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STUDIA MATHEMATICA, T. XXXVIII. (1970)

Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

Small operators between Banach and Hilbert spaces

by

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1. Let B be a Banach space and let H be a Hilbert space with inner product $(\cdot,\cdot)$. We shall study operators from B into H which are sufficiently "small" to decompose continuous linear random processes on H.

Here are some definitions. Let P be a probability measure on a space Ω. Let $L^0(\Omega, P)$ be the space of all real P-measurable functions on Ω modulo functions vanishing P-almost everywhere. $L^0(\Omega, P)$ has its usual topology of convergence in measure. Given a real topological linear space S, a *continuous linear process* on S is a continuous linear map L from S into some $L^0(\Omega, P)$. S' denotes the (dual) space of continuous linear real-valued forms on S. L is called *decomposable* iff (i.e. if and only if) there is a mapping $\omega \to L_\omega$ from Ω into S' such that for every x in S, $L_\omega(x) = L(x)(\omega)$ for almost all ω. A continuous linear map C from another topological vector space X into S will be called *L^0-decomposing* iff $L\circ C$ is decomposable on X for every continuous linear process L on S.

The following result has been stated by S. Kwapień:

THEOREM 1. *An operator A from B into H is L^0-decomposing iff $A = J\circ C$ for some Hilbert-Schmidt operator J from H into itself and bounded operator C from B into H.*

The proof to be given here uses the following probabilistic result which may be of independent interest. (I do not know what would be the largest possible function of α in place of $\alpha^2/4$.)

LEMMA 1. *Let A_j, $j = 1, 2, \ldots$, be independent events, and $\alpha > 0$. Let $B_j \subset A_j$ for all j and $P(B_j) \geqslant \alpha P(A_j)$. Then*

$$P\Big(\bigcup_j B_j\Big) \geqslant \alpha^2 P\Big(\bigcup_j A_j\Big)/4.$$

Proof. If $P(A_j) = 0$ for all j, there is nothing to prove. Otherwise we have $\alpha \leqslant 1$. If, for some j, $P(A_j) > \alpha/2$, then

$$P(B_j) > \alpha^2/2 > \alpha^2 P\Big(\bigcup A_j\Big)/4.$$

So we assume $P(A_j) \leqslant \alpha/2$ for all j. Let N be such that

$$P\Big(\bigcup_{j=1}^{N-1} A_j\Big) \leqslant \alpha/2$$

(e.g. $N = 1, 2$). Then

$$P\Big(B_N \sim \bigcup_{j=1}^{N-1} B_j\Big) \geqslant P(B_N) - P\Big(A_N \cap \Big(\bigcup_{j=1}^{N-1} A_j\Big)\Big) \geqslant (\alpha - \alpha/2)P(A_N).$$

So

$$P\Big(\bigcup_{j=1}^{N} B_j\Big) \geqslant \alpha \sum_{j=1}^{N} P(A_j)/2.$$

If for some N, the latter quantity is at least $\alpha^2/4$, then

$$P\Big(\bigcup B_j\Big) \geqslant \alpha^2/4 \geqslant \alpha^2 P\Big(\bigcup A_j\Big)/4.$$

Otherwise, we let $N \to \infty$ and obtain

$$P\Big(\bigcup B_j\Big) \geqslant \alpha P\Big(\bigcup A_j\Big)/2 \geqslant \alpha^2 P\Big(\bigcup A_j\Big)/4, \qquad \text{q.e.d.}$$

LEMMA 2. *Let A be an L^0-decomposing operator from B into H, and A^t its transpose from H into the dual space B^*. Then*

$$\sum \|A^t \varphi_n\|_{B^*}^2 < \infty$$

for any orthonormal set $\{\varphi_n\}$ in H.

