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Perturbation theory and strictly singular operators in locally convex spaces*

by

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Introduction. In this paper we present some results on perturbations of Fredholm operators in locally convex spaces. We are interested in what happens to the index of such an operator when another operator is added to it. In particular, we want to establish conditions that guarantee that the index remains invariant. These problems have been studied extensively in the case of Fredholm operators acting in Banach spaces. Since non-normable and even non-metrizable spaces abound in analysis, it seems worthwhile to investigate what can be done in a more general context. The natural limit to which we can hope to generalize the theory is indicated by the fact that the closed graph theorem plays an essential role in it. Since the work of Ptak [8], however, it has become known that the validity of this theorem is rather wide. We shall make ample use of the following generalized closed graph theorem, due to Ptak: any closed linear operator mapping all of a barreled space into a Ptak (= fully complete = B-complete) space is continuous.

Of the three chapters in which this paper is divided, Chapter I contains the preliminaries. Chapter II and Chapter III can be read independently.

In Chapter II we study perturbations of Fredholm operators T by weakly continuous operators B which are "small with respect to T". In the case where the spaces involved are Banach spaces and B is continuous, the smallness condition takes the form $\|B\| < \gamma(T)$, where $\gamma(T)$ is some constant depending on T. In the non-normable case in which we are interested here, a corresponding condition can be formulated in terms of seminorms (conditions (P) and (P*)).

Chapter III is devoted to strictly singular operators. First we obtain

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a characterization of strictly singular operators in Ptak spaces. This characterization, which is of interest in itself, leads to the more restrictive notion of super strictly singular operators. We then go on to prove a result on perturbations of Fredholm operators by super strictly singular operators. We conclude with some theorems on the algebraic structure of the spaces of strictly singular and super strictly singular operators, respectively.

The system of internal references is as follows. Theorem 2.1 of Chapter II is referred to as II. 2.1 if the reference is made outside of Chapter II, and as 2.1 otherwise.

I. PRELIMINARIES

For purposes of reference we list in this chapter some definitions and theorems, most of which are generally known. E and F will denote arbitrary locally convex spaces (l.c.s.), unless otherwise specified.

A linear operator T with domain $D(T)$ a linear subspace of E and range $R(T)$ a linear subspace of F will be briefly denoted by $T: E \to F$. We emphasize that this notation does not imply that $D(T) = E$ or that $R(T) = F$. The null space of T is denoted by $N(T)$.

A linear operator $T: E \to F$ is called *closed* if its graph $G(T) = \{(x, Tx): x \in D(T)\}$ is a closed linear subspace of the product space $E \times F$.

The kernel index $\alpha(T)$ and the deficiency index $\beta(T)$ of a linear operator $T: E \to F$ are defined by

$$\alpha(T) = \dim N(T), \qquad \beta(T) = \dim F/R(T).$$

$\beta(T)$ is called the *deficiency* of $R(T)$ in F. In general, a linear subspace $M \subset E$ is said to have *finite* (*infinite*) *deficiency in* E if $\dim E/M < \infty$ $(= \infty)$. Whenever $\alpha(T)$ and $\beta(T)$ are not both infinite, the number

$$\varkappa(T) = \alpha(T) - \beta(T)$$

is defined. $\varkappa(T)$ is called the *index* of T.

A closed linear operator with closed range is called *normally solvable*. A closed linear operator which has a finite index is called a *Fredholm operator*. It is known that if E and F are Banach spaces, any Fredholm operator has a closed range and is therefore normally solvable. This is not true in general.

Let E' be the dual of E. We consider the following topologies on E':

$\sigma(E', E)$: the topology of pointwise convergence (or the weak topology);

$\beta(E', E)$: the topology of uniform convergence on the bounded subsets of E (or the strong topology).

The l.c.s. E' equipped with these topologies are denoted by E'_σ and E'_β, respectively. Analogously one defines $\sigma(E, E')$, $\beta(E, E')$, E_σ, E_β.

LEMMA 1. *The graph $G(T)$ of a linear operator $T: E \to F$ is closed in $E \times F$ if and only if it is closed in $E_\sigma \times F_\sigma$.*

Proof. $(E \times F)_\sigma$ is topologically isomorphic to $E_\sigma \times F_\sigma$ (cf. [9]). The lemma follows from this and from the well known fact that a convex subset of a l.c.s. is closed if and only if it is weakly closed.

If $T: E \to F$ is a linear operator, it is sometimes convenient to consider the associated 1-1 operator $\hat{T}: E/N(T) \to F$, where $E/N(T)$ has the quotient topology. We collect some facts relating T and $\hat{T}$ in the following theorem:

THEOREM 2. *Let $T: E \to F$ be a linear operator with domain $D(T)$ and null space $N(T)$. Let φ be the quotient map $E \to E/N(T)$ and let $\hat{T}$ be the 1-1 operator from $\varphi(D(T))$ into F associated with T. Then the following holds:*

(i) *T is continuous if and only if $\hat{T}$ is continuous.*

(ii) *T is open if and only if $\hat{T}$ is open.*

(iii) *$G(T)$ is closed in $E \times F$ if and only if $G(\hat{T})$ is closed in $E/N(T) \times F$.*

(iv) *If $D(T)$ is dense in E, then $D(\hat{T})$ is dense in $E/N(T)$.*

(v) *If T is closed, then $N(T)$ is closed.*

A continuous linear operator $T: E \to F$ which is open, is called a (*topological*) *homomorphism*. If, in addition, T is 1-1, T is called a (*topological*) *isomorphism*.

A linear operator $T: E \to F$ is called *nearly open* if for each 0-neighborhood U in E, TU is dense in some 0-neighborhood in TE.

Before stating the generalized closed graph and open mapping theorems we need to define Ptak spaces.

Definition 3. A l.c.s. E is called a *Ptak space* (or fully complete, or B-complete) if a linear subspace Q of E' is closed in E'_σ, whenever $Q \cap A$ is $\sigma(E', E)$-closed in A for each equicontinuous set $A \subset E'$.

It is an immediate consequence of the Krein-Šmulian Theorem (cf. [9]) that every Fréchet space is a Ptak space.

We also need the following two results:

THEOREM 4. *Every Ptak space is complete.*

THEOREM 5. *Every closed linear subspace and every separated quotient of a Ptak space is a Ptak space.*

We now proceed to formulate the generalized open mapping and closed graph theorems.

THEOREM 6 (generalized open mapping theorem). *If E is a Ptak* *space and F is barreled, then every continuous linear operator $T: E \to F$ with $D(T) = E$ and $R(T) = F$ is a topological homomorphism.*

Another version of the open mapping theorem which deals with closed and not necessarily continuous operators is the following:

THEOREM 7. *Let E be a Ptak space and let $T: E \to F$ be a nearly open linear operator with $D(T)$ dense in E and with closed graph. Then T is open.*

A corollary of Theorem 7 is

THEOREM 8 (generalized closed graph theorem). *Let E be barreled and let F be a Ptak space. If $T: E \to F$ is a closed linear operator with $D(T) = E$, then T is continuous.*

The next theorem states what relations there exist between a linear operator T and its adjoint T', as well as their respective ranges and null spaces. It was proved by Browder [1]. To formulate it conveniently, we need some definitions which are also due to Browder.

Definition 9. A l.c.s. E is called *fully barreled* if every closed linear subspace of E is barreled.

Definition 10. A l.c.s. E is said to satisfy condition (t) if for every linear subspace M of E the Mackey topology $\tau(M, M')$ coincides with the topology induced on M by $\tau(E, E')$.

THEOREM 11. *Let $T: E \to F$ be a linear operator with domain $D(T)$ dense in E and range $R(T)$ in F. Let $G(T)$ be the graph of T considered as a subspace of $E \times F$, T' the adjoint of T with domain $D(T')$ in F' and range $R(T')$ in E', $N(T)$ and $N(T')$ the null spaces of T and T', respectively. Consider the following properties of T and T'.*

(1) *$R(T)$ is closed in F.*

(2) *$R(T')$ is closed in E'_β.*

(3) *$R(T')$ is closed in E'_σ.*

(4) *T is open.*

(5) *T is weakly open (i.e., open with respect to the topologies $\sigma(E, E')$ and $\sigma(F, F')$).*

(6) *T' is weakly open (i.e., open with respect to $\sigma(F', F)$ and $\sigma(E', E)$).*

(7) *$R(T) = N(T')^0$.*

(8) *$R(T') = N(T)^0$.*

(9) *$R(T)$ is barreled.*

Then the following relations hold between these properties:

(I) $(1) \Leftrightarrow (7) \Leftrightarrow (6)$
$(2) \Leftarrow (3) \Leftrightarrow (8) \Leftrightarrow (5) \Leftarrow (4)$

(II) *If $G(T)$ is a Ptak space and F is fully barreled,*

$$\begin{array}{c} (9) \Leftrightarrow (1) \Leftrightarrow (7) \Leftrightarrow (6) \Leftrightarrow (4) \\ \Downarrow \\ (2) \Leftarrow (3) \Leftrightarrow (8) \Leftrightarrow (5) \end{array}$$

(III) *If $G(T)$ is a Ptak space, F is fully barreled and satisfies condition (t), then*

$$\begin{array}{c} (9) \Leftrightarrow (1) \Leftrightarrow (7) \Leftrightarrow (6) \Leftrightarrow (4) \\ \Updownarrow \\ (2) \Leftarrow (3) \Leftrightarrow (8) \Leftrightarrow (5) \end{array}$$

Of these many relations we need only a few which we collect in a corollary.

COROLLARY 12.

(a) *If $T: E \to F$ is a linear operator with $D(T)$ dense in E, then*

$$(4) \Rightarrow (5) \Leftrightarrow (3) \Leftrightarrow (8).$$

(b) *Let $E \times F$ be a Ptak space and let F be fully barreled. If $T: E \to F$ is a linear operator with $D(T)$ dense in E and with closed graph, then*

$$(1) \Leftrightarrow (4) \Rightarrow (3).$$

(c) *Let $E \times F$ be a Ptak space and let F be fully barreled and satisfy condition (t). If $T: E \to F$ is a closed linear operator with $D(T)$ dense in E, then*

$$(5) \Leftrightarrow (1) \Leftrightarrow (3)$$

Remark. Of course in (b) and (c) the relations $(1) \Leftrightarrow (4)$ and $(5) \Leftrightarrow (1)$, respectively, also hold if $D(T)$ is not dense in E. The denseness of $D(T)$ only serves to guarantee the existence of the adjoint.

We conclude this chapter with some facts about topological direct sums.

Let E be an l.c.s. and let M and N be two linear subspaces of E with $M + N = E$ and $M \cap N = \{0\}$. We then call E the *algebraic direct sum* of the subspaces M and N. By the definition of an l.c.s. the map $\psi: M \times N \to E$ defined by

$$\psi(m, n) = m + n \quad (m \in M,\ n \in N)$$

is continuous. If ψ is a topological isomorphism, E is called the *topological direct sum* of M and N and we write $E = M \oplus N$. From now on, we reserve the notation $E = M \oplus N$ exclusively for a topological direct sum, although it is used by many authors to denote an algebraic direct sum as well.

