Fortunately, in the n-dimensional case, the theorem of F. and M. Riesz could be replaced by the following result [12]:

THEOREM 6. *Let μ be an A-measure on $\mathscr{B}_n$. Then every measure ν which is absolutely continuous with respect to μ, is an A-measure.*

In paper [13], Theorem 6 is generalized to the case of an arbitrary strictly pseudo-convex domain in $\mathscr{C}^n$. This generalization has been used to obtain an essential strengthening of Theorem 5 (see [13], Theorem 1.6).

Concluding this paper the author would like to say, that his interest in the topics discussed above is due to the fact that they combine general ideas and metods of the theory of Banach spaces with interesting and quite difficult analytical problems concerning concrete functional spaces.

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Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

A characterization of inner product spaces using absolutely 2-summing operators

by

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Introduction. Pietsch [4] has recently introduced the notion of an absolutely p-summing operator between normed linear spaces. A linear operator T mapping a normed space E into a normed space F is absolutely *p-summing* if there exists a constant $C \geqslant 0$, such that for all finite sets $x_1, \ldots, x_n$ in E, the inequality

$$\Big(\sum_{i=1}^{n} \|Tx_i\|^p\Big)^{1/p} \leqslant C \sup_{\|x'\|\leqslant 1} \Big(\sum_{i=1}^{n} |\langle x_i, x'\rangle|^p\Big)^{1/p}$$

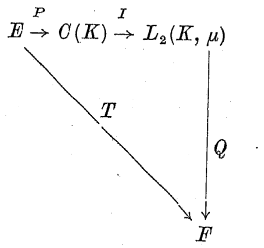
is satisfied. The smallest constant C such that the above inequality is satisfied is called the *absolutely p-summing norm* of T and is denoted by $\Pi_p(T)$. The normed space of absolutely p-summing operators from E into F is denoted by $\Pi_p(E, F)$.

The absolutely p-summing operators are not closed under conjugation. For example, Pietsch ([4], p. 338) has shown that the identity operator I from l_1 into l_2 is absolutely 2-summing, but the conjugate operator I' mapping l_2 into l_∞ is not absolutely 2-summing. In this note we discuss a relationship between the structure of the domain space E and the conjugation of absolutely 2-summing operators (Theorem 1.1). In Section 2, we present a reformulation of this result using the tensor norms introduced by P. Saphar ([6], p. 125).

1. Characterization of inner product spaces. In this section we present a characterization of inner product spaces using the absolutely 2-summing operators and their conjugates. A normed linear space E is an *inner product space* if there is an inner product defined in E such that $\|x\|^2 = (x, x)$.

THEOREM 1.1. *Let E be a normed linear space. Then, E is an inner product space if and only if for all Banach spaces F and for all absolutely 2-summing operators T mapping E into F, the conjugate operator T' is absolutely 2-summing and $\Pi_2(T') \leqslant \Pi_2(T)$.*

Proof. Let E be an inner product space, F a Banach space and T an absolutely 2-summing operator mapping E into F. Since T is absolutely 2-summing there exists a compact Hausdorff space K and a positive Radon measure μ on K, with $\|\mu\| = 1$, such that T can be factored



([4], p. 345). In the above diagram $\|P\| = 1$, $\|Q\| = \Pi_2(T)$, and the formal identity map $C(K) \xrightarrow{I} L_2(K, \mu)$ is absolutely 2-summing with $\Pi_2(I) = 1$ ([4], Satz 6 and Satz 15). Let $\tilde{E}$ be the normed space completion of E and let $\tilde{P}$ be the canonical extension of P to $\tilde{E}$. Since I is absolutely 2-summing, the operator $I\tilde{P}$ is an absolutely 2-summing operator between the Hilbert Spaces $\tilde{E}$ and $L_2(K, \mu)$ ([4], Satz 4). However, for Hilbert Spaces, the class of absolutely 2-summing operators coincides with the class of Hilbert-Schmidt operators ([4], p. 339). Therefore, $I\tilde{P}$ is a Hilbert-Schmidt operator and the Hilbert-Schmidt norm $\sigma(I\tilde{P})$ is given by

$$\sigma(I\tilde{P}) = \Pi_2(I\tilde{P}).$$

Furthermore, since the Hilbert-Schmidt operators are closed under conjugation ([2], p. 37), it follows that $(I\tilde{P})'$ is absolutely 2-summing with

$$\Pi_2\big((I\tilde{P})'\big) = \sigma\big((I\tilde{P})'\big) = \sigma(I\tilde{P}) = \Pi_2(I\tilde{P}).$$

