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Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

Integration in function spaces and differential equations with functional derivatives

by

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1. Various reasons impose the necessity of development of analysis on certain infinite-dimensional structures, in particular, of development of a theory of differential equations. Those reasons are due to the intrinsic progress in mathematics as well as to applications in physics. Construction of such a theory is certainly impossible without a knowledge of several methods of integration in infinite-dimensional vector spaces and non-linear differentiable manifolds.

The simplest and best known are Gauss measures in a Hilbert space $\mathfrak{H}$ (see, e.g., [1]). One, usually starts with a finitely additive non-negative set function $\tilde{\mu}$ defined on the algebra $\mathfrak{A}_0$ of cyllindric sets in $\mathfrak{H}$ (a weak distribution). This set function is fully determined by its characteristic functional

$$\chi_\mu(x) = \int_{\mathfrak{H}} e^{i(x,y)}\, \tilde{\mu}(dy),$$

which, in the case of a Gauss measure with mean-value zero, has the form

$$\chi_\mu(x) = \exp\left(-\tfrac{1}{2}(Bx, x)\right),$$

where B is a positive operator in $\mathfrak{H}$ called the *correlation operator* of the measure $\tilde{\mu} = \tilde{\mu}_B$. As is well known, the nuclearity of B is necessary and sufficient in order that the weak distribution $\tilde{\mu}_B$ be extendible to a (countably additive) measure μ_B defined on the σ-algebra $\mathfrak{A}$ of Borel sets in $\mathfrak{H}$. In other cases such an extension can be realized only in spaces containing $\mathfrak{H}$.

Consider, for instance, the space $\mathfrak{H}_-$ defined as the completion of $\mathfrak{H}$ in the norm $\|x\|_- = \|Sx\|$, where S is a positive Hilbert-Schmidt operator. Any Gauss measure defined in the σ-algebra $\mathfrak{A}_-$ of Borel subsets of $\mathfrak{H}_-$ is countably additive, provided its correlation operator B is bounded (e.g. $B = I$).

It turns out practical to consider also the Hilbert space $\mathfrak{H}_+$ obtained by means of the norm $\|x\|_+ = \|S^{-1}x\|$ defined in the domain $D_{S^{-1}}$ of S^{-1}. Extend the operator S by continuity to the whole of $\mathfrak{H}_-$ and denote the resulting operator by $\tilde{S}$. The formula

$$\langle x_+, x_-\rangle = (S^{-1}x_+, \tilde{S}x_-) \qquad (x_\pm \in \mathfrak{H}_\pm)$$

realizes the duality between the spaces $\mathfrak{H}_+$ and $\mathfrak{H}_-$. For any fixed $x_+ \in \mathfrak{H}_+$, the functional $\langle x_+, x_-\rangle$ is continuous in $\mathfrak{H}_-$. The correspondence

$$\int_{\mathfrak{H}_-} |\langle x_+, \xi\rangle|^2 \mu_B(d\xi) = (Bx_+, x_-)$$

gives a method of a definition of measurable linear functionals $\langle x, x_-\rangle$ on $\mathfrak{H}_-$ for all x's in $\mathfrak{H}$. This is done by means of a suitable limit process in the Hilbert space $L^2(\mathfrak{H}_-, \mu_B)$.

2. We consider first the case where $\mathfrak{H}$ is a finite-dimensional real vector-space. Let an operator B be given by a matrix $B = \|b_{jk}\|$. As is well known, the function

$$u(x, t) = \int_{\mathfrak{H}} f(x-y)\mu_{Bt}(dy) \tag{1}$$

is, under suitable assumptions about $f(x)$, a solution of the Cauchy problem for the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sum_{j,k=1}^{n} b_{jk}\frac{\partial^2 u}{\partial x_j \partial x_k} \qquad (t \geqslant 0) \tag{2}$$

with the condition

$$u(x, 0) = f(x). \tag{3}$$

This result can be generalized to the infinite-dimensional case if equation (2) is rewritten in the form

$$\frac{\partial u}{\partial t} = \mathrm{Sp}(Bu''), \tag{4}$$

where u'' stands for the second order Fréchet derivative of the function $u(x)$, i.e. the operator occurring in the Taylor expansion

$$u(x+h) - u(x) = \big(u'(x), h\big) + \frac{1}{2}\big(u''(x)h, h\big) + o(\|h\|^2).$$

