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Unconditional and normalised bases

by

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1. Introduction. A Schauder basis (x_n) of a locally convex space E is unconditional if, whenever $\sum_{i=1}^{\infty} \alpha_i x_i$ converges, the convergence is unconditional. In [16], Pełczyński and Singer proved that every Banach space with a basis possesses a conditional (i.e. not unconditional) basis. In this paper I shall generalise this theorem using the concept of normalisation introduced in [12].

A sequence (x_n) is *regular* if there is a neighbourhood V of zero with $x_n \notin V$ for all n ; a regular bounded sequence is said to be *normalised*. If there exists a scalar sequence (α_n) with $(\alpha_n x_n)$ normalised, then (x_n) is said to be *normal*; otherwise (x_n) is abnormal.

If (x_n) is a Schauder basis of E , then (f_n) will always denote its dual sequence in E' ; if $(f_n)_{n=1}^{\infty}$ is equicontinuous, then (x_n) is *equi-regular*, and hence regular; if E is barrelled, then any regular basis is equi-regular.

The sequence space of all a such that $\sum_{i=1}^{\infty} \alpha_i x_i$ converges will be denoted by λ_x , and μ_x is the sequence space $\{(f(x_n))_{n=1}^{\infty}; f \in E'\}$. If E is sequentially complete, then (x_n) is unconditional if and only if λ_x is solid (see [4]), that is if $\alpha \in \lambda_x$ and $|\theta_n| \leq 1$ for all n , then $(\theta_n \alpha_n) \in \lambda_x$. If E is also barrelled, it can be shown that the topology on E may be given by a collection of solid semi-norms p such that

$$p(x) = \sup_{|\theta_i| < 1} p\left(\sum_{i=1}^{\infty} \theta_i f_i(x) x_i\right).$$

A sequentially complete barrelled space with a Schauder basis is complete (see [10]); in this paper I shall restrict attention almost exclusively to complete barrelled spaces.

2. Reflexivity and unconditional bases. A Schauder basis (x_n) is γ -complete or boundedly-complete if whenever $(\sum_{i=1}^n \alpha_i x_i; n = 1, 2, \dots)$ is

bounded, then $\sum_{i=1}^{\infty} \alpha_i x_i$ converges; it is shrinking if (f_n) is a basis for E' in its strong topology. It is shown in [20] that E is semi-reflexive if and only if (x_n) is γ -complete and shrinking. The following results generalise those of [9] and [19]:

THEOREM 2.1. *Let E be a complete barrelled space with an unconditional Schauder basis (x_n) ; if (x_n) is not γ -complete, then E contains a complemented subspace G isomorphic to c_0 .*

There exists a sequence (α_n) such that $(\sum_{i=1}^n \alpha_i x_i)_{n=1}^{\infty}$ is bounded, but does not converge; thus there exists an increasing sequence (n_j) with $n_0 = 0$, and a neighbourhood V of zero such that, if

$$y_j = \sum_{n_{j-1}+1}^{n_j} \alpha_n x_n,$$

then $y_j \notin V$. Let p be a solid semi-norm, then

$$p\left(\sum_{i=1}^n \beta_i y_i\right) \leq \|\beta\|_{\infty} p\left(\sum_{i=1}^n y_i\right),$$

where $\|\beta\|_{\infty} = \sup_n |\beta_n|$. However, $(\sum_{i=1}^n y_i; n = 1, 2, \dots)$ is bounded, and thus

$$p\left(\sum_{i=1}^n \beta_i y_i\right) \leq K \|\beta\|_{\infty}.$$

Therefore $c_0 \subset \lambda_y$; but as (y_n) is regular $\lambda_y \subset c_0$, so that $\lambda_y = c_0$.

Let q be a solid continuous semi-norm on E such that $q(y_j) \geq 1$ for all j ; if $E_j = \text{lin}(x_{n_{j-1}+1}, \dots, x_{n_j})$ then there exists a linear functional h_j on E_j such that $h_j(y_j) = 1$, and $|h_j(x)| \leq q(x)$ for $x \in E_j$. Define $g_j \in E'$ by

$$g_j(x) = h_j\left(\sum_{n_{j-1}+1}^{n_j} f_i(x) x_i\right);$$

then $|g_j(x)| \leq q(x)$. Then $(g_j)_{j=1}^{\infty}$ is equicontinuous, and so possesses a $\sigma(E', E)$ cluster point g ; obviously $g(x_j) = 0$ for all j , and so $g = 0$. As zero is the sole cluster point of (g_j) it follows that $\lim_{j \rightarrow \infty} g_j = 0$ weakly.