Since $T' = (I\tilde{P})'Q'$, it follows that T' is absolutely 2-summing and

$$\begin{aligned}\Pi_2(T') &= \Pi_2\big((I\tilde{P})'Q'\big) \leqslant \Pi_2\big((I\tilde{P})'\big)\|Q'\| \\ &= \Pi_2(I\tilde{P})\|Q\| \leqslant \Pi_2(I)\|\tilde{P}\|\|Q\| \\ &= \|Q\| = \Pi_2(T).\end{aligned}$$

To prove the converse we shall show E' is a Hilbert space by showing the norm in E' satisfies the parallelogram law ([1], p. 115). Let l_2^2 be

a 2-dimensional inner product space and let $\{e_1, e_2\}$ be an orthonormal basis for l_2^2. Consider the operator T mapping E into l_2^2 defined by

$$T(x) = x_1'(x)e_1 + x_2'(x)e_2.$$

Since T has finite-dimensional range, it is absolutely 2-summing and

$$\begin{aligned}\Pi_2(T) &\leqslant (\|x_1'\|^2 + \|x_2'\|^2)^{1/2} \sup_{\|a\| \leqslant 1} \big(|(a\cdot e_1)|^2 + |(a\cdot e_2)|^2\big)^{1/2} \\ &\leqslant (\|x_1'\|^2 + \|x_2'\|^2)^{1/2}\end{aligned} \tag{1.1}$$

([5], p. 242). Furthermore, by our assumption, T' is absolutely 2-summing and $\Pi_2(T') \leqslant \Pi_2(T)$. Therefore

$$\begin{aligned}(\|T'e_1\|^2 + \|T'e_2\|^2)^{1/2} &\leqslant \Pi_2(T') \sup_{\|a\| \leqslant 1} \big(|(a\cdot e_1)|^2 + |(a\cdot e_2)|^2\big)^{1/2} \\ &\leqslant \Pi_2(T') \leqslant \Pi_2(T).\end{aligned}$$

However, since $T'e_i = x_i'$, $i = 1, 2$, we have

$$(\|x_1'\|^2 + \|x_2'\|^2)^{1/2} \leqslant \Pi_2(T). \tag{1.2}$$

Combining equations (1.1) and (1.2) we obtain

$$\Pi_2(T) = (\|x_1'\|^2 + \|x_2'\|^2)^{1/2}. \tag{1.3}$$

Now let

$$w_1' = \frac{x_1' + x_2'}{\sqrt{2}}, \quad w_2' = \frac{x_1' - x_2'}{\sqrt{2}},$$

$$f_1 = \frac{e_1 + e_2}{\sqrt{2}}, \quad f_2 = \frac{e_1 - e_2}{\sqrt{2}}.$$

Proceeding as in the first part of the proof we obtain

$$\begin{aligned}\Pi_2(T) &= (\|w_1'\|^2 + \|w_2'\|^2)^{1/2} \\ &= \left(\left(\frac{\|x_1' + x_2'\|^2}{2}\right) + \left(\frac{\|x_1' - x_2'\|^2}{2}\right)\right)^{1/2}.\end{aligned} \tag{1.4}$$

Combining equations (1.3) and (1.4) we obtain

$$2\|x_1'\|^2 + 2\|x_2'\|^2 = \|x_1' + x_2'\|^2 + \|x_1' - x_2'\|^2,$$

which proves E' is a Hilbert space. It follows immediately that E is an inner product space.