In the case where the function $u(x)$ is defined on $\mathfrak{H}_-$, the form $\langle x_+, x_-\rangle$ may be used in this expansion instead of the scalar product (x, y). It then turns out that $u''(x) \in L(\mathfrak{H}_-, \mathfrak{H}_+)$. The restriction of such an operator

to $\mathfrak{H}$ is nuclear and, consequently, for a bounded operator B the estimate $\mathrm{Sp}(Bu'') < \infty$ holds. In general, it can be shown that equation (4) makes sense provided the function $u(x)$ has continuous bounded derivatives up to the second order ($u \in C_2(\tilde{\mathfrak{H}})$) in that extension $\tilde{\mathfrak{H}}$ of $\mathfrak{H}$ where the measure μ is σ-additive. Then the formula

$$u(x, t) = \int_{\tilde{\mathfrak{H}}} f(x-y)\mu_{Bt}(dy) \tag{1'}$$

gives a solution of the Cauchy problem (4)-(3) for $f \in C_2(\tilde{\mathfrak{H}})$(cf. [2]-[4])

Gross [5] has independently considered equation (4) with $B = I$. He obtained, for this case, more precise results under weaker assumptions concerning the smoothness of $f(x)$ and developed the potential theory. His methods have been recently employed by Piech [6] who has dealt with a wider class of equations, including certain equations in which the operator B depends on $x: B = B(x)$.

3. We now pass to the discussion of equations with non-constant coefficients. The solution of the Cauchy problem can in this case be expressed by a formula of type (1′) in which the integration is performed with respect to a non-Gauss measure. This measure can be constructed with use of the Ito stochastic integral equations.

In the space $\mathfrak{H}_-$ there exists a Wiener stochastic process $w(t)$: the Gauss process with independent increasements $w(t)-w(\tau)$ whose mean value is zero and whose correlation operator is $I(t-\tau)$.

Now let H be a Hilbert space and let $a(t, x)$, $A(t, x)$ be a vector-valued, resp. an operator-valued function:

$$a: [t_0, T]\times H \to H, \qquad A: [t_0, T]\times H \to L(\mathfrak{H}_-, H).$$

The Ito equation has the form

$$\xi(t) = \xi_0 + \int_{t_0}^{t} a\big(s, \xi(s)\big)ds + \int_{t_0}^{t} A\big(s, \xi(s)\big)dw(s). \tag{5}$$

Similar equations have been examined by K. Ito, I. I. Gichman and A. V. Skorokhod in the finite-dimmensional case (see, e.g., [7]); for the infinite-dimensional case refer to the papers [2]-[4]. Suppose that the functions $a(s, x)$ and $A(s, x)$ fulfil the Lipschitz condition with respect to x uniformly in $[t_0, T]\times H$. Then equation (5) has a unique (up to stochastic equivalence) solution $\xi(t)$ which is a stochastic process with values in H. Its transition probabilities

$$\mu^t_{\tau,x}(M) = P\{\xi(t) \in M | \xi(\tau) = x\}$$

are σ-additive measures on the σ-algebra of Borel subsets of H.

Let $f \in C_2(H)$. It can be shown, under some additional assumptions, concerning the smoothness of $a(s, x)$ and $A(s, x)$, that the function

$$(6) \qquad u(\tau, x) = \int_H f(y)\, \mu^t_{\tau,x}(dy)$$

represents the unique in $C_2(H)$ solution of the Cauchy problem for the equation

$$(7) \qquad -\frac{\partial u}{\partial \tau} = \frac{1}{2}\operatorname{Sp}[A^*(\tau, x)u''A(\tau, x)] + \big(a(\tau, x), u'\big) \qquad (t_0 \leqslant \tau \leqslant t)$$

with the condition

$$u(\tau, x)|_{\tau=t} = f(x).$$

The solution of this problem, as well as that of equation (5), may be obtained by passing to the limit with the solutions of appropriate finite-dimensional problems resulting under projections of H onto finite-dimensional subspaces.

A solution of the Cauchy problem for an equation which differs from (7) by an additional summand $V(\tau, x)u$ can be written in the form of a function-valued integral over the space of trajectories with values in the space H. This is done by means of the well known method due to Kac [9].

Equation (7) is just the backward Chapman-Kolmogorov equation for the process $\xi(t)$. The negative sign on its left-hand side occurs in connection with the fact that the Cauchy problem is formulated for the left half-axis. The usual form is regained after an inversion of the direction of time.

In the finite-dimensional case the foreward Chapman-Kolmogorov equation (the Fokker-Planck equation) for the density of the measure $\mu^t_{\tau,x}$ with respect to the Lebesgue measure may be considered together with the backward equation. In the infinite-dimensional case such a setting of the problem has no sense. However, an analogue of the foreward equation is obtained, following an idea of Fomin [8], by taking into account equations in which set functions occur.

4. Let μ_B be a Gauss measure with a bounded correlation operator, supported by $\mathfrak{H}_-$. As is well known, the translated measure $\mu_{B,x_0}(M) = \mu_B(x_0+M)$ (for $x_0 \in \mathfrak{H}$) is equivalent to μ_B.