Let

$$T_k x = \sum_{i=1}^k g_i(x) y_i;$$

then each T_k is continuous and $\lim_{k \rightarrow \infty} T_k x = T x$ exists for each x . Therefore by the Banach-Steinhaus Theorem for barrelled spaces, T is a continuous

projection of E onto $G = \overline{\text{lin}}(y_j)$. As G is complemented in E , G is barrelled; and in (y_j) is a Schauder basis of G with $\lambda_y = c_0$, it follows that $\mu_y = c_0^{\beta} = l_1$, and so $G \cong c_0$.

THEOREM 2.2. *Let E be a complete barrelled space with an unconditional Schauder basis (x_n) ; if (x_n) is not shrinking, then E has a complemented subspace $G \cong l^1$.*

As (x_n) is not shrinking, there exists $f \in E'$, and a bounded block basic sequence (y_j) such that $f(y_j) = 1$ (see [12], Theorem 5.4). If $\sum_{j=1}^{\infty} \alpha_j y_j$ converges, then $\sum_{j=1}^{\infty} |\alpha_j|$ converges, and as (y_j) is bounded $\lambda_y = l^1$. Let $G = \overline{\text{lin}}(y_n)$, and define the norm p on G by

$$p\left(\sum_{i=1}^{\infty} \alpha_i y_i\right) = \sum_{i=1}^{\infty} |\alpha_i|,$$

then

$$p\left(\sum_{i=1}^{\infty} \alpha_i y_i\right) = \sum_{i=1}^{\infty} |\alpha_i f(y_i)| \leq \sum_{i=1}^{\infty} |f_i(y) f(x_i)|,$$

where $y = \sum_{i=1}^{\infty} \alpha_i y_i$. As the topology on E may be given by solid semi-norms if

$$q(x) = \sum_{i=1}^{\infty} |f_i(x) f(x_i)|,$$

then q is continuous; thus p is continuous, and $G \cong l^1$.

For $x \in E$, let

$$T x = \sum_{j=1}^{\infty} f\left(\sum_{n_{j-1}+1}^{n_j} f_i(x) x_i\right) y_j, \quad \text{where } y_j = \sum_{n_{j-1}+1}^{n_j} \beta_i x_i;$$

for

$$\sum_{i=1}^{\infty} \left| f\left(\sum_{n_{j-1}+1}^{n_j} f_i(x) x_i\right) \right| \leq q(x).$$

Then T is a projection of E onto G , and $p(Tx) \leq q(x)$ so that T is continuous.

As E is semi-reflexive if and only if (x_n) is shrinking and γ -complete, the following theorem is immediate:

THEOREM 2.3. *If E is a complete barrelled space with an unconditional Schauder basis, then E is reflexive if and only if E possesses no complemented subspace isomorphic to c_0 or l^1 .*

3. Symmetric bases. Two basic sequences (x_n) and (y_n) are said to be equivalent if $\lambda_x = \lambda_y$; it can easily be seen that if (x_n) is a Schauder

basis of E , and (y_n) is a Schauder basis of F , and both E and F are barrelled, then E and F are isomorphic. Suppose (x_n) is a Schauder basis of E , such that for every permutation σ of the positive integers Z , $(x_{\sigma(n)})$ is a Schauder basis of E equivalent to (x_n) ; then (x_n) is said to be *symmetric*. Symmetric bases of Banach spaces were introduced and studied by Singer [17] and [18]; in locally convex spaces they have been studied by Garling [6] and [7]. The definition here corresponds to condition SB_3 of [7] and [18].

A symmetric basis is necessarily unconditional, as for all $x \in E$, and all permutations σ , $\sum_{i=1}^{\infty} f_{\sigma(i)}(x) x_{\sigma(i)}$ converges to x . The following lemma is essentially established in [3]:

LEMMA 3.1. *If (x_n) is a symmetric Schauder basis of E , then either (x_n) is bounded, or (x_n) is a Hamel basis of E .*

THEOREM 3.2. *If E is a complete barrelled space with a symmetric Schauder basis (x_n) , then either $E \cong \omega$, or $E \cong \varphi$ or (x_n) is normalised.*

(ω is the space of all sequences, and φ is the dual sequence space of all sequences eventually equal to zero; ω has the topology $\beta(\omega, \varphi)$ ($= \sigma(\omega, \varphi)$) and φ has the topology $\beta(\varphi, \omega)$).