2. Tensor Product Formulation. Theorem (1.1) can be easily reformulated in the language of tensor products. In [6], P. Saphar has introduced the norms g_2 and d_2 in the tensor product $E\otimes F$. For u in $E\otimes F$ we have

$$g_2(u) = \inf\Big\{\Big(\sum_{i=1}^{n}\|x_i\|^2\Big)^{1/2}\sup_{\|y'\|\leqslant 1}\Big(\sum_{i=1}^{n}|\langle y_i, y'\rangle|^2\Big)^{1/2}\Big\},$$

$$d_2(u) = \inf\Big\{\sup_{\|x'\|\leqslant 1}\Big(\sum_{i=1}^{n}|\langle x_i, x'\rangle|^2\Big)^{1/2}\Big(\sum_{i=1}^{n}\|y_i\|^2\Big)^{1/2}\Big\}$$

([6], p. 125). In each case the infimum is taken over all representations of

$$u = \sum_{i=1}^{n} x_i\otimes y_i \quad \text{in } E\otimes F.$$

The tensor product with the norm $g_2(u)$ (resp. $d_2(u)$) is denoted by $E\otimes_{g_2}F$ (resp. $E\otimes_{d_2}F$).

Recall that a linear operator $T\colon E\to F'$ can be considered as a linear form on $E\otimes F$ according to the formula

$$T\Big(\sum_{i=1}^{n} x_i\otimes y_i\Big) = \sum_{i=1}^{n}\langle Tx_i, y_i\rangle.$$

Similarly, an operator $T\colon F''\to E'$ defines a linear form on $E\otimes F$ according to the formula

$$T\Big(\sum_{i=1}^{n} x_i\otimes y_i\Big) = \sum_{i=1}^{n}\langle x_i, Ty_i\rangle. \tag{2.1}$$

We shall use the following proposition ([6], Theorem 2) to reformulate Theorem (1.1) in the framework of tensor products:

Proposition 2.1. *The space $(E\otimes_{d_2}F)'$ (resp. $(E\otimes_{g_2}F)'$) can be identified with $\Pi_2(E, F')$ (resp. $\Pi_2(F, E')$).*

In a similar way one can prove

Proposition 2.2. *Let $A_2(F'', E')$ denote the subspace of $\Pi_2(F'', E')$ consisting of those operators which are conjugates of operators which map E into F'. Then, the canonical injection of $A_2(F'', E')$ into $(E\otimes_{g_2}F)'$ defined by equation (2.1) has norm equal to 1.*

Theorem 2.4. *Let E be a normed linear space. The following statements are equivalent:*

(1) *E is an inner product space.*

(2) *For all Banach spaces F, the canonical injection of $E\otimes_{g_2}F$ into $E\otimes_{d_2}F$ has norm $\leqslant 1$.*

(3) *The canonical injection of $E\otimes_{g_2}l_2^2$ into $E\otimes_{d_2}l_2^2$ has norm $\leqslant 1$.*

Proof (1) ⇒ (2). It is sufficient to show the canonical injection of $(E\otimes_{d_2}F)'$ into $(E\otimes_{g_2}F)'$ has norm $\leqslant 1$. If T belongs to $(E\otimes_{d_2}F)'$, then by Proposition 2.1, T is an absolutely 2-summing operator from E into F' with $\Pi_2(T) = \|T\|_{d_2'}$. (The symbol $\|T\|_{d_2'}$ (resp. $\|T\|_{g_2'}$) denotes the norm of T in $(E\otimes_{d_2}F)'$ (resp. $(E\otimes_{d_2}F)'$).) Furthermore, since E is an inner product space, it follows from Theorem 1.1 that the conjugate operator $T'\colon F''\to E'$ is absolutely 2-summing with $\Pi_2(T')\leqslant\Pi_2(T)$.