Let $P_1: E \to M$ and $P_2: E \to N$ be the projections defined by

$$P_1(m+n) = m \quad \text{and} \quad P_2(m+n) = n \quad (m \in M,\ n \in N)$$

and let the 1-1 map $\varphi: M \to E/N$ be defined by

$$\varphi(m) = m+N \quad (m \in M).$$

Since φ is continuous and the quotient map $E \to E/N$ is continuous and open, the following two theorems hold.

THEOREM 13. *Let an l.c.s. E be the algebraic direct sum of the linear subspaces M and N. Then $E = M \oplus N$ if and only if P_1 and P_2 are continuous.*

Remark. If P_1 is continuous, then so is $P_2 = I - P_1$.

THEOREM 14. *Let E be the algebraic direct sum of the linear subspaces M and N. Then $E = M \oplus N$ if and only if φ is a topological isomorphism.*

The next theorem is a consequence of Theorem 14.

THEOREM 15. *If E is the algebraic direct sum of M and N and if* $\dim M < \infty$ *and N is closed, then $E = M \oplus N$.*

With the aid of the Hahn-Banach theorem one can prove the following result:

THEOREM 16. *For any finite-dimensional linear subspace M of an l.c.s. E there exists a closed linear subspace N of E such that $E = M \oplus N$.*

We express this by saying that M has a *closed complement* in E.

THEOREM 17. *Let E be a barreled Ptak space. If E is the algebraic direct sum of two closed linear subspaces M and N, then $E = M \oplus N$.*

Proof. Let P_1 be the projection of E onto M with null space N. Since M is a Ptak space it suffices, by Theorem 8, to prove that P_1 is closed. This is quickly seen to be equivalent with the closedness of N.

To conclude this chapter, we prove the following easy but extremely useful result:

THEOREM 18. *Let M be a closed and N be a finite-dimensional subspace of an l.c.s. E. Then $M+N$ is closed.*

Proof. The quotient map $\varphi: E \to E/M$ is continuous and φN is finite-dimensional and therefore closed in E/M. Then $\varphi^{-1}\varphi N = M+N$ is closed in E.

II. PERTURBATIONS BY WEAKLY CONTINUOUS OPERATORS

In this chapter we consider perturbations of normally solvable operators by weakly continuous operators and we establish conditions under which the indices α, β and $\varkappa$ remain invariant.

1. Definition of the conditions [P] **and** [P*]. We begin by proving a lemma which we shall need throughout this chapter.

LEMMA 1.1. *Let E be an l.c.s. and let $M \subset E$ be a closed linear subspace of finite deficiency in E. Then the following holds:*

(i) *For any linear subspace L of E there exists a finite-dimensional linear subspace N contained in L such that*

$$\bar{L} = (\bar{L} \cap M) \oplus N.$$

(ii) *If L is dense in E, then $L \cap M$ is dense in M.*

Proof. The space $\bar{L}/(\bar{L} \cap M)$ is finite-dimensional, since it is algebraically isomorphic to a subspace of E/M. Hence there exists a finite-dimensional linear subspace K of L such that

$$\bar{L} = (\bar{L} \cap M) \oplus K. \tag{1}$$

That this direct sum is topological, follows from Theorem I.15. The projection P of $\bar{L}$ onto K with null space $\bar{L} \cap M$ is therefore continuous. Since $P\bar{L}$ is finite-dimensional, it follows that

$$K = P\bar{L} \subset \overline{PL} = PL. \tag{2}$$

Let $y_1, \ldots, y_n$ be a basis for K. Then by (2) there exist elements $x_1, \ldots, x_n \in L$ such that $Px_i = y_i$ $(i = 1, \ldots, n)$. Clearly, the linear hull $N = \mathrm{sp}\{x_1, \ldots, x_n\}$ is an n-dimensional subspace of L on which P is 1-1. In other words,

$$N \subset L \quad \text{and} \quad N \cap (\bar{L} \cap M) = \{0\}. \tag{3}$$

Since $\dim K = \dim N = n$, it follows from (1) and (3) that

$$\bar{L} = (\bar{L} \cap M) \oplus N.$$

This proves (i).

We now suppose that $\bar{L} = E$. Then (i) implies that $E = M \oplus N$, where N is a finite-dimensional linear subspace of L. Hence $L = (L \cap M) \oplus N$. By Theorem I.14, M is topologically isomorphic to E/N under the map η defined by $\eta(m) = m+N$ $(m \in M)$. Under the same map η, $L \cap M$ is isomorphic to L/N. The last space is dense in E/N, because L is dense in E and the quotient map $\varphi: E \to E/N$ is continuous. Hence $\eta^{-1}(L/N) = L \cap M$ is dense in $\eta^{-1}(E/N) = M$ and the lemma is proved.

Throughout the rest of this chapter we consider two l.c.s. E and F, two linear operators $T, B: E \to F$ and we always assume that the following conditions are satisfied:

(A) $E \times F$ is a Ptak space, F is fully barreled; T is closed, $D(T)$ dense in E, $R(T)$ closed in F, $\alpha(T) < \infty$; B is weakly continuous (i.e., continuous with respect to $\sigma(E, E')$ and $\sigma(F, F')$), and $D(B) \supset D(T)$.

In some of the following lemmas and theorems more assumptions will be necessary. Only the extra conditions will be explicitly stated.

0-neighborhoods in E and F will always be assumed closed and absolutely convex.

Definition 1.2. The pair (T, B) is said to satisfy condition (P) if there exists a closed complement M of $N(T)$ in E and a constant $\gamma < 1$ such that for every 0-neighborhood U in E there exists a 0-neighborhood V in F with $T(U \cap M) \supset V \cap R(T)$ and with the property that

(1) $p_V(Bx) \leqslant \gamma p_V(Tx)$ for all $x \in M \cap D(T)$ (p_V is the gauge of V).

Remark 1. Note that by Corollary I.12(b) T is open, because $R(T)$ is closed. It is easy to see that T_M (the restriction of T to M) is also open. Hence for every 0-neighborhood U in E there always exists a 0-neighborhood V in F with $T(U \cap M) \supset V \cap R(T)$. Condition (P) says that this 0-neighborhood V can be so chosen that (1) holds.

Remark 2. Loosely speaking, one could express the meaning of condition (P) for the pair (T, B) by saying that B is "bounded with respect to T" on a closed complement of $N(T)$.

Remark 3. In order to check if condition (P) holds, it clearly suffices to establish the existence of a V with property (1) for every U out of a 0-neighborhood base in E.

THEOREM 1.3. *If the pair* (T, B) *satisfies condition* (P), *then* $T+B$ *is open.*

Proof. Let γ and M be as in Definition 1.2. Let U be an arbitrary 0-neighborhood in E. We choose a 0-neighborhood V in F with $T(U \cap M) \supset V \cap R(T)$ and such that

$$p_V(Bx) \leqslant \gamma p_V(Tx) \quad \text{for all } x \in M \cap D(T).$$

Then

(1) $$p_V\big((T+B)x\big) \geqslant (1-\gamma) p_V(Tx) \quad \text{for all } x \in M \cap D(T).$$

Indeed, for any $x \in M \cap D(T)$ we have, by the convexity of p_V,

$$p_V\big((T+B)x\big) \geqslant p_V(Tx) - p_V(Bx) \geqslant (1-\gamma) p_V(Tx).$$

We now show that

(2) $$(T+B)(U \cap M) \supset (1-\gamma) V \cap (T+B)M.$$

Let $y \in (1-\gamma) V \cap (T+B) M$ be arbitrary, so $y = (T+B)x$ with $x \in M$ and $p_V\big((T+B)x\big) \leqslant 1-\gamma$. We prove that $x \in U$. Suppose $x \notin U$. Since $T(U \cap M) \supset V \cap R(T)$ and T is 1-1 on M we then must have $Tx \notin V$ and therefore $p_V(Tx) > 1$. (1) now implies that $p_V\big((T+B)x\big) > 1-\gamma$ and this contradicts the assumption. Thus we have proved (2).

Since the 0-neighborhood U in E is arbitrary, it follows from (2) that the restricted operator $(T+B)_M$ is open. We show that this implies that $T+B$ is open. First we observe that $T+B$ is closed. This is proved by applying Lemma I.1 twice and noting that the sum of a closed and a continuous operator is closed. Since M is a closed subspace of E, the restricted operator $(T+B)_M \colon M \to F$ is then also closed, while $D((T+B)_M) = M \cap D(T)$ is dense in M by Lemma 1.1. Furthermore, as we have just proved, $(T+B)_M$ is open. By Corollary I.12(b), $(T+B)M = R((T+B)_M)$ is then closed in F. $(T+B)M$ being a subspace of finite deficiency in $R(T+B)$, $R(T+B)$ is closed by Theorem I.18. Again applying Corollary I.12, we see that $T+B$ is open.

Let us consider for a moment the case where E and F are Banach spaces and B is continuous. It is well-known (cf. [3]) that in that case the adjoint T', which is closed, has a closed range (in the norm topology) and that $\gamma(T) = \gamma(T')$. Hence $\|B\| < \gamma(T)$ implies $\|B'\| < \gamma(T')$. This plays an important role in the proofs of perturbation theorems.

In the present case the condition (P) for the pair (T, B) serves the same purpose as the requirement $\|B\| < \gamma(T)$ in the Banach space context. Therefore, we would like to prove, analogously, that the pair (T', B') satisfies condition (P) (with respect to the strong duals F'_β and E'_β) if the pair (T, B) does. Unfortunately, we can only prove a weaker statement.

THEOREM 1.4. *Let* F *be a Banach space and suppose that* $\beta(T) < \infty$. *If the pair* (T, B) *satisfies condition* (P) *and if the 0-neighborhoods* V *in* F, *occurring in Definition* 1.2 *can be chosen out of the 0-neighborhood base* $\{n^{-1}C_F\}_{n=1}^{\infty}$ (C_F *is the unit ball in* F), *then there exists a* $\varrho > 0$ *such that the pair* $(T', \lambda B')$ *satisfies condition* (P) *with respect to* F'_β *and* E'_β, *for every* $\lambda \in C$ *with* $|\lambda| < \varrho$.

Proof. A 0-neighborhood base for F'_β is formed by the polars in F' of $\{nC_F\}_{n=1}^{\infty}$. Let U be an arbitrary 0-neighborhood in F'_β. It is no restriction to suppose that $U = C_F^0$. Let M be a closed complement of $N(T)$ in E as in Definition 1.2 and put $K = (T_M)^{-1}C_F$. As we noted in Remark 1 following Definition 1.2, T_M is open and so $(T_M)^{-1}$ is continuous. Hence K is bounded in M. Then K is bounded in E and K^0 is therefore a 0-neighborhood in E'_β. We shall show that

(1) $$T'(C_F^0) \supset K^0 \cap R(T').$$

Suppose $T'y' \in K^0$ for some $y' \in D(T')$. Then

$$|\langle x, T'y'\rangle| = |\langle Tx, y'\rangle| \leqslant 1 \quad \text{for all } x \in K.$$

Hence

$$|\langle y, y'\rangle| \leqslant 1 \quad \text{for all } y \in TK = C_F \cap R(T).$$

By the Hahn-Banach theorem there exists a $y'' \in F'$ such that $\langle y, y'\rangle = \langle y, y''\rangle$ for all $y \in R(T)$, while

$$|\langle y, y''\rangle| \leqslant 1 \quad \text{for all } y \in C_F, \text{ i.e., } y'' \in C_F^0.$$

From the fact that $\langle y, y'\rangle = \langle y, y''\rangle$ for all $y \in R(T)$, together with the relation $y' \in D(T')$, we infer that $y'' \in D(T')$ and $T'y'' = T'y'$. This proves (1).

