}} (U^{-1}AU)^\wedge(m)\,d\mu_1(m)$$

$$= (P_{S_{\eta_i}}U^{-1}AU\eta_i, \eta_i) = ([UPU^{-1}]AU\eta_i, U\eta_i).$$

On the other hand,

$$(AU\eta_i, U\eta_i) = \int_{M_2} A^\wedge(m)\varphi_{U\eta_i}(m),$$

so that we need to show $UP_{S_{\eta_i}}U^{-1} = P_{S_{U\eta_i}}$ or, which is the same, $UP_{S_{\eta_i}} = P_{S_{U\eta_i}}U$.

Let $x = X_1 + x_2 \in H$, where $P_{S_{\eta_i}}x_1 = x_1$ and $P_{S_{\eta_i}}x_2 = 0$. Then $UP_{S_{\eta_i}}x = Ux_1$. Also,

$$P_{S_{U\eta_i}}x = P_{S_{U\eta_i}}x_1 + P_{S_{U\eta_i}}x_2.$$

Suppose $B \in (E^2)'$. Then $U^{-1}BU \in (E^1)'$ and $(BU\eta_i, Ux_2) = (U^{-1}BU\eta_i, x_2) = 0$ inasmuch as $K_i = P_{S_{\eta_i}}H$ is an invariant subspace for $(E^1)'$ (cf. 9, Theorem 2.1). Hence, $P_{S_{U\eta_i}}Ux_2 = 0$. Since $\eta_i$ is cyclic for $E^1$ on $K_i$, there exists $A_n \in E^1$, $n = 1, 2, \ldots$, such that $A_n\eta_i \to x_1$ as $n \to \infty$. Hence, $UA_nU^{-1}U\eta_i \to Ux_1$ as $n \to \infty$ and, therefore, $P_{S_{U\eta_i}}Ux_1 = Ux_1$ since $(UA_nU^{-1}) \in E^2 \subset (E^2)'$ and $(UA_nU^{-1})U\eta_i \in P_{S_{U\eta_i}}H$. Hence, $P_{S_{U\eta_i}}Ux = P_{S_{U\eta_i}}Ux_1 = Ux_1 = UP_{S_{\eta_i}}x$ for all $x \in H$, so that $P_{S_{U\eta_i}}U = UP_{S_{\eta_i}}$.

Hence, we have shown that the mapping $A \to UAU^{-1}$, $A \in E^1$, takes canonical decomposition systems onto canonical decomposition systems. Also, this mapping induces in a natural way a homeomorphism $\Phi\colon M_1 \to M_2$, via $A^\wedge(m) = (UAU^{-1})^\wedge(\Phi(m))$, of $M_1$ onto $M_2$ such that

$$\{\Phi(m) \mid m \in N_1^{\wedge -1}(D) \cap S_{\eta_i}\} = N_2^{\wedge -1}(D) \cap S_{U\eta_i}.$$



Consequently, $N_1$ and $N_2$ have the same spectral classes with the same respective multiplicities.

This completes the proof of Theorem 3.3.

Remark 3.3. Our assumption that each of $E^1$ and $E^2$ contain the identity of $B(H)$ is not a serious restriction. In fact, suppose $P_i$ is the principal identity of $E^i$ (cf. 1, Theorem 2.1), then our theorem tells us when $N_1|_{P_1(H)}$ is unitarily equivalent to $N_2|_{P_2(H)}$. Inasmuch as $N_ix = 0$ if and only if $P_ix = 0$ $(i = 1, 2)$; we see that if $N_1|_{P_1(H)}$ is unitarily equivalent to $N_2|_{P_2(H)}$, then $N_1$ is unitarily equivalent to $N_2$ provided their respective null spaces have the same dimension.

Remark 3.4. It may aid our intuition if we look into the relationship between the spectral classes and eigenvalues of a normal operator and the corresponding multiplicities of each. Hence, suppose that $N$, $E$ and $M$ have the properties listed in Definition 3.1 and that the Borel set consisting of the complex number $\{\lambda_0\}$ is not equivalent to the empty set (in the sense of the equivalent relation defined in Definition 3.1). This can occur if and only if there exists an isolated point $m_0 \in M$ such that $N^\wedge(m_0) = \lambda_0$. In this case $\lambda_0$ is an eigenvalue of $N$, the multiplicity of its spectral class is the same as its multiplicity as an eigenvalue which is $d(m_0)$, i.e. the local dimension at $m_0$. Finally, we note that if $m \in M$, then the functional $f_m(A) = A^\wedge(m)$, $A \in E$, is a norm continuous functional, and is weakly continuous if and only if $m$ is an isolated point of $M$.