Proof. Let $\|A^t\varphi_j\|_{B^*} = a_j$. If $\sum a_j^2 = \infty$, we can assume $a_j > 0$ for all j. We choose $b_j > 0$ such that $\sum b_j^2 = \infty$ and $b_j/a_j \to 0$ as $j \to \infty$. (Let $n_k \uparrow$, $\sum_{j=n_k+1}^{n_{k+1}} a_j^2 > k^2$, and $b_j = a_j/k$ for $n_k < j \leqslant n_{k+1}$.) Let X_j be independent random variables such that

$$P(X_j = 1/b_j) = P(X_j = -1/b_j) = P(X_j \neq 0)/2 = b_j^2/2.$$

Then we can let $L(\varphi_j) = X_j$ for all j and extend L to a continuous linear process on H. Choose x_j in B with $\|x_j\|_B = 1$ and $(\psi_j, \varphi_j) > a_j/2$, where $\psi_j = A(x_j)$. Then

$$P(|L(\psi_j)| \geqslant a_j/2b_j) \geqslant P(|L(\varphi_j)| = 1/b_j)/2$$

since $L(\psi_j) = c_j X_j + Y$, where $c_j > a_j/2$ and Y is a symmetric random variable independent of X_j. Thus we can apply Lemma 1 with $\alpha = \frac{1}{2}$, $A_j = \{|X_j| = 1/b_j\}$, $B_j = A_j \cap \{|L(\psi_j)| \geqslant a_j/2b_j\}$. Hence for any r,

$$P\Big(\bigcup_{j=r}^{\infty} B_j\Big) \geqslant (1/2)^2/4 = 1/16,$$

since $P\Big(\bigcup_{j=r}^{\infty} A_j\Big) = 1$ by the zero-one law $\big(\sum P(A_j) = +\infty\big)$. Thus B_j occurs infinitely often with probability at least $1/16$. Since $\|x_j\| = 1$ and $a_j/2b_j \to \infty$, $L \circ A$ is not decomposable. Lemma 2 is proved.

Proof of Theorem 1. The "if" part follows from the well known results of Sazonov and Minlos. The converse follows from Lemma 2 and a theorem of Sudakov [7] (for another proof see Słowikowski [6]; cf. Kwapień [4]). Sudakov's theorem states that if C is an operator from a Hilbert space H into a normed space X such that for every orthonormal set $\{\varphi_n\}$, $\sum \|C\varphi_n\|^2 < \infty$, then $C = U \circ V$, where $U: H \to X$ is bounded and $V: H \to H$ is Hilbert-Schmidt. But the total length of the proof can apparently be shortened somewhat by exploiting L^0-decomposability further before passing to pure operator theory.

Suppose A is L^0-decomposing from B into H. A sequence $\{\eta_n\} \subset H$ is called *weakly 2-summable* iff $\sum |(\eta_n, \varphi)^2| < \infty$ for each φ in H. Then the operator $\eta: \varphi \to \{(\eta_n, \varphi)\}_{n=1}^{\infty}$ is bounded from H into the Hilbert space l_2 of square-summable sequences, by the Banach-Steinhaus theorem. Thus $\eta \circ A$ is L^0-decomposing. η^t applied to the standard orthonormal basis of l_2 yields $\{\eta_n\}_{n=1}^{\infty}$. Hence, by Lemma 2, $\sum \|A^t \eta_n\|^2 < \infty$. So A^t is a "2-absolutely summing" operator. It can be suitably approximated by operators with finite rank. Thus according to Pietsch [5], p. 243-244, we have y_j in B^* and f_j in H such that for every φ in H,

$$A^t \varphi = \sum_{j=1}^{\infty} (\varphi, f_j) y_j,$$

where $\sum \|f_j\|_H^2 < \infty$ and for any $x \in B$, $\sum |y_j(x)|^2 < \infty$. Now $\varphi \to \{(\varphi, f_j)\}_{j=1}^{\infty}$ is a Hilbert-Schmidt operator from H into l_2, and $\{u_j\}_{j=1}^{\infty} \to \sum u_j y_j$ is a bounded operator from l_2 into B^*. Thus A^{tt} from B^{**} into H is a composition of a Hilbert-Schmidt operator with a bounded operator. We can replace l_2 by a subspace of H (or if H is finite-dimensional, the theorem is trivial). Then, restricting A^{tt} to $B \subset B^{**}$, we obtain A and Theorem 1 as stated, q. e.d.

Sudakov [7] obtains the following result using his deeper theorem. So a simple, direct proof may be of some interest. (It seems like a "compactness and continuity" fact, but the proof seems not to work that way.)