Since $\beta(T) < \infty$ and $R(T)$ is closed, there exists by Theorem I.15 a finite-dimensional linear subspace $L \subset F$ such that $F = R(T) \oplus L$. Clearly, the Banach space F'_β is the algebraic direct sum of $R(T)^0$ and L^0. Since both $R(T)^0$ and L^0 are closed in F'_β, Theorem I.17 implies that $F'_\beta = R(T)^0 \oplus L^0$. Furthermore, by the definition of the adjoint we have $R(T)^0 = N(T')$. Thus $F'_\beta = N(T') \oplus L^0$. Also $\dim N(T') < \infty$, since $N(T') = R(T)^0$ and $\beta(T) < \infty$.

The projection P_1 of F'_β onto L^0 with null space $N(T')$ is continuous. Putting $\|P_1\| = k$, we may assume that $k \geqslant 1$. Indeed, the other possibility $k = 0$ implies that $L^0 = \{0\}$. Hence $R(T) = 0$, so $T = 0$. In this case the theorem trivially holds. We have

$$P_1(C_F^0) \subset k(C_F^0 \cap L^0).$$

From this and (1) we conclude that

$$T'\big(k(C_F^0 \cap L^0)\big) \supset K^0 \cap R(T')$$

or, written somewhat differently,

(2) $$T'(C_F^0 \cap L^0) \supset (kK)^0 \cap R(T').$$

We write again $U = C_F^0$ and we put $V = (kK)^0$. Then (2) takes the form

$$T'(U \cap L^0) \supset V \cap R(T') \quad \text{(cf. Definition 1.2).}$$

We will show that there exists a $\varrho > 0$ such that for every $\lambda \in C$ with $|\lambda| < \varrho$ there is a constant $\gamma' = \gamma'(\lambda) < 1$ such that

(3) $$p_V\big((\lambda B')y'\big) \leqslant \gamma' p_V(T'y') \quad \text{for all } y' \in L^0 \cap D(T').$$

Equivalently, we prove that there exists a $c > 0$ such that

(4) $$p_{K^0}(B'y') \leqslant c\, p_{K^0}(T'y') \quad \text{for all } y' \in L^0 \cap D(T').$$

This last inequality can also be written in the form

(5) $$\sup_{x \in K} |\langle Bx, y'\rangle| \leqslant c \sup_{x \in K} |\langle Tx, y'\rangle| \quad \text{for all } y' \in L^0 \cap D(T').$$

The proof of (5) proceeds as follows. By assumption we have $p_{C_F}(Bx) \leqslant \gamma p_{C_F}(Tx)$ for all $x \in M \cap D(T)$ or, equivalently, $\|Bx\| < \gamma \|Tx\|$ for all $x \in M \cap D(T)$, where γ is the constant < 1 that occurs in the definition of condition (P) for the pair (T, B). Hence $BK \subset \gamma C_F$, since $x \in K$ implies $x \in M \cap D(T)$ and $Tx \in C_F$. Let P_2 be the projection of F onto $R(T)$ with null space L. For every $y' \in L^0 \cap D(T')$ we have

$$\sup_{x \in K} |\langle Bx, y'\rangle| = \sup_{y \in BK} |\langle y, y'\rangle| \leqslant \sup_{y \in \gamma C_F} |\langle y, y'\rangle| = \sup_{y \in \gamma P_2 C_F} |\langle y, y'\rangle|.$$

Putting $\|P_2\| = \alpha$, we may again assume that $\alpha \geqslant 1$. We have $P_2 C_F \subset \alpha C_F \cap R(T)$. Hence

$$\begin{aligned}\sup_{x \in K} |\langle Bx, y'\rangle| &\leqslant \sup_{y \in \gamma P_2 C_F} |\langle y, y'\rangle| \leqslant \sup_{y \in \gamma\alpha C_F \cap R(T)} |\langle y, y'\rangle| \\ &= \alpha\gamma \sup_{y \in C_F \cap R(T)} |\langle y, y'\rangle| = \alpha\gamma \sup_{y \in TK} |\langle y, y'\rangle| \\ &= \alpha\gamma \sup_{x \in K} |\langle Tx, y'\rangle|\end{aligned}$$

for all $y' \in L^0 \cap D(T')$. Thus we have proved (4) with $c = \alpha\gamma$. Now the constant $\varrho = 1/\alpha\gamma$ satisfies the assertion of the theorem. Indeed, for any λ with $|\lambda| < \varrho = 1/\alpha\gamma$ we have $p_{K^0}(\lambda B'y') = |\lambda| p_{K^0}(B'y') \leqslant |\lambda| \alpha\gamma p_{K^0}(T'y')$ for all $y' \in L^0 \cap D(T')$, while $|\lambda|\alpha\gamma < 1$. This means that (3) holds for arbitrary $\lambda \in C$, $|\lambda| < \varrho = 1/\alpha\gamma$, if we take $\gamma' = \gamma'(\lambda) = |\lambda|\alpha\gamma$.

Remark. If F is a Hilbert space, we can take L to be the orthogonal complement $R(T)^\perp$ of $R(T)$ in F. Then $\|P_2\| = 1$ (if $T \neq 0$). Consequently, in this case we may take $\varrho = 1/\gamma > 1$. Hence the pair (T', B') satisfies condition (P) with respect to F'_β and E'_β.

We now define a condition (P*) which is somewhat stronger that (P).

Definition 1.5. The pair (T, B) is said to satisfy *condition* (P*) if there exists a closed complement M of $N(T)$ in E and a constant $\gamma < 1$ such that there is a 0-neighborhood base $\mathscr{V}$ of 0-neighborhoods V in F with the property that $p_V(Bx) \leqslant \gamma p_V(Tx)$ for all $x \in M \cap D(T)$ and for all $V \in \mathscr{V}$.

Remark. If F is a Banach space, condition (P*) can be formulated more easily: there exists a closed complement M of $N(T)$ in E, a constant $\gamma < 1$ and a bounded 0-neighborhood V in F such that

$$p_V(Bx) \leqslant \gamma p_V(Tx) \quad \text{for all } x \in M \cap D(T).$$

Theorem 1.4 now has the following corollary:

Corollary 1.6. *Let F be a Banach space and suppose that $\beta(T) < \infty$. If the pair (T, B) satisfies condition* (P*), *then there exists a $\varrho > 0$ such that the pair $(T', \lambda B')$ satisfies condition* (P) *with respect to F'_β and E'_β, for every $\lambda \in C$ with $|\lambda| < \varrho$.*

Proof. By the preceding remark there is a closed complement M

of $N(T)$ in E, a $\gamma<1$ and a bounded 0-neighborhood V in F such that

$$p_V(Bx)<\gamma p_V(Tx) \quad \text{for all } x\in M\cap D(T).$$

There exist positive constants a_1 and a_2 such that

$$a_1\|y\|\leqslant p_V(y)\leqslant a_2\|y\| \quad \text{for all } y\in F.$$

Hence

$$a_1\|Bx\|<\gamma a_2\|Tx\| \quad \text{for all } x\in M\cap D(T)$$

or, equivalently,

$$\left\|\frac{a_1}{a_2}Bx\right\|\leqslant\gamma\|Tx\| \quad \text{for all } x\in M\cap D(T).$$

Thus the pair $\left(T,\frac{a_1}{a_2}B\right)$ satisfies all requirements of Theorem 1.4.

2. Perturbation theorems. Making use of the results of section 1, we now derive some theorems concerning the indices of T and $T+B$. We still assume that the conditions (A) hold.

Lemma 2.1. *Suppose that the pair* (T,B) *satisfies condition* (P*). *If* $a(T)=0$ *and* $R(T)=F$, *then*

(i) $a(T+B)=a(T)=0$,

(ii) $R(T+B)=R(T)=F$.

Proof. For an arbitrary 0-neighborhood V in F we consider the linear space $F_V=F/p_V^{-1}(0)$, equipped with the norm $\|\hat{x}\|=p_V(x)$ $(x\in\hat{x},\hat{x}\in F_V)$. The canonical map $\varphi_V\colon F\to F_V$ is clearly continuous for every V. Now let $\mathscr{V}$ be a 0-neighborhood base in F as in Definition 1.5 and let $V\in\mathscr{V}$ be arbitrary. Then

(1) $$p_V(Bx)\leqslant\gamma p_V(Tx) \quad \text{for all } x\in D(T).$$

By hypothesis T^{-1} exists. We consider the operators BT^{-1}, $I+BT^{-1}$: $F\to F$ (I is the identity).

It follows from (1) that BT^{-1} and $I+BT^{-1}$ may also be considered as operators in F_V. Moreover, (1) implies that $BT^{-1}\colon F_V\to F_V$ has norm $\leqslant\gamma<1$. Let $\tilde{F}_V$ be the completion of F_V. Then BT^{-1} can be extended continuously to an operator $BT^{-1}\colon\tilde{F}_V\to\tilde{F}_V$ defined on $\tilde{F}_V$, also with norm $\leqslant\gamma\leqslant1$. By a well-known theorem $I+BT^{-1}\colon\tilde{F}_V\to\tilde{F}_V$ is then surjective. Since this operator is also continuous, it follows that $(I+BT^{-1})F_V$ is dense in $\tilde{F}_V$, and therefore dense in F_V. Also $(I+BT^{-1})F_V=\varphi_V(I+BT^{-1})F=\varphi_V(I+BT^{-1})TE=\varphi_V(T+B)E=\varphi_VR(T+B)$. Hence

(2) $$\varphi_VR(T+B) \text{ is dense in } F_V.$$

Since V was arbitrarily chosen from the 0-neighborhood base $\mathscr{V}$, (2) is true for every $V\in\mathscr{V}$. Furthermore, by Theorem 1.3 and Corollary I.12, $R(T+B)$ is closed in F. These last two facts imply that $R(T+B)=F$.

Finally, $a(T+B)=0$, since (1) holds for all $V\in\mathscr{V}$ and some $\gamma<1$.

Remark 1. If in Lemma 2.1, T is continuous (and therefore defined on E), it is an isomorphism, since T is also open. In this case the condition (P) is equivalent to the condition (P*). In general, however, condition (P) does not suffice to prove the lemma. Indeed, the collection of the V for which (1) is true, may not be a base for F. Hence we cannot conclude that $R(T+B)=F$.

Remark 2. With a view to what follows we observe that for the proof of the equality $a(T+B)=0$ in Lemma 2.1, the condition (P) suffices. Indeed, let $x\in D(T)$, $x\neq0$ be arbitrary. Choose a 0-neighborhood U in E such that $x\notin U$. Since T is 1-1, $Tx\notin TU$. Let V be any 0-neighborhood in F with $V\subset TU$ and such that (1) of Definition 1.2 holds. Then $p_V(Bx)<p_V(Tx)$, so $(T+B)x\neq0$.

We now drop the requirement that $a(T)=0$.

Lemma 2.2. *Suppose that the pair* (T,B) *satisfies condition* (P*). *If* $a(T)<\infty$ *and* $R(T)=F$, *then*

(i) $a(T+B)=a(T)$,

(ii) $R(T+B)=R(T)=F$.