A similar result is also valid in the more general case considered above (cf. [7], [4]). Let μ_a and μ_b be measures in the space of H-valued functions; suppose that those measures are generated by the solutions of a pair of stochastic equations (5) with the same diffusion operator A and different translation vectors a and b. Suppose that the function $\alpha(t, x) = A^{-1}(t, x)[b(t, x) - a(t, x)]$ has values in $\mathfrak{H}$ and that the linear estimate $\|\alpha\| \leqslant c_1 + c_2\|x\|$ holds. Then the measures μ_a and μ_b are equivalent and their relative density fulfils the equality

$$\log\frac{d\mu_b}{d\mu_a}[x(\cdot)] = \frac{1}{2}\int_{t_0}^{T} \|\alpha(\tau; \xi(\tau))\|^2 d\tau + \int_{t_0}^{T} \langle \alpha(\tau, \xi(\tau)), dw(\tau)\rangle.$$

Hence it follows that the measures obtained as the transition probabilities of the Markov processes in question are equivalent. It would be of an interest to obtain some conditions for the equivalence of such transition probabilities, analogous to those concerning Gauss measures, also in that case when the occurring diffusion operators are different and the measures μ_a and μ_b are not equivalent.

5. Further development of the above results is required by a proceeding from a linear Hilbert space to a non-linear differentiable Hilbert manifold. A way to construct the diffusion on a finite-dimensional differentiable manifold has been pointed out by Ito [10] and developed by Gangolli [11]. The results referred to below have been obtained by Belopolska (Schneiderman) and the author [12]. Let X be a Hilbert manifold without boundary, let an affine connection be defined on X, finally, let a_x and A_x be a vector- and an operator-field on X such that $\operatorname{Sp} A_x^* A_x < \infty$. An analogue of the Ito equation has the form

$$(8) \qquad \xi(t+dt) = \exp_{\xi(t)}[a_{\xi(t)}dt + A_{\xi(t)}dw(t)].$$

Let $\varphi\colon U \to H_\varphi$ be a coordinate map of a neighborhood U of φ in X into the corresponding Hilbert space H_φ, let $\dot\varphi_x$ be the derivative of this map. Write

$$a_x^\varphi = \dot\varphi_x a_x, \qquad A_x^\varphi = \dot\varphi_x A_x \dot\varphi^{-1}, \qquad dw_x^\varphi = \dot\varphi_x dw_x.$$

Let Γ_x^φ denote the Christofel symbol, i.e. the bilinear operator in H_φ which enjoys the transformation property

$$\dot F^{\varphi\psi}_{\psi(x)}\Gamma^{\psi}_{\psi(x)}(\dot F^{\psi\varphi}_{\varphi(x)}f^\varphi, \dot F^{\psi\varphi}_{\varphi(x)}g^\varphi) = \Gamma^\varphi_{\varphi(x)}(f^\varphi, g^\varphi) + \ddot F^{\varphi\psi}_{\psi(x)}(\dot F^{\psi\varphi}_{\varphi(x)}f^\varphi, \dot F^{\psi\varphi}_{\varphi(x)}g^\varphi),$$

where $F^{\psi\varphi} = \psi\circ\varphi^{-1}$. Let

$$\operatorname{Sp} A^{\varphi^*}\Gamma^\varphi A^\varphi = \sum_k \Gamma^\varphi(A^\varphi e_k, A^\varphi e_k)$$

hold for an orthonormal basis $\{e_k\}_{k=1}^\infty$ in H_φ. Then φ carries the process $\xi(t)$ into a process $\varphi(\xi(t))$ in H, which satisfies the equation analogous to (5):

$$d\varphi(\xi(t)) = [a^\varphi_{\xi(t)} - \frac{1}{2}\operatorname{Sp} A^{\varphi*}_{\xi(t)}\Gamma^\varphi_{\varphi(\xi(t))}A^\varphi_{\xi(t)}]dt + A^\varphi_{\xi(t)}dw^\varphi_{\xi(t)}.$$

Applying the measure generated by the process $\xi(t)$ one obtains a solution of the Cauchy problem for the parabolic differential equation on X,

$$\frac{\partial u}{\partial t} = \tfrac{1}{2}(\mathrm{Sp}\nabla . A_x A_x^* \nabla .)u + (a_x, u'),$$

where ∇_f is the symbol of covariant differentiation and the operator $l(u) = \nabla . A_x A_x^* \nabla . u(x)$ is defined by $\big(l(u)f, f\big)_x = (\nabla_f A_x A_x^* \nabla_f)\, u(x)$.

A condition for the equivalence of a pair of measures corresponding to a pair of equations (8) and the formula expressing their relative density are much similar to those formulated in Section 4.