Let σ be a permutation of Z , and let $\tau = \sigma^{-1}$; suppose $\sum_{i=1}^{\infty} \alpha_i f_i$ converges weakly to f . Let

$$g_n = \sum_{i=1}^{\infty} \alpha_{\tau(i)} f_i;$$

then

$$g_n(x) = \sum_{i=1}^n f(x_{\tau(i)}) f_i(x) = f\left(\sum_{i=1}^n f_i(x) x_{\tau(i)}\right);$$

as (x_n) is symmetric, $\lim_{n \rightarrow \infty} g_n(x) = g(x)$ exists, and by the Banach-Steinhaus Theorem, g is continuous. Obviously $g(x_i) = \alpha_{\tau(i)}$, and so

$$g = \sum_{i=1}^{\infty} \alpha_{\tau(i)} f_i;$$

as (f_n) is an unconditional basis of $(E', \sigma(E', E))$,

$$g = \sum_{i=1}^{\infty} \alpha_i f_{\sigma(i)}$$

and so (f_n) is a symmetric basis of E' in its weak topology.

By Lemma 3.1, either (f_n) is a Hamel basis of E' or is bounded; in the latter case (x_n) is equi-regular. If E' has countable dimension, then $E \cong \varphi$; if E has countable dimension $E \cong \omega$; otherwise (x_n) is regular and bounded, i.e. normalised.

Eliminating the cases of ω and φ , symmetric bases of complete barrelled spaces may be treated much like symmetric bases of Banach spaces.

The following theorem follows from the results of Cac [3]:

THEOREM 3.3. *If E is a complete barrelled space with a normalised symmetric Schauder basis (x_n) , then the topology on E may be given by a collection of symmetric norms p , satisfying*

$$p(x) = \sup_{\{\theta_i \leq 1\}} \sup_{\tau \in \Pi} p\left(\sum_{i=1}^{\infty} \theta_i f_i(x) x_{\tau(i)}\right),$$

where π is the group of all permutations of Z .

Cac's result is essentially that for fixed x

$$\left\{ \sum_{i=1}^{\infty} \theta_i f_i(x) x_{\tau(i)}; \tau \in \Pi, |\theta_i| \leq 1 \right\}$$

is a bounded set. For each sequence (θ_i) with $|\theta_i| \leq 1$ and $\tau \in \Pi$, the map

$$x \rightarrow \sum_{i=1}^{\infty} \theta_i f_i(x) x_{\tau(i)}$$

is continuous by an application of the Banach-Steinhaus theorem, and so this collection of maps is equicontinuous (E is barrelled). The result then follows at once.

If (x_n) is a symmetric Schauder basis of E , where E is complete and barrelled, then a k -block is an element $u(K) = \sum_{i \in K} x_i$, where K is a subset of Z with k members; two blocks $u(K_1)$ and $u(K_2)$ are disjoint if $K_1 \cap K_2 = \emptyset$. Then the following theorems generalise results of Lindenstrauss and Zippin [15]:

THEOREM 3.4. *If $(u(K_n))$ is a sequence of disjoint k_n -blocks, the averaging projection*

$$Tx = \sum_{n=1}^{\infty} \frac{1}{k_n} \left\{ \sum_{i \in K_n} f_i(x) \right\} u(K_n)$$

is a well-defined continuous operator on E .

THEOREM 3.5. *If (k_n) is a sequence with $k_n > 1$ for all n , then E possesses an unconditional Schauder basis (y_n) with a subsequence (y_{n_j}) of disjoint k_j -blocks.*

The proofs of both these theorems are almost identical to the proofs of the original results for Banach spaces in [15]; all the calculations may be carried out with individual symmetric norms.