Proof. Let M be a closed complement of $N(T)$ in E as in Definition 1.5. We may apply Lemma 2.1 to the restricted operators $T_M,B_M\colon M\to F$, since clearly the pair (T_M,B_M) also satisfies condition (P*) as well as conditions (A). It follows then that $a(T_M+B_M)=a(T_M)=0$ and $R(T_M+B_M)=R(T_M)=F$. Since $F=R(T_M+B_M)\subset R(T+B)\subset F$, it follows that $R(T+B)=F$. Furthermore, $a(T_M+B_M)=0$, whence $N(T+B)\cap M=\{0\}$. Let $x\in D(T)$ be arbitrary. Then $(T+B)x\in F=R(T_M+B_M)$. Consequently, there exists an $x_0\in M\cap D(T)$ such that $(T+B)x=(T_M+B_M)x_0=(T+B)x_0$. Hence $x=(x-x_0)+x_0$ with $x_0\in M\cap D(T)$ and $x-x_0\in N(T+B)$. Thus $N(T+B)\oplus(M\cap D(T))=D(T)$. We also have $N(T)\oplus(M\cap D(T))=D(T)$. Hence $a(T+B)=a(T)$.

Lemma 2.3. *Suppose that* F *is a Banach space and the pair* (T,B) *satisfies condition* (P*). *If* $a(T)=0$ *and* $\beta(T)<\infty$, *then*

(i) $a(T+B)=a(T)=0$,

(ii) $\beta(T+B)=\beta(T)$ *for all* $\lambda\in C$ *with* $|\lambda|<\varrho$, *for certain* $\varrho>0$.

Proof. As in Lemma 2.1 the relation $a(T)=0$ and condition (P*) imply that $a(T+B)=0$. Of course we also have $a(T+\lambda B)=0$ for all $\lambda\in C$, $|\lambda|\leqslant1$, since any pair $(T,\lambda B)$, $|\lambda|\leqslant1$ also satisfies condition (P*). We now consider T'. The operator T is open. Hence, by the implication (4) ⇒ (8) of Corollary I.12, $R(T')=N(T)^0$ and therefore $\beta(T')=a(T)=0$.

Thus T' is surjective. We also have $R(T)^0 = N(T')$, whence $\alpha(T') = \beta(T) < \infty$.

By Theorem 1.3, $T+\lambda B$ is open for $|\lambda| \leqslant 1$. Using the same arguments as above, we find that $\beta(T'+\lambda B') = \alpha(T+\lambda B) = 0$ and $\alpha(T'+\lambda B') = \beta(T+\lambda B)$ for $|\lambda| \leqslant 1$.

If Lemma 2.2 were applicable to the pair $(T', \lambda B')$ for small λ, we could conclude that $\alpha(T'+\lambda B') = \alpha(T')$ and therefore $\beta(T+\lambda B) = \beta(T)$. However, there are some difficulties. In the first place we do not know if the pair $(T', \lambda B')$ satisfies condition (P*) with respect to F'_β and E'_β and secondly, the conditions (A) may not hold for $(T', \lambda B')$.

A close examination of the proof of Lemma 2.2 shows, however, that these conditions are needed only to prove that for sufficiently small λ we have $\alpha\big((T')_K+\lambda(B')_K\big) = 0$ and $R\big((T')_K+\lambda(B')_K\big) = E'$, K being a closed complement of $N(T')$ in F'_β. Now it follows from Corollary 1.6 that a $\varrho' > 0$ exists such that for every $\lambda \in C$ with $|\lambda| < \varrho'$ the pair $(T', \lambda B')$ satisfies condition (P) with respect to F'_β and E'_β, and therefore also $\big((T')_K, \lambda(B')_K\big)$ satisfies (P) with respect to K and E'_β, where K is a closed complement of $N(T')$ in F'_β, as in Definition 1.2. This implies that $\alpha\big((T')_K+\lambda(B')_K\big) = 0$ for $|\lambda| < \varrho'$. (Cf. Remark 2 following Lemma 2.1.)

It remains to be proved that $R\big((T')_K+\lambda(B')_K\big) = E'$ for λ sufficiently small. To this end we introduce an auxiliary topology on $D(T)$. Since T is 1-1 and $R(T)$ is a Banach space, we may place on $D(T)$ the unique Banach topology for which T is a topological isomorphism. To avoid confusion we denote $D(T)$ equipped with this Banach space topology by $D(T)_b$. By T_1 and B_1 we denote the operators $T, B\colon D(T)_b \to F$, respectively.

Since $T\colon D(T) \to F$ is open, $D(T)_b$ has a finer topology than $D(T)$. Therefore we have $\big(D(T)_b\big)' \supset E'$. It is an easy matter to check that $T_1, B_1\colon D(T)_b \to F$ satisfy conditions (A) and (P*). An examination of the proof of Theorem 1.4 shows that, since $T_1\colon D(T)_b \to F$ is a topological isomorphism, the pair $(T'_1, \lambda B'_1)$ satisfies condition (P*) with respect to F'_β and $\big(D(T)_b\big)'_\beta$, for every $\lambda \in C$ with $|\lambda| < \varrho'$. Clearly T'_1 is surjective, since T_1 is an isomorphism. Also $\big((T'_1)_K, \lambda(B'_1)_K\big)$, $|\lambda| < \varrho'$, satisfies (P*) with respect to K and $\big(D(T)_b\big)'_\beta$. It follows now from Lemma 2.1 that $(T'_1+\lambda B'_1)K = \big(D(T)_b\big)'$ for $|\lambda| < \varrho'$. Hence $\big((T'_1+\lambda B'_1)K\big) \cap E' = E'$ for $|\lambda| < \varrho'$. Evidently $\big((T'_1+\lambda B'_1)K\big) \cap E' = (T'+\lambda B')K$ and we have therefore shown that $R\big((T')_K+\lambda(B')_K\big) = E'$ for $|\lambda| < \varrho'$.

If $|\lambda| < \varrho = \min(1, \varrho')$, we have both $\alpha\big((T')_K+\lambda(B')_K\big) = 0$ and $R\big((T')_K+\lambda(B')_K\big) = E'$. The proof can now be completed as that of Lemma 2.2.

Finally we again drop the condition that $\alpha(T) = 0$.

THEOREM 2.4. *Let F be a Banach space and suppose that the pair*

(T, B) satisfies condition (P*). *If $\alpha(T) < \infty$, $\beta(T) < \infty$, i.e., if T is a Fredholm operator, then a $\varrho > 0$ exists such that*

(i) $\alpha(T+\lambda B) \leqslant \alpha(T)$,

(ii) $\beta(T+\lambda B) \leqslant \beta(T)$,

(iii) $\varkappa(T+\lambda B) = \varkappa(T)$,

for all $\lambda \in C$ with $|\lambda| < \varrho$.

Proof. Again we consider the restricted operators $T_M, T_M+B_M\colon M \to F$ (M a closed complement of $N(T)$ as in Definition 1.5). In virtue of the condition (P*), $\alpha(T_M+B_M) = \alpha(T_M) = 0$. Applying Lemma 2.3 to T_M and B_M, we find that a $\varrho' > 0$ exists such that

(1) $$\beta(T_M+\lambda B_M) = \beta(T_M) = \beta(T) < \infty \quad \text{if } |\lambda| < \varrho'.$$

Also, because the pair $(T_M, \lambda B_M)$ satisfies (P*) for $|\lambda| \leqslant 1$,

(2) $$\alpha(T_M+\lambda B_M) = \alpha(T_M) = 0 \quad \text{if } |\lambda| \leqslant 1.$$

Hence, by (1) and (2),

(3) $$\varkappa(T_M+\lambda B_M) = \varkappa(T_M) \quad \text{for } |\lambda| < \varrho = \min(1, \varrho').$$

(3) implies that

(4) $$\varkappa(T+\lambda B) = \varkappa(T) \quad \text{for } |\lambda| < \varrho,$$

since $\varkappa(T+\lambda B) = \varkappa(T_M+\lambda B_M)+\alpha(T)$ and $\varkappa(T) = \varkappa(T_M)+\alpha(T)$. From formula (2) it follows that $T+\lambda B$ is 1-1 on M for $|\lambda| < \varrho$, therefore

(5) $$\alpha(T+\lambda B) \leqslant \alpha(T) \quad \text{for } |\lambda| < \varrho.$$

Finally, (4) and (5) imply

$$\beta(T+\lambda B) \leqslant \beta(T) \quad \text{for } |\lambda| < \varrho.$$

This completes the proof.

To end this chapter we wish to prove a result that allows us to draw conclusions similar to those of Theorem 2.4 in the case that F is not a Banach space. We then cannot appeal to Theorem 1.4 and Corollary 1.6 for a guarantee that the condition (P*) for the pair (T, B) is preserved in the weaker form (P) by the adjoints $(T', \lambda B')$, λ sufficiently small. It is therefore not surprising that we need two conditions, one for the pair (T, B) and another for the pair (T', B').

THEOREM 2.5. *Suppose that F satisfies condition* (t) *and that T is a Fredholm operator. We also suppose that $N(T)$ has a closed complement M and $N(T')$ has a weakly closed complement K ($E = M \oplus N(T)$, $F'_\sigma = K \oplus N(T')$) such that the following conditions hold:*

(a) $$\big((T_M)'+(B_M)'\big)K \supset (T_M)'K,$$

(b) $$(T+B)\,M+K^0 \supset TM.$$

Then $R(T+B)$ is closed and

(i) $$\alpha(T+B) \leqslant \alpha(T),$$
(ii) $$\beta(T+B) \leqslant \beta(T),$$
(iii) $$\varkappa(T+B) = \varkappa(T).$$

Remark. (a) and (b) can be less elegantly but more usefully formulated as follows:

(a′) For every $y' \in K \cap D(T')$ there is a $y_1' \in K \cap D(T')$ such that for all $x \in M \cap D(T)$ we have $\langle (T+B)x, y_1'\rangle = \langle Tx, y'\rangle$.

(b′) For every $x \in M \cap D(T)$ there is an $x_1 \in M \cap D(T)$ such that for all $y' \in K \cap D(T')$ we have $\langle (T+B)x_1, y'\rangle = \langle Tx, y'\rangle$.

Proof of Theorem 2.5. (a) We note first that, by Lemma 1.1, $K \cap D(T')$ is weakly dense in K and $M \cap D(T)$ is dense in M. Furthermore, $T+B$ is closed, since T is closed and B is weakly continuous. This is proved by applying Lemma I.1 twice.

(b) Next we prove that $\alpha(T+B) \leqslant \alpha(T)$. Let $x_0 \in N(T+B)$, $x_0 \in M$, $x_0 \neq 0$. Then $Tx_0 \neq 0$. Hence, since $K \cap D(T')$ is weakly dense in K, there exists a $y' \in K \cap D(T')$ with $\langle Tx_0, y'\rangle \neq 0$. But $\langle (T+B)x_0, y_1'\rangle = 0$ for all $y_1' \in K \cap D(T')$. Thus (a′) is not fulfilled and therefore $N(T+B) \cap M = \{0\}$, so that $\alpha(T+B) \leqslant \alpha(T)$.

