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Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

A linear topological characterization of inner-product spaces

by

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A result due to J. S. Cohen in [1] suggests the following

THEOREM. *Let E be a Banach space; then the following conditions are equivalent:*

(i) *E is isomorphic (= linearly homeomorphic) to an inner product space,*

(ii) *if $u \in \Pi_2(E, l_2)$, then $u^* \in \Pi_2(l_2, E^*)$.*

Notation. $B(X, Y)$ denotes the space of bounded linear operators from a Banach space X into a Banach space Y. The class $\Pi_2(X, Y)$ of absolutely 2-summing operators from X into Y is defined by

$$\Pi_2(X, Y) = \Big\{u \in B(X, Y)\colon \sum \|ux_i\|^2 < C(u)$$
$$\text{for } x_i \in X \ (i = 1, 2, \ldots) \text{ with } \Big(\sum_i |x^*(x_i)|^2\Big)^{1/2} \leqslant \|x^*\| \text{ for } x^* \in X^*\Big\}.$$

X^* denotes the dual of X and u^* denotes the adjoint operator of u. By $l_\infty(A)$ we denote the space of bounded scalar-valued functions on a set A. We admit

$$\|f\| = \sup_{a \in A} |f(a)|$$

for f in $l_\infty(A)$. Finally, $u \in B(X, Y)$ is called *hilbertian* if there are a Hilbert space H and operators $v \in B(X, H)$, $w \in B(H, Y)$ such that $u = wv$.

Proof of the Theorem. (i) $\Rightarrow$ (ii). This follows from the fact that if E is (isomorphic to) a Hilbert space; then the class $\Pi_2(E, l_2)$ coincides with the class of Hilbert-Schmidt operators (cf. [3], Theorem 6.3)

(ii) $\Rightarrow$ (i). Let $u \in B(E, l_\infty(A))$ be an isometrically isomorphic embedding (Take A the unit ball of E^* and put $ue(e^*) = e^*(e)$ for $e \in E$ and $e^* \in A$). Since $B(l_\infty(A), l_2) = \Pi_2(l_\infty(A), l_2)$ (cf. [2] and [3], Theorem 4.3), we infer that $vu \in \Pi_2(E, l_2)$ for each $v \in B(l_\infty(A), l_2)$. Thus (ii) implies that

(1) $$u^* v^* \in \Pi_2(l_2, E^*) \quad \text{for every } v \in B(l_\infty(A), l_2).$$

Now pick for $i = 1, 2, \ldots, x_i^* \in (l_\infty(A))^*$ so that $\sum |x^{**}(x_2^*)|^2 < \|x^{**}\|^2$ for every x^{**} in the second dual $(l_\infty(A))^*$. Define $v \in B(l_\infty(A), l_2)$ by $vf = (x_i^*(f))$ for $f \in l_\infty(A)$ and denote by d_i^* the i-th coordinate functional in l_2. Clearly, $\sum |d_i^*(d)|^2 = \|d\|^2$ for every $d \in l_2 = (l_2)^{**}$. Thus (1) implies that

$$\sum \|u^* v^* d_i^*\|^2 < C(u^* v^*).$$

Hence $\Sigma \|u^* x_i^*\|^2 < C(u^* v^*)$, because $v^* d_i^* = x_i^*$ for $i = 1, 2, \ldots$ Therefore $u^* \in \Pi_2((l_\infty(A))^*, E^*)$. Hence, by the Pietsch Factorization Theorem (cf., [3], p. 285), u^* is a hilbertian operator. Thus, by [3], Proposition 5.1, u is hilbertian. Since u is an isometrically isomorphic embedding of E and u is hilbertian, E is isomorphic to an inner product space (because the Banach space $u(E)$ is the range of a bounded linear operator from a Hilbert space). This completes the proof.

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Colloquium on
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On a class of operators in Hilbert space*

by

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0. Introduction. One version of the spectral theorem for Hermitian and normal operators in Hilbert space is a consequence of the Gelfand representation of the uniform closure R_T of the algebra generated by the normal operator T, its adjoint and the identity. The essential fact is that R_T is commutative and isometrically $*$-isomorphic to the algebra $C(X)$ of all complex-valued continuous functions on a compact Hausdorff space ([4]-[6]).

On the other hand, If T is *any* operator on *any* Hilbert space H, then R_T is isometrically $*$-isomorphic to *some* algebra $C(X, A)$ of all continuous A-valued functions on X, where X is a compact Hausdorff space and A is a C^*-algebra. Indeed, we may use for A the algebra R_T and for X any one-point space $\{x\}$. Clearly, what is desirable for any extension of spectral theory is the choice of a canonical or minimal algebra A and of a usefully simple topological space X so that the isometric $*$-isomorphism $R_T \overset{*}{\cong} C(X, A)$ permits some analysis of T.

To pursue these ideas the author has discussed various aspects of a natural and fruitful generalization of the notion of commutative Banach algebra ([4]-[6]). Indeed, since a commutative Banach algebra A is one such that all its quotients by regular maximal ideals are isomorphic (to C, the field of complex numbers), the generalization in question is a so-called Q-uniform Banach algebra defined as follows:

An algebra A is a *Q-uniform algebra* if:

a. Q is a simple Banach algebra with identity;

b. A is a Q-bimodule such that for $a_1, a_2 \in A$, $q_1, q_2 \in Q$,

$$(q_1 a_1) q_2 = q_1(a_1 q_2), \quad (q_1 a_1) a_2 = q_1(a_1 a_2), \quad (a_1 a_2) q_1 = a_1(a_2 q_1),$$
$$(q_1 q_2) a_1 = q_1(q_2 a_1), \quad a_1(q_1 q_2) = (a_1 q_1) q_2, \quad |a_1 q_1|, |q_1 a_1| \leqslant |a_1|\,|q_1|,$$

and where the left and right actions of Q on A are unitary;

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