PROPOSITION 1. *Let T be a bounded operator from a Hilbert space H into a Banach space such that for every orthonormal set $\{\varphi_j\}$ in H, $\sum \|T\varphi_j\|^2 < \infty$. Then*

$$\sup\Big\{\sum \|T\varphi_j\|^2 : \{\varphi_j\} \text{ orthonormal}\Big\} < \infty.$$

Proof. We can assume T has norm 1. If the supremum is $+\infty$, it suffices to show that for each n and orthonormal $\psi_1, \ldots, \psi_n$,

$$\sup\Big\{\sum \|T\varphi_j\|^2 \colon \{\varphi_j\} \text{ orthonormal}, \ \varphi_j = \psi_j, 1 \leqslant j \leqslant n\Big\} = \infty,$$

for then we could chooose orthonormal φ_j inductively to make

$$\sum_{j=1}^{n_k} \|T\varphi_j\|^2 > k, \qquad k = 1, 2, \ldots$$

And, further, we need only treat $n = 1$.

Given ψ_1 and $M > 0$, choose orthonormal η_j such that

$$\sum_{j=1}^{\infty} \|T\eta_j\|^2 > (M + 2^{3/2})^2.$$

Let U be a unitary operator (rotation) such that $U(\eta_1) = \psi_1$ and $U(\varphi) = \varphi$ whenever $(\varphi, \eta_1) = (\varphi, \psi_1) = 0$. Let α_1 and α_2 form an orthonormal basis of the linear span of ψ_1 and η_1 (if ψ_1 and η_1 are proportional, there is no problem). Let I be the identity operator. Then since $I - U$ has its range in this span,

$$\sum \|(I - U)\eta_j\|^2 \leqslant \sum_j \sum_{k=1}^{2} |(\eta_j, (I - U^*)\alpha_k)|^2 \leqslant 8,$$

and

$$\begin{aligned} M + 2^{3/2} &\leqslant \Big(\sum \|T\eta_j\|^2\Big)^{1/2} \leqslant \Big[\sum \big(\|TU\eta_j\| + \|T(I-U)\eta_j\|\big)^2\Big]^{1/2} \\ &\leqslant \Big(\sum \|TU\eta_j\|^2\Big)^{1/2} + \Big(\sum \big(\|T\| \cdot \|(I-U)\eta_j\|\big)^2\Big)^{1/2} \\ &\leqslant \Big(\sum \|TU\eta_j\|^2\Big)^{1/2} + 2^{3/2}. \end{aligned}$$

So we can let $\varphi_j = U\eta_j$, orthonormal, with $\varphi_1 = \psi_1$ and $\sum \|T\varphi_j\|^2 \geqslant M$, q.e.d.

2. Epsilon-entropy and the Gaussian process. Once again let A be a bounded operator from a Banach space B into a Hilbert space H. Let B_1 be the unit ball of B. Then we may say A is a "small" operator if $A(B_1)$ is a "small" set. One way to measure smallness is by way of ε-entropy. For any set $C \subset H$ and $\varepsilon > 0$, let

$$\operatorname{diam}(C) = \sup\{\|x - y\| \colon x, y \in C\},$$

$$N(C, \varepsilon) = \inf\Big\{n \colon C \subset \bigcup_{j=1}^{n} C_j \text{ for some } C_j \text{ with } \operatorname{diam} C_j \leqslant 2\varepsilon, \ j = 1, \ldots, n\Big\},$$

$$r(C) = \limsup_{\varepsilon \downarrow 0} [\log\log N(C, \varepsilon)/\log(1/\varepsilon)].$$

Let $r(A) = r(A(B_1))$. The transpose operator A^t takes H into B^*; let $r(A^t) = r(A^t(H_1))$, where H_1 is the unit ball of H, and we use the norm in B^* in defining r. Chevet [1] considers diagonal operators between l_p-spaces and obtains that $r(A) = r(A^t)$ for such operators. It would be of interest to know whether this relation holds generally.

There is a normalized Gaussian linear process G on H. G maps H into $L^2(\Omega, P)$ for some probability space (Ω, P). $EG(x)G(y) = (x, y)$, and if $(x, x) = 1$, and E is a Borel set in the real line,

$$P\big(G(x) \in E\big) = (2\pi)^{-1/2} \int_E e^{-t^2/2}\, dt.$$

There are operators A with $G \circ A$ decomposable which are far from being L^0-decomposing. It is known ([2], Sudakov [8]) that $G \circ A$ is decomposable if $r(A) < 2$ and not if $r(A) > 2$, while, if $r(A) = 2$, the numbers $N(A(B_1), \varepsilon)$ do not always determine whether $G \circ A$ is decomposable. In some cases, however, precise criteria for decomposability of $G \circ A$ can be found by other methods. L. A. Shepp has recently proved several interesting results on Gaussian processes.