(c) We now show that $R(T+B)$ is closed. Consider the restricted operator T_M. It is closed because T is closed and M is a closed linear subspace of E. Also $D(T_M) = M \cap D(T)$ is dense in M and $R(T_M) = R(T)$ is closed. Furthermore, $M \times F$ is a Ptak space by Theorem I.5. Applying Corrollary I.12(c) to T_M we find that T_M is weakly open. By Corollary I.12(a) this implies that $N(T_M)^0 = R\big((T_M)'\big)$. Now $N(T_M) = \{0\}$ and so the adjoint operator $T_M'\colon F' \to M'$ is surjective. Thus for every $x' \in M'$ a $y' \in D(T')$ can be found such that $x' = (T_M)'y'$. We may even assume that $y' \in K \cap D(T')$, since $N\big((T_M)'\big) = N(T')$.

By (a) we have $\big((T_M)' + (B_M)'\big)K \supset (T_M)'K$. Since $(T_M)'K = M'$ by what was proved above, it follows that $(T_M)' + (B_M)' = (T_M + B_M)'$ is surjective. Because $(T+B)_M$ is closed, $D\big((T+B)_M\big)$ is dense in M and $M \times F$ is a Ptak space, Corollary I.12(c) may be applied to $(T+B)_M$. It follows that $R\big((T+B)_M\big) = (T+B)\,M$ is closed in F. Since $(T+B)\,M$ is a linear subspace of finite deficiency in $R(T+B)$, Theorem I.18 then implies that $R(T+B)$ is closed in F.

(d) We show next that $\beta(T+B) \leqslant \beta(T)$. First we observe that the conditions (a′) and (b′) can also be formulated as follows:

(a″) For every $y' \in K \cap D(T')$ there is a $y_1' \in K \cap D(T')$ such that for all $x \in M \cap D(T)$ we have $\langle x, (T'+B')y_1'\rangle = \langle x, T'y'\rangle$.

(b″) For every $x \in M \cap D(T)$ there is an $x_1 \in M \cap D(T)$ such that for all $y' \in K \cap D(T')$ we have $\langle x_1, (T'+B')y'\rangle = \langle x, T'y'\rangle$.

Upon comparing (a″) and (b″) with (a′) and (b′) it is easily seen that (a′) goes into (b″) and (b′) into (a″) if one makes the substitutions $T \to T'$, $T' \to T$, $T+B \to T'+B'$, $K \to M$, $M \to K$, $F' \to E$, $E' \to F$, $F \to E'$, $E \to F'$.

Since $R(T)$ and $R(T+B)$ are closed, $\beta(T) = \alpha(T')$ and $\beta(T+B) = \alpha(T'+B')$. If we repeat the argument of (b), making the proper substitutions, we find that $\alpha(T'+B') \leqslant \alpha(T')$ and therefore $\beta(T+B) \leqslant \beta(T)$.

The equality $\varkappa(T+B) = \varkappa(T)$ we prove in two steps.

(e) First we show $\varkappa(T+B) \geqslant \varkappa(T)$. Consider the restricted operators T_M and $(T+B)_M$. We showed in (b) that

(1) $$\alpha(T_M) = \alpha\big((T+B)_M\big) = 0.$$

If we apply the result in (d) to T_M and $(T+B)_M$, we find that

(2) $$\beta\big((T+B)_M\big) \leqslant \beta(T_M).$$

From (1) and (2) it follows that

$$\varkappa\big((T+B)_M\big) \geqslant \varkappa(T_M),$$

and therefore

$$\varkappa(T+B) \geqslant \varkappa(T),$$

since $\varkappa(T+B) = \varkappa\big((T+B)_M\big) + \alpha(T)$ and $\varkappa(T) = \varkappa(T_M) + \alpha(T)$.

(f) Finally we prove that $\varkappa(T+B) \leqslant \varkappa(T)$. If we repeat the arguments of (b), (d) and (e), making the substitutions indicated in (d), we find that

$$\varkappa(T'+B') \geqslant \varkappa(T').$$

It is well-known that $\varkappa(T) = -\varkappa(T')$ and $\varkappa(T+B) = -\varkappa(T'+B')$. Hence $-\varkappa(T+B) \geqslant -\varkappa(T)$, or $\varkappa(T+B) \leqslant \varkappa(T)$. This completes the proof.

III. PERTURBATIONS BY STRICTLY SINGULAR OPERATORS

1. Characterization of precompact and strictly singular operators.

Kato [4] defined strictly singular operators in Banach spaces. An obvious generalization is the following

Definition 1.1. Let E and F be l.c.s. A continuous linear operator $B\colon E \to F$ is called *strictly singular* (s.s.) if it is not a topological isomorphism on any infinite-dimensional linear subspace of its domain.

The most important examples of s.s. operators are the compact operators. In cases where the range space is not assumed to be complete, it is convenient to consider also precompact operators.

Definition 1.2. A linear operator $B: E \to F$ is called *precompact* if there exists a 0-neighborhood U in E such that BU is totally bounded.

Remark. If F is complete, the compact and the precompact operators from E into F coincide. Indeed, if U is a 0-neighborhood in E such that BU is totally bounded, then $\overline{BU}$ is totally bounded and complete, and therefore compact.

For X and Y normed linear spaces, the precompact and strictly singular operators from X into Y have been characterized in a way that clearly shows the relationship between them.

THEOREM 1.3. *A linear operator $B: X \to Y$ is precompact if and only if for every $\varepsilon > 0$ there exists a linear subspace $N \subset D(B)$ with finite deficiency in $D(B)$ such that the restricted operator B_N has norm not exceeding ε.*

THEOREM 1.4. *For a continuous linear operator $B: X \to Y$ the following statements are equivalent.*

(i) *B is strictly singular.*

(ii) *Given $\varepsilon > 0$ and given an infinite-dimensional linear subspace M of $D(B)$, there exists an infinite-dimensional linear subspace $N \subset M$ such that the restricted operator B_N has norm not exceeding ε.*

We refer to Goldberg [3] for the proofs of the last two theorems.

Since these theorems play a role in perturbation theory, we shall attempt to generalize them for non-normable spaces. It is interesting to see that in dealing with s.s. operators, Ptak spaces enter the picture quite naturally.

E and F are arbitrary l.c.s. and 0-neighborhoods are always assumed to be absolutely convex and closed.

THEOREM 1.5. *A continuous linear operator $B: E \to F$ defined on E is precompact if and only if there exists a 0-neighborhood U in E with the property that for every 0-neighborhood V in F there is a closed linear subspace $M \subset E$ of the form*

$$M = \bigcap_{i=1}^{n} N(x_i'),$$

with $x_i' \in E'$ and x_i' bounded on U $(i = 1, \ldots, n)$ such that $B(U \cap M) \subset V$.

Proof. (a) Suppose that B is precompact, and let U be a 0-neighborhood in E such that BU is totally bounded. We choose a 0-neighborhood V in F arbitrarily and proceed to construct a linear subspace $M \subset E$ with the required properties.

There exist $x_1, \ldots, x_n \in E$ such that

$$BU \subset \bigcup_{i=1}^{n} \left(Bx_i + \frac{1}{2} V\right),$$

since BU is totally bounded. We set $p_{V/2}(Bx_i) = \lambda_i$ $(i = 1, \ldots, n)$, where $p_{V/2}$ is the gauge of $\frac{1}{2}V$. By the Hahn-Banach theorem we can select $y_1', \ldots, y_n' \in F'$ with

$$\langle Bx_i, y_i' \rangle = \lambda_i \quad \text{and} \quad |\langle y, y_i' \rangle| \leqslant p_{V/2}(y)$$

for all $y \in F$ $(i = 1, \ldots, n)$.

Let $x_i' = y_i' B$ $(i = 1, \ldots, n)$ and

$$M = \bigcap_{i=1}^{n} N(x_i').$$

Since BU is bounded and the y_i' are continuous, the x_i' are bounded on U $(i = 1, \ldots, n)$. For any $x \in U \cap M$ we have $Bx \in Bx_i + \frac{1}{2}V$ for some i, whence $B(x - x_i) \in \frac{1}{2}V$. Furthermore,

$$|\langle B(x - x_i), y_i' \rangle| = |\langle Bx_i, y_i' \rangle| = \lambda_i,$$

since $x \in M \subset N(y_i' B)$. Also

$$|\langle B(x - x_i), y_i' \rangle| \leqslant p_{\frac{1}{2}V}(B(x - x_i)) \leqslant 1,$$

since $B(x - x_i) \in \frac{1}{2}V$. Hence $\lambda_i \leqslant 1$ or, equivalently, $Bx_i \in \frac{1}{2}V$. It follows that $Bx = Bx_i + B(x - x_i) \in V$ for every $x \in U \cap M$. This proves that $B(U \cap M) \subset V$.

(b) Suppose now that U is a 0-neighborhood in E with the property stated in the theorem. V being an arbitrary 0-neighborhood in F, let

$$M = \bigcap_{i=1}^{n} N(x_i')$$

be such that $B(U \cap M) \subset V$, while x_i' is bounded on U $(i = 1, \ldots, n)$. We may assume that $x_1', \ldots, x_n'$ are linearly independent. Then we can select $b_1, \ldots, b_n \in E$ such that $\langle b_j, x_i' \rangle = \delta_{ij}$ $(i, j = 1, \ldots, n)$. Evidently $b_1, \ldots, b_n$ are linearly independent and $E = M \oplus \mathrm{sp}\{b_1, \ldots, b_n\}$. Therefore every $x \in E$ can be written in the form

$$x = \bar{x} + \sum_{i=1}^{n} \langle x, x_i' \rangle b_i, \quad \text{with } \bar{x} \in M.$$

Hence

$$Bx = B\bar{x} + \sum_{i=1}^{n} \langle x, x_i' \rangle Bb_i$$

and

$$p_V(Bx) \leqslant p_V(B\bar{x}) + \sum_{i=1}^{n} |\langle x, x_i' \rangle| p_V(Bb_i). \tag{1}$$

Also

$$(2)\qquad p_U(\bar{x}) \leqslant p_U(x) + \sum_{i=1}^{n} |\langle x, x_i'\rangle| p_U(b_i).$$

Since $B(U \cap M) \subset V$,

$$(3)\qquad p_V(B\bar{x}) \leqslant p_U(\bar{x}).$$

It follows from (1), (2) and (3) that

$$p_V(Bx) \leqslant p_U(x) + \sum_{i=1}^{n} |\langle x, x_i'\rangle| p_U(b_i) + \sum_{i=1}^{n} |\langle x, x_i'\rangle| p_V(Bb_i).$$

If K is the maximum of the numbers $p_U(b_i)$ and $p_V(Bb_i)$ $(i = 1, \ldots, n)$, then

$$(4)\qquad p_V(Bx) \leqslant p_U(x) + 2K \sum_{i=1}^{n} |\langle x, x_i'\rangle|.$$

Now since the x_i' are bounded on U $(i = 1, \ldots, n)$, the map $x \to (\langle x, x_1'\rangle, \ldots, \langle x, x_n'\rangle)$ of E into C^n maps U onto a bounded set in C^n, therefore onto a totally bounded set, since C^n is finite-dimensional. This means that for any $\eta > 0$ we can select $x_1, \ldots, x_m \in U$ such that for every $x \in U$ an x_k exists with

$$\sum_{i=1}^{n} |\langle x, x_i'\rangle - \langle x_k, x_i'\rangle| < \eta.$$

This implies, together with (4), that for every $x \in U$ an $x_k \in U$ exists such that

$$(5)\qquad p_V(Bx - Bx_k) \leqslant 2 + 2K\eta = \varrho.$$

We may assume that $\varrho < 3$, since this can be achieved by choosing η sufficiently small. Then (5) implies

$$BU \subset \bigcap_{k=1}^{m} (Bx_k + 3V).$$

Hence B is precompact.