**4. Multiplicity and positive functionals.** We shall now show that it is possible to derive a multiplicity theory using the weakly continuous positive functionals on $E$. One of the basic tools in this approach is the Dixmier-Sakai Theorem that asserts that a $C^*$ algebra has a representation as a von Neumann algebra if and only if it has a pre-dual as a Banach space.

THEOREM 4.1. *Suppose $H$, $N_i$, $E^i$, $M^i$, $\xi_i$, and $\mu_i$ $(i = 1, 2)$ satisfy the hypothesis of Theorem 3.2. Then the following two statements are equivalent:*

(1) *$N_1$ is unitarily equivalent to $N_2$;*

(2) *there exists an isometric isomorphism $\Phi\colon E_*^1 \to E_*^2$ such that $\Phi f \geqslant 0$ if and only if $f \geqslant 0$ and $(\Phi f)(N_2) = f(N_1)$ for all $f \in E_*^1$.*

Proof. The proof that (1) implies (2) is left to the reader. Suppose, therefore, (2) holds. In view of the Dixmier-Sakai Theorem, we may regard $E^i$, as a Banach space, to be $(E_*^i)^*$; hence, the mapping $\Phi$ induces an adjoint mapping, say $\theta$, defined by $[\theta(A)](f) = A(\Phi(f))$ for all $f \in E_*^1$ and $A \in E^2$. Also,

$$\|\theta(A)\| = \sup\{|[\theta(A)](f)|;\ \|f\| = 1\}$$

$$= \sup\{|A(\Phi(f))|;\ \|f\| = 1\} = \sup\{|A(g)|;\ \|g\| = 1\}\ \|A\|$$

so that $\theta$; $E^2 \to E^1$ and, in fact, is an isometry. Also, $[\theta(N_2)](f) = N_2(\Phi(f)) = \Phi(f)(N_1) = f(N_1) = N_1(f)$ so that $\theta(N_2) = N_1$. Finally, if $f \in E^1_*$, then $f \geqslant 0$ if and only if $F(A) = (Ax, x)$ for some $x \in H$ (cf. [4]); hence, $\theta(A) \geqslant 0$ if and only if $A \geqslant 0$.

The positivity of $\theta$ allows us to establish that $\theta$ is an algebraic isomorphism. We shall do this in several steps. We first show $\theta$ preserves order. Suppose $A \geqslant 0$ and $B \geqslant 0$. Then $A \wedge B \leqslant A$, $\theta(A \wedge B) \leqslant \theta(A)$, and, also, $\theta(A \wedge B) \leqslant \theta(B)$. Consequently, $\theta(A \wedge B) \leqslant \theta(A) \wedge \theta(B)$. Inasmuch as $E^2$ is a weakly closed subring of $B(H)$, a typical Zorn's Lemma argument shows that there exists $C \in E^2$ such that $C \geqslant 0$ and $\theta(C) = \theta(A) \wedge \theta(B)$. But $\theta(C) \leqslant \theta(A)$ implies $C \leqslant A$ and, similarly, $C \leqslant B$ so that $C \leqslant A \wedge B$ and $\theta(C) = \theta(A) \wedge \theta(A \wedge B)$. Thus, $\theta(A \wedge B) = \theta(A) \wedge \theta(B)$. Note that if $A$ and $B$ are projections in $E^2$, then $A \wedge B = AB$ and $\theta(A \wedge B) = \theta(AB)$.

Also note that $\theta(I) = I$.

Inasmuch as $\theta$ is an isometry, we see that if $A$ and $B$ are projections such that $AB \neq 0$, then $\|\theta(AB)^\wedge(m)\|_\infty = 1$ and this together with $\theta(A) \wedge \theta(B) = \theta(AB) = \theta(A \wedge B)$ implies that supports of $\theta(A)^\wedge$ and $\theta(B)^\wedge$ intersect. Hence, there exists $m_0 \in M_1$ such $\theta(A)^\wedge(m_0) = 1 = \theta(B)^\wedge(m_0)$. Therefore, $\|[\theta(A)\theta(B)]^\wedge(m)\|_\infty = 1$. Also, from $0 \leqslant \theta(A)^\wedge(m) \leqslant 1$ and $0 \leqslant \theta(B)^\wedge(m) \leqslant 1$ we see that $\theta(A)\theta(B) \leqslant \theta(A)$ and $\theta(A)\theta(B) \leqslant \theta(B)$ or, what is the same, $\theta(A)\theta(B) \leqslant \theta(A) \wedge \theta(B) = \theta(AB)$. Hence, if $B = I - A$, then $\theta(A) - [\theta(A)]^2 \leqslant \theta(A(I-A))$, so that $\theta(A)$ is a projection. It follows now that if $A$ and $B$ are projections, that $\theta(A)\theta(B) = \theta(A) \wedge \theta(AB)$ so that $\theta$ is multiplicative on projections and, by linearity, on clopen step functions. Since $\theta$ is an isometry in the $L_\infty$-norm, $\theta(AB) = \theta(A)\theta(B)$ for all $A$ and $B$ in $E^2$. Also, it is easy to verify that $\theta(A^*) = \theta(A)^*$ for $A \in E^2$.