3. Volumes. If C is any convex set in a Hilbert space, let

$$V_n(C) = \sup \lambda_n(P_n C),$$

where λ_n is n-dimensional Lebesgue measure and the supremum extends over all orthogonal projections P_n with n-dimensional range. Then we define the exponent of volume of C by

$$EV(C) = \limsup_{n \to \infty} \big(\log V_n(C)\big)/n \log n.$$

In [2], I considered compact, convex symmetric sets C and showed that $G \circ A$ is decomposable if $EV\big(A(B_1)\big) < -3/2$, and not if $EV\big(A(B_1)\big) > -1$. I conjectured that

$$r(C) = -2/\big(1 + 2EV(C)\big) \quad \text{if } EV(C) < -\tfrac{1}{2}.$$

Sets C satisfying the above relation are called *volumetric*. $A(B_1)$ is volumetric whenever A is a diagonal map $\{x_n\} \to \{a_n x_n\}$ from l_p into l_2; in [2] this was proved for $p = 1, 2$, and ∞, and for other values of p by Chevet [1]. Here we will consider some natural injections into L^2.

Let I be the unit interval $[0, 1]$ and I^k the corresponding k-dimensional cube. Let $q > 0$, $q = r + \alpha$, where r is an integer and $0 < \alpha \leqslant 1$. We consider the space of all real-valued functions f on I^k which have continuous partial derivatives of all orders $\leqslant r$ and for which the partial derivatives of order r satisfy a Hölder condition of order α, so that the following norm is finite:

$$\|f\|_q = \sup_{|p|\leqslant r, x\in I^k} |D^p f(x)| + \sup_{\substack{|p|=r,\\ x,y\in I^k}} |D^p f(x) - D^p f(y)|/|x-y|^\alpha,$$

where $D^p = \partial^{|p|}/\partial x_1^{p_1}\ldots\partial x_k^{p_k}$, $|p| = p_1+\ldots+p_k$.

Let $C_{q,k} = \{f: \|f\|_q \leqslant 1\}$. $C_{q,k}$ is naturally a subset of the Hilbert space $H = L^2(I^k, \lambda)$, where λ is Lebesgue measure on I^k.

Proposition 2. *$C_{q,k}$ is volumetric for all q and k, with $EV(C_{q,k}) = -\frac{1}{2} - q/k$.*

Proof. Kolmogorov and Tikhomirov ([3], Theorem XIV) showed that the exponent of entropy $r(C_{q,k})$ for the *supremum* norm is k/q. Hence in H, $r(C_{q,k}) \leqslant k/q$ and by [2], Proposition 5.8, $EV(C_{q,k}) \leqslant -\frac{1}{2} - q/k$.

For the converse inequality, let f be some C^∞-function on I^k with $f(x) = 0$ for $|x-c| \geqslant \frac{1}{4}$, where c is the center of I^k, $f(c) \neq 0$, and for which $\|f\|_q = 1$. Let

$$\int |f|^2 d\lambda = \varepsilon^2 > 0.$$

For each $n = 1, 2, \ldots$ we divide I^k into n^k parallel cubes of side $1/n$. Let $g_1(x) = f(nx)/n^q$ for $nx \in I^k$. Let $g_2, \ldots, g_{n^k}$ be the functions obtained by translating g_1 on I^k/n to the other small cubes. Then $\|g_j\|_q \leqslant 1$ for all j. A linear combination $\Sigma a_j g_j$ belongs to $C_{q,k}$ whenever $|a_j| \leqslant 1$ for each $j = 1, \ldots, n^k$. Thus $C_{q,k}$ in H includes a cube of dimension n^k and side $\varepsilon n^{-q-k/2}$. Letting $m = n^k$, we have

$$V_m(C_{q,k}) \geqslant \varepsilon^m m^{m(-\frac{1}{2}-q/k)}.$$

Hence $EV(C_{q,k}) \geqslant -\frac{1}{2} - q/k$. So equality holds and $C_{q,k}$ is volumetric, q.e.d.

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