Remark. Theorem 1.3 is a corollary of Theorem 1.5.

Next we characterize strictly singular operators.

THEOREM 1.6. *Let E be a Ptak space and let F be an arbitrary l.c.s. Then, for a continuous linear operator $B\colon E \to F$ defined on E the following statements are equivalent.*

(i) *B is strictly singular.*

(ii) *For every closed infinite-dimensional linear subspace $M \subset E$ there exists a 0-neighborhood U in E with the property that for every 0-neighborhood V in F there exists an infinite-dimensional linear subspace $N \subset M$ such that $N \not\subset U$ and $B(U \cap N) \subset V$.*

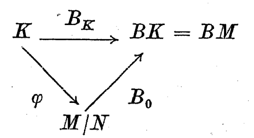
Proof. (a) We suppose first that B is strictly singular. Having chosen an arbitrary closed infinite-dimensional linear subspace $M \subset E$, we determine a 0-neighborhood U in E with the desired property. We may assume that the restricted operator B_M has a finite-dimensional null space. Indeed, if $\alpha(B_M) = \infty$, then any 0-neighborhood U in E with $N(B_M) \not\subset U$ satisfies (ii): given an arbitrary 0-neighborhood V in F we can always take $N = N(B_M)$. We also observe that, by Theorem I.5, M is a Ptak space.

First we prove that B_M is not nearly open.

Indeed, suppose that B_M is nearly open. It then follows from Theorem I.7 that B_M is a homomorphism. Hence, putting $N = N(B_M)$, the associated 1-1 operator

$$B_0\colon M/N \to BM$$

is a topological isomorphism. Since N is finite-dimensional by hypothesis, N has a closed complement K in M by Theorem I.16. Also, by Theorem I.14, K is topologically isomorphic to M/N under the quotient map φ. Thus we find that in the diagram



B_K is a topological isomorphism. Since $\dim K = \infty$, this contradicts the strict singularity of B. Hence B_M is not nearly open.

A consequence of this is that a 0-neighborhood U in E exists such that

$$(1)\qquad \overline{B(U \cap M)}^{(BM)} \text{ is not a 0-neighborhood in } BM.$$

We shall prove that this U has the required property. Suppose that V is an arbitrary 0-neighborhood in F. Then

$$B(U \cap M) \not\supset \delta V \cap BM \quad \text{for all } \delta > 0,$$

since $\overline{B(U \cap M)}^{(BM)}$ and therefore surely $B(U \cap M)$ is not a 0-neighborhood in BM. Hence there exists an $x_1 \in M$ with $p_U(x_1) = 1$ and $p_V(Bx_1) < 3^{-1}$. By the Hahn-Banach theorem we can select an $x_1' \in E'$ such that

$$\langle x_1, x_1'\rangle = 1 \text{ and } |\langle x, x_1'\rangle| \leqslant p_U(x) \quad \text{for all } x \in E.$$

If $N(x_1')$ is the null space of x_1', then $N_1 = N(x_1') \cap M$ is closed in M and has deficiency 1 in M.

We now proceed to show that

(2) $$\overline{B(U \cap N_1)}^{(BN_1)} \text{ is not a 0-neighborhood in } BN_1.$$

We distinguish two cases.

1° BN_1 is closed in BM. We have $M = N_1 \oplus \mathrm{sp}\{x_1\}$, by Theorem I.15. Suppose now that $\overline{B(U \cap N_1)}^{(BN_1)}$ is a 0-neighborhood in BN_1. If $Bx_1 \in BN_1$, then $BN_1 = BM$ and our assumption contradicts (1). We may therefore assume that $Bx_1 \notin BN_1$. Then $BM = BN_1 \oplus \mathrm{sp}\{Bx_1\}$, again by Theorem I.15. Since $B(U \cap \mathrm{sp}\{x_1\})$ is clearly a 0-neighborhood in $\mathrm{sp}\{Bx_1\}$, it follows from this and from our assumption that $\overline{B(U \cap M)}^{(BM)}$ contains a 0-neighborhood of each of the spaces BN_1 and $\mathrm{sp}\{Bx_1\}$. Then $\overline{B(U \cap M)}^{(BM)}$ is a 0-neighborhood in $BM = BN_1 \oplus \mathrm{sp}\{Bx_1\}$, which contradicts (1).

2° BN_1 is dense in BM. Then $\widetilde{BN_1} = \widetilde{BM}$. Suppose again that $V = \overline{B(U \cap N_1)}^{(BN_1)}$ is a 0-neighborhood in BN_1. Then $\tilde{V} = \overline{B(U \cap N_1)}^{(\widetilde{BM})}$ is a 0-neighborhood in $\widetilde{BM}$. Since $\overline{B(U \cap M)}^{(BM)} = \overline{B(U \cap M)}^{(\widetilde{BM})} \cap BM \supset \overline{B(U \cap N_1)}^{(\widetilde{BM})} \cap BM$, $\overline{B(U \cap M)}^{(BM)}$ is then a 0-neighborhood in BM. This again contradicts (1). The proof of (2) is now complete.

(2) implies that

$$B(U \cap N_1) \not\supset \delta V \cap BN_1 \quad \text{for all } \delta > 0.$$

In particular, there exists an $x_2 \in N_1$ with

$$p_U(x_2) = 1 \quad \text{and} \quad p_V(Bx_2) < 3^{-2}.$$

Using the Hahn-Banach theorem, we can select an $x_2' \in E'$ such that

$$\langle x_2, x_2' \rangle = 1 \text{ and } |\langle x, x_2' \rangle| \leqslant p_U(x) \quad \text{for all } x \in E.$$

If $N(x_2')$ is the null space of x_2', then $N_2 = N(x_1') \cap N(x_2') \cap M$ is closed in M and has deficiency 1 in N_1.

Using similar arguments as before we can prove that

(3) $$\overline{B(U \cap N_2)}^{(BN_2)} \text{ is not a 0-neighborhood in } BN_2.$$

We only note that the following two possibilities have to be treated.

1° BN_2 is closed in BN_1. If $Bx_2 \in BN_2$, then $BN_2 = BN_1$ and the assumption that $\overline{B(U \cap N_2)}^{(BN_2)}$ is a 0-neighborhood in BN_2 contradicts (2). If $Bx_2 \notin BN_2$, then the proof is based on the fact that $BN_1 = BN_2 \oplus \mathrm{sp}\{Bx_2\}$.

2° BN_2 is dense in BN_1. Then we use that $\widetilde{BN_1} = \widetilde{BN_2}$.

(3) implies that

$$B(U \cap N_2) \not\supset \delta V \cap BN_2 \quad \text{for all } \delta > 0.$$

Hence there exists an $x_3 \in N_2$ with

$$p_U(x_3) = 1 \quad \text{and} \quad p_V(Bx_3) < 3^{-3}.$$

We then select an $x_3' \in E'$ with $\langle x_3, x_3' \rangle = 1$ and $|\langle x, x_3' \rangle| \leqslant p_U(x)$ for all $x \in E$ and set

$$N_3 = N(x_1') \cap N(x_2') \cap N(x_3') \cap M.$$

Inductively, sequences (x_k) and (x_k') are selected in M and E', respectively, such that

(4) $$p_U(x_k) = \langle x_k, x_k' \rangle = 1, \quad p_V(Bx_k) < 3^{-k},$$
$$|\langle x, x_k' \rangle| \leqslant p_U(x) \quad \text{for all } x \in E \ (k = 1, 2, \ldots),$$

(5) $$x_k \in \bigcap_{i=1}^{k-1} N(x_i') \quad \text{or, equivalently, } \langle x_k, x_i' \rangle = 0 \quad \text{for } i < k.$$

It is easily verified that the sequence (x_k) is linearly independent. Thus the space $N = \mathrm{sp}\{x_1, \ldots, x_n \ldots\}$ is infinite-dimensional.

We show that N has the required property. Suppose

$$x = \sum_{i=1}^{m} a_i x_i.$$

Then $|a_1| = |\langle x, x_1' \rangle| \leqslant p_U(x)$. By induction we prove that, more generally,

(6) $$|a_k| \leqslant 2^{k-1} p_U(x) \quad (k = 1, \ldots, m).$$

Assuming that (6) holds for $k \leqslant j < m$, it follows from

$$\langle x, x_{j+1}' \rangle = \sum_{i=1}^{j} a_i \langle x_i, x_{j+1}' \rangle + a_{j+1}$$

that

$$|a_{j+1}| \leqslant |\langle x, x_{j+1}' \rangle| + \sum_{i=1}^{j} |a_i| \, |\langle x_i, x_{j+1}' \rangle|$$
$$\leqslant p_U(x) + \sum_{i=1}^{j} 2^{i-1} p_U(x) p_U(x_i)$$
$$= p_U(x) + \sum_{i=1}^{j} 2^{i-1} p_U(x) = 2^j p_U(x).$$

This proves (6). Hence for every $x \in N$

$$p_V(Bx) \leqslant \sum_{i=1}^{m} |a_i| p_V(Bx_i) \leqslant \sum_{i=1}^{m} 2^{i-1} p_U(x) 3^{-i} \leqslant p_U(x),$$

or, equivalently, $B(U \cap N) \subset V$.

Finally, it is evident that $N \not\subset U$, since the points x_k are on the boundary of U, as well as in N.

(b) Suppose now that B is not strictly singular. Then there is a closed infinite-dimensional linear subspace $M \subset E$ such that B_M is a topological isomorphism. Assume that for this M a 0-neighborhood U in E exists with the property mentioned in (ii). This will lead to a contradiction. Indeed, since B_M is an isomorphism, $B(U \cap M)$ is a 0-neighborhood in BM. Theorefore we can choose a 0-neighborhood V in E such that

(1) $$2V \cap BM = B(U \cap M).$$

By assumption there is an infinite-dimensional subspace $N \subset M$ such that $N \not\subset U$ and

(2) $$B(U \cap N) \subset V.$$

It follows from (1) that $2V \cap BN = B(U \cap N)$, since B_M is 1-1. Now take a point $x \in N$ with $x \in (2U) \setminus U$. Then $Bx \in BN$ and $Bx \notin B(U \cap N) = 2V \cap BN$, because B_N is 1-1. Hence $Bx \notin 2V$. However, (2) implies that, in contradiction to this,

$$Bx \in B(2U \cap N) \subset 2V.$$

Thus (ii) is not fulfilled.

Remark 1. Note that (6) implies $p_U(x) > 0$ for every $x \in N$, $x \neq 0$. Hence we know not only that $N \not\subset U$, but even that U does not contain any linear subspace of N.

Remark 2. Note that in the proof of the implication (ii) $\Rightarrow$ (i) we have not used that E is a Ptak space.