We now define a measure $\alpha$ on $M_2$ in the following manner. If $V$ is a clopen subset of $M_2$, set

$$\alpha(V) = \int_{M_1} \theta(P_V)^\wedge(m)\, d\mu_1(m).$$

If $K$ is a Borel subset of $M_2$, then there exists a clopen set $V$ such that $V \simeq K$ (recall $V \simeq$ means $V \Delta K$ has void interior) and we set $\alpha(K) = \alpha(V)$. The positivity of $\theta$ implies $\alpha \geqslant 0$, $\theta(I) = I$ and $\mu(M_1) = 1$ implies $\alpha$ is supported on $M_2$ and $\alpha(M_2) = 1$, and $\mu_2(K) = 0$ implies $K$ has void interior (therefore, $K \sim 0$) and $\alpha(K) = 0$. Hence, $\alpha$ is absolutely continuous with respect to $\mu_2$ and there exist $\xi(m) \in L_1(M_2, \mu_2)$ and $\xi_2' \in H$ (cf. 11 and 8 of Th. 2.1) such that, for $A \in E^2$,

$$\int_{M_2} A^\wedge(m)\, d\alpha(m) = \int_{M_2} A^\wedge(m)\, \xi(m)\, d\mu_2(m) = (A\xi_2', \xi_2').$$



$\alpha(M_2) = I$ implies $\|\xi_2'\| = 1$. Note that if $\xi_2'$ were not cyclic for $E^2$, then $S_\xi$ would miss a clopen set and $\alpha(M_2) < 1$ which would be a contradiction.

We now define an operator $U$ by $U\theta(A)\xi_1 = A\xi_2'$ for $A \in E^2$. Hence

$$\|U\theta(A)\xi_1\|^2 = \|A\xi_2'\|^2 = \int_{M_2} (A^*A)^\wedge(m)\, d\alpha(m)$$
$$= \int_{M_1} \theta(A^*A)^\wedge(m)\, d\mu_1(m) = \big(\theta(A)^*\theta(A)\xi_1, \xi_1\big) = \|\theta(A)\xi_1\|^2.$$

Hence, $U$ can be extended to an unitary operator and for $A \in E^2$, $UN_1\theta(A)\xi_1 = U\theta(N_2)\theta(A)\xi_1 = U\theta(N_2A)\xi_1 = N_2A\xi_2' = N_2U\theta(A)\xi_1$. Hence $UN_1 = UN_2$ on a dense subset of $H$ and, therefore, $N_1$ is unitarily equivalent to $N_2$. Note that $U^{-1}AU = \theta(A)$.

This completes the proof of Theorem 4.1.

We shall now consider the case where $E^1$ and $E^2$ are not maximal commutative, i.e. do not have cyclic vectors. Again we introduce a multiplicity function in order to reduce this more general case to the special case considered in the previous Theorem. First of all we note that in Theorem 4.1 we could achieved the same result by assuming $\varphi$ was an isometric isomorphism from the cone of positive functionals in $E^1_*$ to the cone of positive functionals in $E^2_*$, for such a map could be extended to an isometric isomorphism of $E^1_*$ to $E^2_*$. We shall denote the cone of positive functionals of $E^i_*$ by $(E^i_*)^+$ $(i = 1, 2)$.

Definition 4.1. Suppose (1) $E$ is a commutative, symmetric and weakly closed ring acting on a separable Hilbert space $H$ and containing the identity of $B(H)$; (2) $E'$ has a cyclic vector $\xi$ and $\mu$ is the Borel measure on the maximal ideal space $M$ of $E$ such that $\mu$ corresponds to $\xi$ (cf. 4 and 5 of Theorem 2.1); (3) $\{(K_i, \eta_i)\}$ a cannonical decomposition system for $E$; and (4) $T$ a positive function on $E$ and $\varphi \in L_1(M, \mu)$, where $T$ and $\varphi$ satisfy 11 of Theorem 2.1. The multiplicity of $T$ is the least positive integer $n$ such that $\{m \mid \varphi(m) > 0\} \cap S_{\eta_{n+1}}$ is empty provided such an $n$ exists. If no such $n$ exists, we say $T$ has infinite multiplicity.

