[3] A. Grothendieck, *Espaces vectoriels topologiques*, Sao Paulo 1954.

[4] N. J. Kalton, *Schauder decompositions and completeness*, Bulletin of London Math. Soc. 2 (1970), p. 34-36.

[5] — *Schauder decompositions of locally convex spaces*, to be published in Proc. Cambridge Phil. Soc.

[6] — *Normalisation properties of Schauder bases*, to be published in Proc. London Math. Soc.

[7] A. Pełczyński and I. Singer, *On non-equivalent and conditional bases in Banach spaces*, Stud. Math. 25 (1964), p. 5-25.

[8] A. Pełczyński and W. Szlenk, *An example of a non-shrinking basis*, Rev. Roumaine Math. Pures et Appl. 10 (1965), p. 961-966.

[9] J. R. Retherford, *Bases, basic sequences and reflexivity of linear topological spaces*, Math. Ann. 248 (1966), p. 280-285.

[10] — *A semi-shrinking basis which is not shrinking*, Proc. Amer. Math. Soc. 19 (1968), p. 766.

[11] I. Singer, *Basic sequences and reflexivity of Banach spaces*, Stud. Math. 21 (1962), p. 351-369.

[12] M. Zippin, *A remark on bases and reflexivity in Banach spaces*, Israel Jour. Math. 6 (1968), p. 74-80.

[13] T. A. Cook, *Schauder decompositions and semi-reflexive spaces*, Math. Ann. 182 (1969), p. 232-235.

STUDIA MATHEMATICA, T. XXXVIII. (1970)

Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

On non-isomorphism of Banach spaces of holomorphic functions

by

G. M. HENKIN (Moscow)

1. The results discussed in this paper are based on two classical theorems of functional analysis. The first of these theorems says that every separable Banach space X is linearly isometric to a subspace of $C = C([0,1])$, the space of continuous functions on the segment $[0,1]$ (see Banach [1], p. 163). The second theorem, stated in a form which is convenient for our purposes, is the following:

Suppose that the space C is a closed linear subspace of a Banach space B. Then there exists a continuous linear operator of extending continuous linear functionals on C to the functionals on B (see Nachbin [4], Lindenstrauss [3], 86-89, and also [8], p. 49).

These results have given to A. A. Miliutin and A. Pełczyński (separately) the idea of trying to establish non-isomorphism of given Banach spaces X and Y by comparing the positions in which these spaces can be embedded into the universal space C. The realisation of this idea has supplied the proof of non-isomorphism of certain Banach spaces which could not as yet be distinguished by any other method.

2. Let X be a Banach space and let $J\colon X\to C$ be an operator of an isomorphic embedding of X into C. Then the conjugate $J^*\colon C^*\to X^*$ is onto X^*. For every subspace M of X^*, let $\chi_J(M)$ denote the infimum of the norms of linear operators $S\colon M\to C^*$ having the property that $J^*S\colon M\to M$ is the identity on M.

In 1964 A. A. Miliutin comunicated to the author the following (unpublished) result:

THEOREM 1. *Let $I\colon X\to C$ and $J\colon X\to C$ be two isomorphic embeddings of the space X into C. Then, for every subspace $M\subset X^*$, we have*

$$(\|J\|\cdot\|I^{-1}\|)^{-1}\leqslant \chi_J(M)/\chi_I(M)\leqslant \|I\|\cdot\|J^{-1}\|,$$

where $\|J\|$ denotes the norm of the operator J, and $\|J^{-1}\|$ the norm of the inverse of J, which is defined on the image of X under the map J.

It turned out (see [11], p. 60) that Theorem 1 is an immediate consequence of the classical results mentioned in section 1.

In our paper [11] the number $\chi(M) = \chi_I(M)$ (where $I\colon X \to C$ is an isomorphically isometric embedding of X into C) was called the *Miliutin characteristic* of the subspace $M \subset X^*$. Theorem 1 just says that the property of a fixed subspace $M \subset X^*$ of possessing finite characteristic $\chi(M)$ is not only an isometrical invariant but also a linear-topological (isometrical) one.

Theorem 1 implies $\chi(C^*) = 1 < \infty$. Hence, if a Banach space X has the property $\chi(M^*) = \infty$, then X is not isomorphic to C.

3. The first proof of non-isomorphism of spaces based on the above argument was given by Pełczyński [6], see also [7], p. 25-28. Let $A(\mathscr{D})$ be the space of functions which are analytic on the open unit disk $\mathscr{D}$ in the complex plane and continuous on $\overline{\mathscr{D}}$, the closure of the disc, under the uniform norm

$$\|f\| = \sup_{z \in \mathscr{D}} |f(z)|.$$

THEOREM 2 (A. Pełczyński). *We have $\chi(A^*(D)) = \infty$, and therefore the space $A(\mathscr{D})$ is not isomorphic to C.*

Let us mention that Pełczyński's proof is based on the "analytic" result of Newman [5], stating that the Hardy space H_1 is non-complemented in the space $L_1(\partial\mathscr{D})$, of absolutely integrable functions on the circle $\partial\mathscr{D}$.