Using Proposition 2.2 and the fact that T and T' are identified with the same linear from on $E\otimes F$, it follows that T belongs to $(E\otimes_{g_2}F)'$ and

$$\|T\|_{g_2'}\leqslant\Pi_2(T')\leqslant\Pi_2(T)\leqslant\|T\|_{d_2'}.$$

This inequality shows the canonical injection of $(E\otimes_{d_2}F)'$ into $(E\otimes_{g_2}F)'$ has norm $\leqslant 1$ which proves the result.

(2) ⇒ (3). This assertion is obvious.

(3) ⇒ (1). We shall prove every absolutely 2-summing operator from E into l_2^2 has a conjugate which is absolutely 2-summing with $\Pi_2(T')\leqslant\Pi_2(T)$. The result will follow from the proof of Theorem 1.1. If T belongs to $\Pi_2(E, l_2^2)$, then, by Proposition 2.1, T belongs to $(E\otimes_{d_2}l_2^2)'$ with $\Pi_2(T) = \|T\|_{d_2'}$. Furthermore, since canonical injection

$$E\otimes_{g_2}l_2^2\to E\otimes_{d_2}l_2^2$$

has norm $\leqslant 1$, it follows that the canonical injection

$$(E\otimes_{g_2}l_2^2)'\to(E\otimes_{d_2}l_2^2)'$$

also has norm $\leqslant 1$. Using the above remarks and Proposition 2.1, it follows that T' is absolutely 2-summing and $\Pi_2(T')\leqslant\Pi_2(T)$. The result will now follow from the proof of Theorem 1.1.

3. Normed linear spaces equivalent to inner product spaces. At the recent conference on nuclear spaces held in Warsaw, Poland, the author conjectured there should be a characterization of normed linear spaces equivalent to inner product spaces which is similar to Theorem (1.1). The following result has been proved by S. Kwapień [3]:

Theorem 3.1. *Let E be a normed linear space. Then E is equivalent to an inner product space if and only if for all Banach spaces F and for all absolutely 2-summing operators T mapping E into F, the conjugate operator T' is also absolutely 2-summing.*

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A linear topological characterization of inner-product spaces

by

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A result due to J. S. Cohen in [1] suggests the following

THEOREM. *Let E be a Banach space; then the following conditions are equivalent:*

(i) *E is isomorphic (= linearly homeomorphic) to an inner product space,*

(ii) *if $u \in \Pi_2(E, l_2)$, then $u^* \in \Pi_2(l_2, E^*)$.*

Notation. $B(X, Y)$ denotes the space of bounded linear operators from a Banach space X into a Banach space Y. The class $\Pi_2(X, Y)$ of absolutely 2-summing operators from X into Y is defined by

$$\Pi_2(X, Y) = \{u \in B(X, Y): \sum \|ux_i\|^2 < C(u)$$
$$\text{for } x_i \in X\ (i = 1, 2, \ldots) \text{ with } \Big(\sum_i |x^*(x_i)|^2\Big)^{1/2} \leqslant \|x^*\| \text{ for } x^* \in X^*\}.$$

X^* denotes the dual of X and u^* denotes the adjoint operator of u. By $l_\infty(A)$ we denote the space of bounded scalar-valued functions on a set A. We admit

$$\|f\| = \sup_{a \in A} |f(a)|$$

for f in $l_\infty(A)$. Finally, $u \in B(X, Y)$ is called *hilbertian* if there are a Hilbert space H and operators $v \in B(X, H)$, $w \in B(H, Y)$ such that $u = wv$.

Proof of the Theorem. (i) $\Rightarrow$ (ii). This follows from the fact that if E is (isomorphic to) a Hilbert space; then the class $\Pi_2(E, l_2)$ coincides with the class of Hilbert-Schmidt operators (cf. [3], Theorem 6.3)

(ii) $\Rightarrow$ (i). Let $u \in B(E, l_\infty(A))$ be an isometrically isomorphic embedding (Take A the unit ball of E^* and put $ue(e^*) = e^*(e)$ for $e \in E$ and $e^* \in A$). Since $B(l_\infty(A), l_2) = \Pi_2(l_\infty(A), l_2)$ (cf. [2] and [3], Theorem 4.3), we