Remark 3. Let E be a Banach space and let $B: E \to F$ be s.s. and defined on E. Then, by Theorem 1.6, for every closed infinite-dimensional linear subspace $M \subset D(B)$ there exists a 0-neighborhood U in E with the property that for every 0-neighborhood V in F an infinite-dimensional linear subspace $N \subset M$ exists such that $N \not\subset U$ and $B(U \cap N) \subset V$. We may assume that N is closed since the last inclusion also holds for the closure of N. Now clearly every 0-neighborhood $U' \subset U$ also has this property, as well as λU for any $\lambda \neq 0$. This implies that the unit ball C_E enjoys the property. Hence $U = C_E$ satisfies (ii) of Theorem 1.6 for any M. It is not difficult now to check that if $B_1, B_2: E \to F$ are both defined on E and s.s. and if E is a Banach space, then $B_1 + B_2$ is also s.s. One only has to show that (ii) of Theorem 1.6 holds for $B_1 + B_2$.

2. A perturbation theorem. The following theorem is due to Kato [4]:

THEOREM 2.1. *Let X and Y be Banach spaces. Suppose $T: X \to Y$ is normally solvable and $\alpha(T) < \infty$. If $B: X \to Y$ is strictly singular and $D(B) \supset D(T)$, then the following holds:*

(i) *$T+B$ is normally solvable.*

(ii) *$\varkappa(T+B) = \varkappa(T)$.*

(iii) *$\alpha(T+\lambda B)$ and $\beta(T+\lambda B)$ have constant values n_1 and n_2, respectively, except perhaps for isolated points. At the isolated points*

$$\infty > \alpha(T+\lambda B) > n_1 \quad \text{and} \quad \beta(T+\lambda B) > n_2.$$

In particular, this theorem implies that the sum of a Fredholm operator and a strictly singular operator is again a Fredholm operator. In this section we prove an analogue of the latter statement in the non-normable case. It appears that Definition 1.1 of a strictly singular operator from one l.c.s. into another, is too general for that purpose. Lacey [5] introduced and studied several more restrictive notions of strict singularity which all coincide in the case that E and F are normed spaces. We define the following new concept, suggested by Theorem 1.6.

Definition 2.2. A continuous linear operator $B: E \to F$ is called *super strictly singular* (s.s.s.) if a 0-neighborhood U in E exists with the property that for every infinite-dimensional linear subspace $M \subset D(B)$ such that $M \cap N(B) = \{0\}$ and for every 0-neighborhood V in F there exists an infinite-dimensional linear subspace $N \subset M$ such that $N \not\subset U$ and $B(U \cap N) \subset V$.

Remark 1. It is easily verified that the condition imposed in Definition 2.2 implies property (ii) of Theorem 1.6. We note the following two differences which make super strict singularity a priori a stronger requirement than property (ii) of Theorem 1.6.

1° In Definition 2.2, M is not assumed to be closed.

2° In Definition 2.2 the 0-neighborhood U in E is required to be independent of M for all M such that $M \cap N(B) = \{0\}$. In (ii) of Theorem 1.6, U may depend on M.

Remark 2. For any linear operator $B: E \to F$, E and F arbitrary l.c.s., the following holds:

$$B \text{ precompact} \Rightarrow B \text{ s.s.s.} \Rightarrow B \text{ s.s.}$$

The first implication follows from Theorem 1.5, the second from the foregoing remark and from Remark 2 at the end of section 1.

Remark 3. It follows from Theorem 1.4 and from Remark 3 at the end of section 1 that the s.s. and the s.s.s. operators concide when E and F are both normed linear spaces.

Before stating a stability theorem for s.s.s. operators, we give the following definition:

Definition 2.3 (Cf. Lacey [5]). An l.c.s. E is called *superprojective* if for every closed linear subspace $M \subset E$ with infinite deficiency in E there exists a closed linear subspace $K \supset M$ with infinite deficiency in E such that K has a closed complement in E.

THEOREM 2.4. *Let $E \times F$ be a Ptak space and let F be fully barreled and superprojective. Then, if $T: E \to F$ is a Fredholm operator with closed range and if $B: E \to F$ is s.s.s. and $D(B) \supset D(T)$, $T+B$ is also a Fredholm operator with closed range.*

Proof. (a) It is clearly no restriction to suppose that $D(T)$ is dense in E. $T+B$ is evidently closed, since T is closed and B is continuous. Also $D(T+B) = D(T)$ is dense in E. Hence, by Corollary I.12, $R(T+B)$ is closed if (and only if) $T+B$ is open. In proving that $R(T+B)$ is closed we may assume that $\alpha(T) = 0$. Indeed, if $\alpha(T) \neq 0$ we can choose a closed complement M of $N(T)$ in E, because $\alpha(T) < \infty$. Then $R(T+B) = R(T_M + B_M) + K$, with $\dim K < \infty$. By Theorem I.18, $R(T+B)$ is closed if $R(T_M+B_M)$ is.

We prove that $T+B$ is open. Let U_0 be a 0-neighborhood in E associated with B according to Definition 2.1. Suppose that $T+B$ is not open. Then $T+B$ is not nearly open, by Theorem I.7. Hence a 0-neighborhood U in E exists such that $\overline{(T+B)\,U}^{(R(T+B))}$ is not a 0-neighborhood in $R(T+B)$. We may assume that $U \subset U_0$. Since T is open by Corollary I.12, there is a 0-neighborhood V in F such that $TU = V \cap R(T)$. $T+B$ not being nearly open, we can repeat the argument in the first part of the proof of Theorem 1.6 and construct an infinite-dimensional linear subspace $M \subset D(T)$ such that

(1) $$(T+B)(U \cap M) \subset \tfrac{1}{3}V.$$

By Remark 1 following Theorem 1.6, we may even assume that U contains no linear subspaces of M. Next, putting $K = M \cap N(B)$, we observe that $K = \{0\}$. Indeed, suppose that $K \neq \{0\}$. (1) then implies that $T(U \cap K) \subset \frac{1}{3}V$. On the other hand, since $TU = V \cap R(T)$ and T is 1-1, we have $T(U \cap K) = V \cap TK$. Hence $TK \subset V$ and therefore $K \subset U$, since $TU = V \cap R(T)$. This contradicts the fact that U contains no linear subspaces of M.

Since B is s.s.s., $U \subset U_0$ and $M \cap N(B) = \{0\}$, there exists an infinite-dimensional linear subspace $N \subset M$ such that

(2) $$B(U \cap N) \subset \tfrac{1}{3}V.$$

From (1) it follows, because $N \subset M$, that

(3) $$(T+B)(U \cap N) \subset \tfrac{1}{3}V.$$

By (2) and (3) we have

(4) $$T(U \cap N) \subset (T+B)(U \cap N) + B(U \cap N) \subset \tfrac{2}{3}V.$$

On the other hand, $TU = V \cap R(T)$. Hence

(5) $$T(U \cap N) = V \cap TN,$$

since $\alpha(T) = 0$. Now (4) and (5) imply that $V \cap TN = TN$. Hence $U \cap N = N$ or $N \subset U$, which cannot be since $N \subset M$ and U contains no linear subspaces of M. Therefore $T+B$ is open or, equivalently, $R(T+B)$ is closed.

(b) We show next that $\alpha(T+B) < \infty$. Since $\dim N(T) < \infty$ and therefore $\dim N(T+B) \cap N(T) < \infty$, there exists by Theorem I.16 a closed linear subspace $N_1 \subset E$ such that

$$N(T+B) = \big(N(T+B) \cap N(T)\big) \oplus N_1.$$

If $T_1 = T_{N_1 \oplus N(T)}$, then T_1 is closed because $N_1 \oplus N(T)$ is a closed linear subspace of E. Also $N(T_1) = N(T)$. We prove that T_1 is open. Let U_1 be an arbitrary 0-neighborhood in $N_1 \oplus N(T)$. We extend U_1 to a 0-neighborhood U in E with $U_1 = U \cap \big(N_1 \oplus N(T)\big)$. Since T is open by Corollary I.12, TU is a 0-neighborhood in $R(T)$. Furthermore, $U \cap R(T_1) = T_1 U_1$. Hence $T_1 U_1$ is a 0-neighborhood in $R(T_1)$. This proves that T_1 is open. Again by Corollary II.3.9, $R(T_1) = TN_1$ is closed. Now consider T_{N_1}. Since T is 1-1 on N_1 and $T = -B$ on N_1, T_{N_1} is continuous. Since T_{N_1} is also open, because $R(T_{N_1}) = R(T_1)$ is closed, $T_{N_1} = -B_{N_1}$ is a topological isomorphism. B being strictly singular by Remark 2, it follows that $\dim N_1 < \infty$. Hence $\alpha(T+B) < \infty$.

(c) Finally, we show that $\beta(T+B) < \infty$. We may again assume that $\alpha(T) = 0$ by restricting T and B to a closed complement of $N(T)$ in E. Suppose $\beta(T+B) = \infty$. Since F is superprojective, there exist closed linear subspaces M and K in F with $\dim K = \infty$ such that $R(T+B) \subset M$ and $M \oplus K = F$. If $L = K \cap R(T)$, then L is infinite-dimensional, since $\beta(T) < \infty$. Let U be a 0-neighborhood in E associated with B according to Definition 2.1. Let $V_1 = 3(T+B)\,U$ and $V_2 = (TU) \cap L$. Because $T+B$ and T are both open by Corollary II.3.9, V_1 is a 0-neighborhood in $R(T+B)$ and V_2 is a 0-neighborhood in L. Then the absolutely convex hull $\Gamma(V_1, V_2)$ of V_1 and V_2 is a 0-neighborhood in $R(T+B) \oplus L$. $R(T+B) \oplus L$ being a subspace of $M \oplus K = F$, we can extend $\Gamma(V_1, V_2)$ to a 0-neighborhood V in F with $V \cap \big(R(T+B) \oplus L\big) = \Gamma(V_1, V_2)$. Clearly $\dim T^{-1}L = \infty$, since $\dim L = \infty$ and T is 1-1. Also $T^{-1}L \cap N(B) = \{0\}$. Indeed, $x \in T^{-1}L \cap N(B)$ implies $Tx = (T+B)x \in L \cap$

$\cap R(T+B) = \{0\}$. Hence $x = 0$. Therefore, since B is s.s.s., there exists an infinite-dimensional linear subspace $N \subset T^{-1}L$ such that $N \not\subset U$ and $B(U \cap N) \subset \frac{1}{3}V$. Also $(T+B)(U \cap N) \subset \frac{1}{3}V$, since $(T+B)U = \frac{1}{3}V \subset \frac{1}{3}V$. Hence

$$T(U \cap N) \subset (T+B)(U \cap N) + B(U \cap N) \subset \tfrac{2}{3}V.$$

On the other hand,

$$(TU) \cap L = V_2 = V \cap L, \quad \text{so} \quad T(U \cap N) = V \cap (TN).$$

It follows then that $TN \subset V$. Hence $N \subset U$, which contradicts the choice of N. Therefore $\beta(T+B) < \infty$ and the proof is complete.

Note. Compact operators are a particularly nice kind of s.s.s. operators. Therefore, better results can be expected for them. Pietsch [7] studied perturbations of "σ-Transformationen" by compact operators. The author [2] proved the following

THEOREM. *Let E be a Ptak space and let F be fully barreled. If $T: E \to F$ is a densely defined Fredholm operator with closed range and if $B: E \to F$ is compact and defined on E, then the following holds:*

(i) *$T+\lambda B$ is a Fredholm operator with closed range for every $\lambda \in C$ and $\varkappa(T+\lambda B) = \varkappa(T)$.*

(ii) *There exists a constant $\varrho > 0$ such that $\alpha(T+\lambda B) \leqslant \alpha(T)$ and $\beta(T+\lambda B) \leqslant \beta(T)$ for all $\lambda \in C$ with $|\lambda| < \varrho$.*

(iii) *If, in addition, $E \times F$ is a Ptak space, then a constant $\varrho_1 > 0$ exists such that $\alpha(T+\lambda B)$ and $\beta(T+\lambda B)$ are constant for all $\lambda \in C$ with $0 < |\lambda| < \varrho_1$.*

In the particular case where T is continuous and defined on all of E, this theorem partially follows from Pietsch's result.