Let $C^{(p)}(I^n)$ denote the space of all p times continously differentiable functions on the cube I^n. In papers [8] and [9] we have proved

THEOREM 3. *If $n \geqslant 2$ and $p \geqslant 1$, then $\chi(C^{(p)}(I^n)^*) = \infty$. Hence the space $C^{(p)}(I^n)$, for $n \geqslant 2$, is not isomorphic to C.*

The proof of Theorem 3 is based on a result of [8] and [9] on non existence of linear projector of the space $C(K)$ onto the space $C^{(p)}(I^n)$ embedded in certain "natural" way into the space $C(K)$, of continuous functions on certain, specially constructed, compact space K. Let us note that a result close to Theorem 3 has been stated without proof by Grothendieck [2].

4. Theorems 2 and 3 give an illustration how certain Banach spaces can be distinguished from the space C. In many instances it is much more difficult to establish non isomorphism of Banach spaces X and Y such that $\chi(X^*) = \chi(Y^*) = \infty$ and non of them is isomorphic to C. Of this kind are, for instance, the spaces $A(\mathscr{D}^k)$, of continuous functions on closed polycylinders, of different dimensions, which are analytic in the interiors of polycylinders. The problem of distinction between spaces $A(\mathscr{D}^k)$ is, in general, unsettled. The following partial result (see [10] and [11]) is known:

THEOREM 4. *Let $k \geqslant 2$. Then the spaces $A(\mathscr{D})$ and $A(\mathscr{D}^k)$ are not isomorphic.*

This theorem is a consequence of the following facts concerning the structure of conjugate spaces:

THEOREM 4. A. *The space $A^*(\mathscr{D})$ can be divided into a direct sum of subspaces L and M such that $\chi(M) = 1$ and the space L is separable.*

THEOREM 4. B. *For $k \geqslant 2$, there is no decomposition of $A^*(\mathscr{D}^k)$ into a direct sum of subspaces L and M with the property that L is separable and $\chi(M) < \infty$.*

The main achievement of the paper [11] is the proof of Theorem 4. B, however the theorem itself is not explicitly formulated; it is formulated in the next paper [12]. An essential role in the proof of this theorem is played by the fact that the polycylinder $\mathscr{D}^k$ (for $k \geqslant 2$) admits uncountably many Gleason parts.

Proof of Theorem 4. A (Lemma 2.I in [11]) is based on a theorem of F. and M. Riesz which describes the measures on $\partial\mathscr{D}$ orthogonal to $A(\mathscr{D})$.

5. After Theorem 4 had been proved, the author has ascertained that a more general fact is true. Let $\mathscr{B}_n$ denote the ball in the complex n-space $\mathscr{C}_n$ given by the inequality $|z_1|^2 + \ldots + |z_n|^2 \leqslant 1$. Let $A(\mathscr{B}_n)$ denote the space of all functions which are analytic on $\mathscr{B}_n$ and continuous on $\overline{\mathscr{B}}_n$, with the sup-norm:

$$\|f(z)\| = \sup_{z \in \mathscr{B}_n} |f(z)|.$$

In paper [12] we have proved

THEOREM 5. *For arbitrary integers $k \geqslant 2$ and $n \geqslant 1$, the spaces $A(\mathscr{D}^k)$ and $A(\mathscr{B}_n)$ are not isomorphic, i.e. there is no linear homeomorphism between these spaces.*

The difficult point in the proof of Theorem 5 is that there is no description of measures on $\mathscr{B}_n$ orthogonal to $A(\mathscr{B}_n)$, which could be used in the proof of the $A(\mathscr{B}_n)$ version of Theorem 4. A.

Let μ be a measure on $\partial\mathscr{B}_n$, the boundary of the ball $\mathscr{B}_n$. We shall say that μ is an *A-measure*, if for every norm-bounded sequence $\{f_i(z)\}$ of functions from $A(\mathscr{B}_n)$ which converge to zero together with all their derivatives in certain point $z_0 \in \mathscr{B}_n$, the following holds:

$$\lim_i \int_{\partial\mathscr{B}_n} f_i(\zeta)\,\mu(d\zeta) = 0.$$

Fortunately, in the n-dimensional case, the theorem of F. and M. Riesz could be replaced by the following result [12]:

THEOREM 6. *Let μ be an A-measure on $\mathscr{B}_n$. Then every measure ν which is absolutely continuous with respect to μ, is an A-measure.*

In paper [13], Theorem 6 is generalized to the case of an arbitrary strictly pseudo-convex domain in $\mathscr{C}^n$. This generalization has been used to obtain an essential strengthening of Theorem 5 (see [13], Theorem 1.6).