Let X be the underlying manifold of an infinite-dimensional nuclear Lie group G. By definition, there exists a subgroup $\tilde{G}$ of G for which an embedding of the tangent space $\tilde{T}_e$ into the tangent space T_e is realized by a Hilbert-Schmidt operator. The above considerations lead to a construction of a quasi-invariant measure on G, i.e. a measure whose all translates by element of $\tilde{G}$ are equivalent. The transition probabilities of the solutions of equations (8) have similar quasi-invariance properties, provided the fields a_x, A_x and the affine connection are translation-invariant.

6. We now turn again to equations in a linear space and suppose, for simplicity, that $H = \mathfrak{H}_-$, i.e. $A(x) \in L(\mathfrak{H}_-, \mathfrak{H}_-)$. Assume, further, that $A(x) \in L(\mathfrak{H}_+, \mathfrak{H}_+)$. There is the differential operator

$$(9) \qquad l(u) = \mathrm{Sp}(AA^*u')' - 2(A^*u', A^*x)$$

well defined for functions $u \in C_2(\mathfrak{H}_-)$. This operator is symmetric and non-positive in the Hilbert space $L^2(\mathfrak{H}_-, \mu_I)$. Applying the results of Section 3 concerning parabolic equations one can show that the closure of this operator is self-adjoint. It would be desirable to develop the spectral theory of such operators with functional derivatives.

7. As soon as the operator (9) is shown to be self-adjoint we are able to consider the Schrödinger-type equation with functional derivatives

$$\frac{1}{i}\frac{\partial u}{\partial t} = l(u) + V(x, t)u.$$

One can attempt, much like this has been done with parabolic equations, to write the solution of the Cauchy problem for this equation as an integral of the functional

$$\exp\int_0^T V\big(x(t), t\big)\, dt$$

over the space of trajectories with values in $\mathfrak{H}_-$.

This leads to an integration procedure called the *Feynman integral*. Several difficulties arrise here already in the case of a finite-dimensional phase space $\mathfrak{H}$. The point is that the Feynman measure is complex-valued and is not σ-additive though its finitedimensional projections are σ-additive if $\dim \mathfrak{H} < \infty$. Cameron [13] has proved the convergence of the Feynman integrals for analytic functionals; his approach is that of an analytic extension of the Wiener integrals to the complex plane with respect to the parameter which occurs in the correlation operator as a factor. Another approach, employed in [14], involves a multiplication formula for the perturbated semigroup

$$(10) \quad \exp[t(A+B)] = \lim_{\max \Delta t_k \to 0} \prod_k \exp(\Delta t_k \cdot A)\exp(\Delta t_k \cdot B) \qquad \Big(\sum_k \Delta t_k = t\Big).$$

This formula has been obtained independently in several forms by Trotter [15] and the author [14], [16]. Those ideas have been continued in the papers by Nelson [17] and Faris [18].

In the case $\dim \mathfrak{H} = \infty$, further difficulties occur in connection with the fact that even the measures analogous to the transition probabilities are not σ-additive and integration on cylindric sets requires a specific definition. Those obstacles can be omitted in several ways, for instance by means of Cameron's argument described above (cf. [13]).

On the other hand, the results of Section 6 give the possibility of an application of a formula of type (10).

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Perturbation theory and strictly singular operators in locally convex spaces*

by

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Introduction. In this paper we present some results on perturbations of Fredholm operators in locally convex spaces. We are interested in what happens to the index of such an operator when another operator is added to it. In particular, we want to establish conditions that guarantee that the index remains invariant. These problems have been studied extensively in the case of Fredholm operators acting in Banach spaces. Since non-normable and even non-metrizable spaces abound in analysis, it seems worthwhile to investigate what can be done in a more general context. The natural limit to which we can hope to generalize the theory is indicated by the fact that the closed graph theorem plays an essential role in it. Since the work of Ptak [8], however, it has become known that the validity of this theorem is rather wide. We shall make ample use of the following generalized closed graph theorem, due to Ptak: any closed linear operator mapping all of a barreled space into a Ptak (= fully complete = B-complete) space is continuous.

Of the three chapters in which this paper is divided, Chapter I contains the preliminaries. Chapter II and Chapter III can be read independently.

In Chapter II we study perturbations of Fredholm operators T by weakly continuous operators B which are "small with respect to T". In the case where the spaces involved are Banach spaces and B is continuous, the smallness condition takes the form $\|B\| < \gamma(T)$, where $\gamma(T)$ is some constant depending on T. In the non-normable case in which we are interested here, a corresponding condition can be formulated in terms of seminorms (conditions (P) and (P*)).

Chapter III is devoted to strictly singular operators. First we obtain

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