3. Algebraic properties of the spaces $S(E,F)$ and $S_s(E,F)$. The question as to whether or not, in general, the collection $S(E,F)$ of all s.s. operators $B: E \to F$ defined on E forms a linear space, constitutes an unsolved problem. The difficulty lies in proving the sum of two s.s. operators to be s.s. In this section we exhibit some special cases in which the answer is affirmative. In contrast to Lacey's approach [5], we do not strengthen the definition of strict singularity, but we impose restrictions on the underlying spaces.

It is easily verified that the s.s.s. operators $B: E \to F$ defined on E form a linear space. We can even prove the following result:

THEOREM 3.1. *Let E be a Ptak space and let $B, C: E \to F$ both be defined on E. If B is s.s. and C is s.s.s., then $B+C$ is s.s.*

Proof. Let U_0 be a 0-neighborhood in E satisfying Definition 2.2 for C and let M be any closed infinite-dimensional linear subspace of E.

By Theorem 1.6 there exists a 0-neighborhood U in E with the property that for any 0-neighborhood V in F there is an infinite-dimensional linear subspace $N \subset M$ such that $N \not\subset U$ and

$$B(U \cap N) \subset \tfrac{1}{2}V. \tag{1}$$

Now let a 0-neighborhood V in F be given arbitrarily and suppose that N is an infinite-dimensional linear subspace of M such that (1) holds. By Remark 1 following Theorem 1.6 we may even suppose that U does not contain any linear subspace of N. We may also assume that $U \subset U_0$. We now distinguish two cases.

1° $\dim N \cap N(C) = \infty$.

Putting $N_1 = N \cap N(C)$, (1) clearly implies

$$(B+C)(U \cap N_1) \subset \tfrac{1}{2}V \subset V.$$

2° $\dim N \cap N(C) < \infty$.

Let K be any linear subspace of N such that $K \cap N(C) = \{0\}$, $\dim K = \infty$. Then by the super strict singularity of C, K contains an infinite-dimensional linear subspace N_1 such that $N_1 \not\subset U_0$ and

$$C(U_0 \cap N_1) \subset \tfrac{1}{2}V. \tag{2}$$

Since $U \subset U_0$, we also have

$$C(U \cap N_1) \subset \tfrac{1}{2}V. \tag{3}$$

Furthermore, from (1) and $N_1 \subset N$ it follows that

$$B(U \cap N_1) \subset \tfrac{1}{2}V. \tag{4}$$

(3) and (4) imply that

$$(B+C)(U \cap N_1) \subset V.$$

We have thus proved in both cases that for an arbitrary closed infinite-dimensional linear subspace $M \subset E$ there exists a 0-neighborhood U with the property that for every 0-neighborhood V in F an infinite-dimensional linear subspace $N_1 \subset M$ exists such that $N_1 \not\subset U$ and $(B+C)(U \cap N_1) \subset V$. By Theorem 1.6, $B+C$ is then s.s.

COROLLARY 3.2. *Let E be a Ptak space and let $B, C: E \to F$ both be defined on E. If B is s.s. and C is precompact, then $B+C$ is s.s.*

Proof. C is s.s.s. by Remark 2 in section 2.

In the following theorem $S_s(E,F)$ denotes the collection of all s.s.s. operators $B: E \to F$ defined on E. As usual $L(E,F)$ is the space of all continuous linear operators mapping all of E into F.

THEOREM 3.3. *Let E, F, G and H be arbitrary l.c.s. Then the following holds:*

1° *If $A \in S_s(E, F)$ and $B \in L(F, G)$, then $BA \in S_s(E, G)$.*

2° *If $C \in S_s(G, H)$ and $B \in L(F, G)$, then $CB \in S_s(F, H)$.*

In particular, $S_s(E, E)$ is a two-sided ideal in $L(E, E)$.

Proof. 1° Let U_0 be a 0-neighborhood in E associated with A according to Definition 2.2. We show that U_0 also satisfies Definition 2.2 for the operator BA. Indeed, let M be an infinite-dimensional linear subspace of E such that $M \cap N(BA) = \{0\}$ and let W be an arbitrary 0-neighborhood in G. Then, in the first place, $M \cap N(A) = \{0\}$ and, secondly, $V = B^{-1}W$ is a 0-neighborhood in F. Hence, by the strict singularity of A, there exists an infinite-dimensional linear subspace $N \subset M$ such that $N \not\subset U$ and $A(U \cap N) \subset V$. The last inclusion clearly implies $BA(U \cap N) \subset W$.

2° Let U_0 be a 0-neighborhood in G associated with C according to Definition 2.2. Then $U_1 = B^{-1}U_0$ is a 0-neighborhood in F. We show that U_1 satisfies Definition 2.2 for the operator CB. Let M be any infinite-dimensional linear subspace of F such that $M \cap N(CB) = \{0\}$ and let V be an arbitrary 0-neighborhood in H. Then $M \cap N(B) = \{0\}$, and therefore $\dim BM = \infty$. Also $BM \cap N(C) = \{0\}$. Hence, by the strict singularity of C there exists an infinite-dimensional linear subspace $K \subset BM$ such that $K \not\subset U_0$ and $C(K \cap U_0) \subset V$. Putting $N = B^{-1}K \cap M$, we then have $\dim N = \infty$, $N \not\subset U_1$ and $CB(N \cap U_1) \subset V$. This completes the proof.

Finally we exhibit two cases in which $S(E, F)$ is a linear space.

THEOREM 3.4. *If E is complete and if either E or F has the property that every infinite-dimensional linear subspace contains an infinite-dimensional normable linear subspace, then $S(E, F)$ is a linear space.*

Proof. Assume that F has the property stated in the theorem. Let $B_1, B_2: E \to F$ be s.s. and defined on E and suppose that B_1+B_2 is not s.s. Then there exists an infinite-dimensional linear subspace $M \subset E$ such that $(B_1+B_2)_M$ is a topological isomorphism. We may assume that M is complete since E is complete. We consider the operators

$$B_1\big((B_1+B_2)_M\big)^{-1},\ B_2\big((B_1+B_2)_M\big)^{-1}: (B_1+B_2)\,M \to F.$$

Clearly both operators are s.s. and their sum is the identity. Also $\dim\,(B_1+B_2)\,M = \infty$, so that by assumption there exists a normed linear subspace $N \subset (B_1+B_2)\,M$. We may assume that N is a Banach space, since $(B_1+B_2)\,M$ is complete. The restrictions of $B_1\big((B_1+B_2)_M\big)^{-1}$ and $B_2\big((B_1+B_2)_M\big)^{-1}$ to N we denote by T_3 and T_4, respectively. T_3 and T_4 are s.s. and by Remark 3 in section 1, their sum I_N is then also s.s., which is absurd.

If E has the property of the theorem, then the proof is even simpler. We leave it to the reader.

THEOREM 3.5. *$S(E, F)$ is a linear space if at least one of the following conditions is satisfied:*

(i) *E is weakly complete, i.e., complete with respect to the topology $\sigma(E, E')$.*

(ii) *E is complete and F is weakly complete.*

Proof. An l.c.s. E is called *minimal* if the topology is minimal, i.e., if there exists no strictly coarser locally convex Hausdorff topology on E. It is known (Cf. Martineau [17]) that E is minimal if and only if E is weakly complete, and also that every closed linear subspace and every separated quotient of a minimal space is minimal.

(a) Suppose that E is weakly complete, therefore minimal. Then every continuous linear operator $B: E \to F$ defined on E is open. Consider now the induced operator $\hat{B}: E/N(B) \to R(B) \subset F$ and assume that $\hat{B}$ is not open. Carrying over the topology of $R(B)$ to $E/N(B)$ by means of $\hat{B}^{-1}$, we then obtain a locally convex Hausdorff topology on $E/N(B)$ which is strictly coarser than the quotient topology. This contradicts the minimality of $E/N(B)$. Hence $\hat{B}$ and therefore B is open. Consequently $S(E, F)$ consists precisely of all continuous linear operators of finite rank. Hence $S(E, F)$ is a linear space.

(b) Suppose now that E is complete and that F is weakly complete. We argue as in the proof of Theorem 3.4. If we assume that $B_1, B_2: E \to F$ are s.s. and that B_1+B_2 is not s.s., it follows that an infinite-dimensional linear subspace $M \subset E$ exists such that $(B_1+B_2)_M$ is a topological isomorphism. We may assume that M is closed. As a closed linear subspace of the minimal space F, $(B_1+B_2)\,M$ is then minimal. The operators

$$B_1\big((B_1+B_2)_M\big)^{-1},\ B_2\big((B_1+B_2)_M\big)^{-1}: (B_1+B_2)\,M \to F$$

are then clearly s.s. and their sum is not s.s. Since $(B_1+B_2)\,M$ is minimal, this contradicts what was proved in (a). Hence $S(E, F)$ is a linear space.

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Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

Equivalent nuclear systems

by

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In this paper we introduce the notion of nuclear system as a means of constructing nuclear Fréchet spaces whose topologies are defined by a family of seminorms which are actually norms. We then show that all such spaces are obtained by this construction. The main result (Theorem 2) is an "intrinsic" characterization of when two nuclear systems are equivalent, that is when the spaces which they construct are isomorphic. This result is then applied to the basis problem for nuclear Fréchet spaces. Finally some examples and open questions are listed.

This method of constructing nuclear Fréchet spaces gives rise to examples which have not previously been discussed as well as providing a new way of studying the familiar spaces. These examples will be discussed in detail in a forthcoming paper.

Let $A_k\colon l_2 \to l_2$, $k = 1, 2, \ldots$, be a sequence of nuclear maps and define the *associated space*,

$$\hat{E} = \hat{E}\{(A_k)\} = \{(x_k)_k\colon x_k \in l_2,\ x_k = A_k x_{k+1},\ k = 1, 2, \ldots\}.$$

Thus $\hat{E}$ is a subspace of the countable product of copies of l_2, and we may equip $\hat{E}$ with the topology induced by the usual product topology. Let $P_k\colon \hat{E} \to l_2$ by $P_k\big((x_k)_k\big) = x_k$. We call $(A_k)_k$ a *nuclear system* if

(i) each A_k has dense range

(ii) each P_k is injective.

THEOREM 1. *The associated space of nuclear system is a nuclear Fréchet space with a fundamental sequence of seminorms which are norms; and, conversely, every such space is the associated space of a nuclear system (up to isomorphism).*

Proof. Clearly, $\hat{E}\{(A_k)\}$ is nothing more than the projective limit of the sequence of maps, $(A_k)_k$ and hence it is a Fréchet space. Evidently, a fundamental system of neighborhoods of 0 is given by the sets

$$V_n = \left\{(x_k) \in \hat{E}\colon \|x_k\| \leqslant \frac{1}{n},\ k = 1, 2, \ldots, n\right\}, \quad n = 1, 2, \ldots,$$