A linear topological characterization
of inner-product spaces
by
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A result due to J. S. Cohen in [1] suggests the following

Theorem. Let \( E \) be a Banach space; then the following conditions are equivalent:

(i) \( E \) is isomorphic (=linearly homeomorphic) to an inner product space.

(ii) if \( \varphi \in \Pi_2(E, l_1) \), then \( \varphi^* \in \Pi_2(l_1, E^*) \).

Notation. \( B(X, Y) \) denotes the space of bounded linear operators from a Banach space \( X \) into a Banach space \( Y \). The class \( \Pi_2(X, Y) \) of absolutely 2-summing operators from \( X \) into \( Y \) is defined by

\[
\Pi_2(X, Y) = \{ \varphi \in B(X, Y) : \sum \|a_n\| < C(\varphi) \}
\]

for \( \varphi \in X (i = 1, 2, \ldots) \) with \( \|\sum |a_i|\| < \|\varphi\| \) for \( \varphi \in X^* \).

\( X^* \) denotes the dual of \( X \) and \( \varphi^* \) denotes the adjoint operator of \( \varphi \).

By \( l_\infty(A) \) we denote the space of bounded scalar-valued functions on a set \( A \). We admit

\[
\|f\| = \sup_{a \in A} |f(a)|
\]

for \( f \) in \( l_\infty(A) \). Finally, \( \varphi \in B(X, Y) \) is called Hilbertian if there are a Hilbert space \( H \) and operators \( \psi \in B(X, H), \omega \in B(H, Y) \) such that \( \varphi = \omega \psi \).

Proof of the Theorem. (i) \( \Rightarrow \) (ii). This follows from the fact that if \( E \) is (isomorphic to) a Hilbert space; then the class \( \Pi_2(E, l_1) \) coincides with the class of Hilbert-Schmidt operators (cf. [3], Theorem 6.3)

(ii) \( \Rightarrow \) (i). Let \( \varphi \in B(E, l_\infty(A)) \) be an isometrically isomorphic embedding (Take \( A \) the unit ball of \( E^* \) and put \( \psi(\varphi^ *)(e) = \varphi^*(e) \) for \( e \in E \) and \( \varphi^* \in A \). Since \( B(l_\infty(A), l_1) = H_2(l_\infty(A), l_1) \) (cf. [9] and [3], Theorem 4.3), we
infer that \( u \in \Pi \), for each \( v \in B(l_2(A), l_2) \). Thus (ii) implies that
\[
(1) \quad u^* v u^* \in \Pi \quad \text{for every } v \in B(l_2(A), l_2).
\]

Now pick for \( i = 1, 2, \ldots, n \) so that \( \sum |\alpha_{i,j}^*|^2 < ||x^*||^2 \)
for every \( x^* \) in the second dual \( (l_2(A))^* \). Define \( v^* \in B(l_2(A), l_2) \)
by \( v^* = \sum_{i,j} \alpha_{i,j}^* f_{i,j} \) and denote by \( d_i \) the \( i \)-th coordinate functional
in \( l_2 \). Clearly, \( \sum |d_i^*|^2 = ||d_i||^2 \) for every \( d \in l_2 = (l_2)^* \). Thus (1) implies
that
\[
\sum |u^* v u^* d_i|^2 < C(u^* v^*).
\]

Hence \( E \subseteq \Pi \subseteq C(l_2(A))^* \), because \( v^* d_i = d_i^* \)
for \( i = 1, 2, \ldots \). Therefore \( u^* \Pi u^* \subseteq C(l_2(A))^* \).
Hence, by the Pietsch Factorization Theorem (cf., [3], p. 283), \( u^* \) is a Hilbertian operator.
Thus, by [3], Proposition 5.1, \( u \) is Hilbertian. Since \( u \) is an isometrically
isomorphic embedding of \( E \) and \( u \) is Hilbertian, \( E \) is isomorphic to an inner
product space (because the Banach space \( u(E) \) is the range of a bounded linear
operator from a Hilbert space). This completes the proof.

References

operators, this volume, p. 271-276.
[2] A. Grothendieck, R閚um馥 de la th閙ie m閚trique des produits tensoriels

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Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

On a class of operators in Hilbert space*

by

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0. Introduction. One version of the spectral theorem for Hermitian
and normal operators in Hilbert space is a consequence of the Gelfand
representation of the uniform closure \( B \) of the algebra generated by
the normal operator \( T \), its adjoint and the identity. The essential fact
is that \( B \) is commutative and isometrically *-isomorphic to the algebra
\( C(X) \) of all complex-valued continuous functions on a compact Haus-
dorff space \( (X) \).

On the other hand, if \( T \) is any operator on any Hilbert space \( H \),
then \( B \) is isometrically *-isomorphic to some algebra \( C(X, A) \) of all
continuous \( A \)-valued functions on \( X \), where \( X \) is a compact Hausdorff
space and \( A \) is a C*-algebra. Indeed, we may use for \( A \) the algebra \( B \)
and for \( X \) any one-point space \( \{a\} \). Clearly, what is desirable for any
extension of spectral theory is the choice of a canonical or minimal
algebra \( A \) and of a usefully simple topological space \( X \) so that the isometric
* -isomorphism \( B \cong C(X, A) \) permits some analysis of \( T \).

To pursue these ideas the author has discussed various aspects of
a natural and fruitful generalization of the notion of commutative Banach
algebra \([4]-[6]\). Indeed, since a commutative Banach algebra \( A \) is one
such that all its quotients by regular maximal ideals are isomorphic
(to \( G \), the field of complex numbers), the generalization in question is
a so-called Q-uniform Banach algebra defined as follows:

An algebra \( A \) is a Q-uniform algebra if:

a. \( A \) is a simple Banach algebra with identity;
b. \( A \) is a Q-bimodule such that for \( a_1, a_2 \in A, g_1, g_2 \in Q \),
\[
(g_1 a_1 a_2 - g_2 a_1 a_2, g_1 a_2 a_1 - g_2 a_2 a_1, a_1 g_1 a_2 - a_1 g_2 a_2, a_2 g_1 a_1 - a_2 g_2 a_1),
\]
where \( \{g_1, g_2 \} \in Q \), and where the left and right actions of \( Q \) on \( A \) are unitary;

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