Concluding this paper the author would like to say, that his interest in the topics discussed above is due to the fact that they combine general ideas and metods of the theory of Banach spaces with interesting and quite difficult analytical problems concerning concrete functional spaces.

I want to express my gratitude to Professor B. S. Mitjagin, who kindly presented, on my behalf, this paper during the Colloquium in Warsaw.

References

[1] S. Banach, *Théorie des opérations linéaires*, Warszawa 1932.
[2] A. Grothendieck, *Erratum au mémoire „Produits tensoriels topologiques et espaces nucléaires"*, Ann. Inst. Fourier 6 117 (1956).
[3] J. Lindenstrauss, *Extension of compact operators*, Memoirs Amer. Math. Soc. 48 (1964).
[4] L. Nachbin, *A theorem of the Hahn-Banach type*, Trans. Amer. Math. Soc. 68 (1950), p. 28-46.
[5] D. J. Newman, *The non-existence of projections from L^1 to H^1*, Proc. Am. Math. Soc. 12 (1961).
[6] A. Pełczyński, *On simultaneous extension of continuous functions*, Studia Math. 24 (1964), p. 285-304.
[7] H. P. Rosenthal, *Projections onto translation-invariant subspaces of $L^p(G)$*, Memoirs Am. Math. Soc. 63 (1966).
[8] Г. М. Хенкин, *Доказательство неизоморфизма пространств гладких функций на отрезке и на квадрате*, ДАН СССР 172, № I (1967), p. 48-51.
[9] — *Отсутствие равномерного гомеоморфизма между пространствами гладких функций от одного и от n переменных* ($n \geqslant 2$), Mat. Sbornik 74, № 4, (1967), p. 595–607.
[10] — *Неизоморфность банаховых пространств аналитических функций от одного и от двух переменных*, ДАН СССР 175 (1967).
[11] — *Неизоморфность некоторых пространств функций от разного числа переменных*, Функц. анализ и его прилож., I, вып. 4 (1967), p. 57–68.
[12] — *Банаховы пространства аналитических функций в шаре и в бицилиндре неизоморфны*, ibidem, 2, вып. 4 (1968), p. 82–91.
[13] — *Интегральное представление функций, голоморфных в строго псевдовыпуклых областях и некоторые приложения*, Мат. сборник 78, № 4 (1969), p. 611–632.

icm©

STUDIA MATHEMATICA, T. XXXVIII. (1970)
Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

A characterization of inner product spaces using absolutely 2-summing operators

by

JOEL S. COHEN (Denver)

Introduction. Pietsch [4] has recently introduced the notion of an absolutely p-summing operator between normed linear spaces. A linear operator T mapping a normed space E into a normed space F is absolutely *p-summing* if there exists a constant $C \geqslant 0$, such that for all finite sets $x_1, \ldots, x_n$ in E, the inequality

$$\Big(\sum_{i=1}^{n} \|Tx_i\|^p\Big)^{1/p} \leqslant C \sup_{\|x'\|\leqslant 1} \Big(\sum_{i=1}^{n} |\langle x_i, x'\rangle|^p\Big)^{1/p}$$

is satisfied. The smallest constant C such that the above inequality is satisfied is called the *absolutely p-summing norm* of T and is denoted by $\Pi_p(T)$. The normed space of absolutely p-summing operators from E into F is denoted by $\Pi_p(E, F)$.

The absolutely p-summing operators are not closed under conjugation. For example, Pietsch ([4], p. 338) has shown that the identity operator I from l_1 into l_2 is absolutely 2-summing, but the conjugate operator I' mapping l_2 into l_∞ is not absolutely 2-summing. In this note we discuss a relationship between the structure of the domain space E and the conjugation of absolutely 2-summing operators (Theorem 1.1). In Section 2, we present a reformulation of this result using the tensor norms introduced by P. Saphar ([6], p. 125).

1. Characterization of inner product spaces. In this section we present a characterization of inner product spaces using the absolutely 2-summing operators and their conjugates. A normed linear space E is an *inner product space* if there is an inner product defined in E such that $\|x\|^2 = (x,x)$.

THEOREM 1.1. *Let E be a normed linear space. Then, E is an inner product space if and only if for all Banach spaces F and for all absolutely 2-summing operators T mapping E into F, the conjugate operator T' is absolutely 2-summing and $\Pi_2(T') \leqslant \Pi_2(T)$.*