THEOREM 4.2. *Suppose $H$, $N_i$, $E^i$, and $M_i$ $(i = 1, 2)$ satisfy the hypothesis of Theorem 3.3. Then $N_1$ is unitarily equivalent $N_2$ if and only if there exists an isometric isomorphism $\Phi\colon (E^1_*)^+ \to (E^2_*)^+$ such that $\Phi(f)(N_2) = f(N_1)$, and the multiplicity of $\Phi(f)$ equals the multiplicity of $f$ for all $f \in (E^1_*)^+$.*

Proof. If $UN_1 = N_2U$, $U$ unitary, then the map $\Phi$ defined by $\Phi(f)(A) = f(U^{-1}AU)$, for all $A \in E^2$, satisfies the requirements.

Conversely, suppose such a map $\Phi$ exists, $\{(K_i, \eta_i)\}$ is a canonical decomposition system for $E^1$ and $\{(L_i, \xi_i)\}$ a canonical decomposition system for $E^2$. The maximal ideal space of $E^1|_{K_i}$ (respectively, $E^2|_{L_i}$)

is $S_{\eta_i}$ (respectively, $S_{\xi_i}$). The positive functionals on $E^1|_{K_i}(E^2|_{L_i})$ arise from non-negative function in $L_1(M_1, \mu_1)$ $(L_1(M_2, \mu_2))$ whose support is contained in $S_{\eta_i}$ $(S_{\xi_i})$ (cf. [4]), where $\mu_i$ has the usual meaning. Our condition implies that there exists a restriction $\Phi_i$ of $\Phi$ such that $\Phi_i$ is an isometric isomorphism from $\big((E^1|_{K_i})_*\big)^+$ to $\big((E^2|_{L_i})_*\big)^+$ satisfying $\Phi_i(N_1|_{K_i}) = N_2|_{L_i}$. From Theorem 4.1, there exists partial isometry from $K_i$ onto $L_i$ such that $V_i N_1 = N_2 V_i$. It easily follows that if $U = \sum_i \oplus V_i$, then $U$ is unitary and $UN_1 = N_2 U$.

This concludes the proof of Theorem 4.2. Also, we note again that the assumption that $I \in E^i$ is not a serious restriction.

**5. Conclusion.** We shall conclude this paper by comparing our results with those presented in [1] and [6]. In our presentation the essential restriction of the normal operators is that they operate on a separable Hilbert space. This is done in order to insure that the commutant of a normal operator has a cyclic vector. In both of [1] and [6] there is no separability requirement on the underlying Hilbert space.

We shall first compare our theory with that in [1] under the assumption that the underlying Hilbert space is separable. Both theories associate with a normal operator a multiplicity function which completely characterizes the operator, however, the domains of definition of the two functions are different. We have assigned a multiplicity to each weakly continuous linear functional on the weakly closed ring, say $E$, generated by $N$ and $N^*$. In [1] the multiplicity is defined for each norm continuous linear function on $E$ which is a much more extensive class of linear functions.

In order to make a sharper comparison let us set the theory in [1] into our context. Hence, $E$, $M$ and $\{(K_i, \eta_i)\}$ have our usual meaning. In [1], a multiplicity $h(\cdot)$ is assigned to each projection $P \in E$ as follows. If $P = 0$, then $h(P) = 0$. If $P = P_U$ for some clopen set $U$, then $h(P_U)$ is the largest integer $n$ such that $U \cap S_{\eta_n} = U$; if $U \cap S_{\eta_n} = U$ for all $n$, the $h(P_u) = \infty$. Actually, in [1] the multiplicities is defined for measures on the spectrum of $N$. Suppose $x \in H$ and $\alpha_x(D) = \big(P(D)x, x\big)$, where $\{P(D)\}$ denotes the spectral decomposition of $N$. If $\sigma$ is a measure on the spectrum of $N$, then $\{x \mid \alpha_x \ll \sigma\}$ is an invariant subspace of $E'$ (the commutant of $E$) and, therefore, the projection on this subspace is given by some $P_U \in E$, $U$ clopen; we define $h(\sigma) = h(P_U)$. If $\sigma = \alpha_x$ for some $X$, then the corresponding $U$ is $S_x$.

Hence, we can compare the two multiplicity functions by $h(\cdot)$ and $l(\cdot)$ (we use $l(\cdot)$ to denote the function developed in this paper) by their action on clopen sets or, what is the same, projections in $E$. One essential difference between $h$ and $l$ is that if $P_i \in E$ $(i = 1, 2)$ and $P_1 \leqslant P_2$, then $h(P_\circ) \leqslant h(P_1)$ and $l(P_1) \leqslant l(P_2)$.



In [6] it is shown that $N_1$ is equivalent to $N_2$ if and only if there exists a star-isomorphism $\beta$ of $(E^1)'$ onto $(E^2)'$ such that $\beta(N_1) = N_2$. One interpretation of our Theorem 4.2, is that we can restrict $\beta$ to $E^1 \subset (E^1)'$ under the assumption that $H$ is separable. From this point of view, Example 3.1, has additional interest.

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