One further property of symmetric norms will be required. Let E be a 2^n -dimensional vector space and let $(x_i)_{i=1}^{2^n}$ be a basis of E ; then following Pełczyński and Singer [16], one may define the Haar system $(y_i)_{i=1}^{2^n}$ of (x_i) by

$$y_1 = \sum_{i=1}^{2^n} x_i, \quad y_{2^k+s} = \sum_{i=1}^{2^n} \beta_i(k, s) x_i,$$

where

$$\beta_i(k, s) = \begin{cases} 1 & \text{if } (2s-2)2^{n-k-1} + 1 \leq i \leq (2s-1)2^{n-k-1}, \\ -1 & \text{if } (2s-1)2^{n-k-1} + 1 \leq i \leq 2s \cdot 2^{n-k-1}, \\ 0 & \text{otherwise.} \end{cases}$$

The Rademacher system $(z_i)_{i=1}^{2^n}$ is given by

$$z_k = \sum_{s=1}^{2^{k-1}} y_{2^{k-1}+s}.$$

The following results are proved in [16]:

PROPOSITION 3.6. (i) $(y_i)_{i=1}^{2^n}$ is a basis of E , and for any norm p which is symmetric with respect to (x_i) , and any sequence $(\alpha_i)_{i=1}^{2^n}$

$$p \left(\sum_{i=1}^k \alpha_i y_i \right) \leq p \left(\sum_{i=1}^{2^n} \alpha_i y_i \right) \quad \text{for } k \leq 2^n.$$

(ii) $(z_i)_{i=1}^{2^n}$ is a block basic sequence with respect to $(y_i)_{i=1}^{2^n}$, and for any symmetric p , and sequence $(\alpha_i)_{i=1}^{2^n}$

$$p \left(\sum_{i=1}^n \alpha_i \frac{z_i}{p(z_i)} \right) \geq \frac{1}{8} \left(\sum_{i=1}^n |\alpha_i|^2 \right)^{1/2}.$$

4. The existence of conditional bases. For convenience, I shall define a J -space as a complete barrelled space with a normalised Schauder basis, and which has the property that any two normalised Schauder bases are equivalent. I do not know whether any J -space exists, although it has been shown in [16] that there is no Banach J -space. The techniques employed in this section stem largely from those of [16].

PROPOSITION 4.1. *If E is a J -space, then:*

- (i) any normalised Schauder basis of E is symmetric,
- (ii) E is reflexive,
- (iii) E' is a J -space in its strong topology.

(i) If (x_n) is a normalised Schauder basis of E , and (θ_n) is any sequence with $|\theta_n| = 1$, then $(\theta_n x_n)$ is a normalised Schauder basis equivalent to (x_n) ; thus $\alpha \in \lambda_x$ if and only if $(\theta_n \alpha_n) \in \lambda_x$, so that λ_x is solid. Thus (x_n)

is unconditional, and for any π , a permutation of Z , $(x_{\pi(n)})$ is a basis equivalent to (x_n) .

(ii) This follows from Theorem 2.3, as otherwise $E = c_0 \oplus F \cong c_0 \oplus c_0 \oplus F \cong c_0 \oplus E$ or $E \cong l^1 \oplus E$, so that by combining a conditional normalised basis of c_0 or l^1 (see [8] and [14]) with a normalised Schauder basis of E , one obtains a conditional normalised Schauder basis of E , contradicting (i).

(iii) Let (f_n) be a normalised Schauder basis of E' , with dual (x_n) in E ; then (x_n) is also normalised (see [12], Theorem 3.4). If (g_n) is any other normalised Schauder basis of E' with dual (y_n) then $\lambda_x = \lambda_y$, and so $\lambda_f = \mu_x = \lambda_x^p = \lambda_y^p = \mu_y = \lambda_g$ as E is barrelled.

If a J -space E exists, then certainly E' is not isomorphic to E , for if it were, $\lambda_x = \mu_x$ for any normalised basis (x_n) of E ; thus $\lambda_x = \lambda_x^p$ and so $\lambda_x = l^p$, and E is a separable Hilbert space, and this is not a J -space [1].

THEOREM 4.2. *Let E be a complete barrelled space with a normalised Schauder basis; then E is a J -space if and only if every normalised Schauder basis of E is unconditional.*

Certainly, by Proposition 4.1, every normalised basis of a J -space is unconditional. Conversely suppose (x_n) is a normalised Schauder basis of E , and let $E_1 = \overline{\text{lin}}(x_{2n-1})_{n=1}^\infty$ and $E_2 = \overline{\text{lin}}(x_{2n})_{n=1}^\infty$; as (x_n) is unconditional, by an application of the Banach-Steinhaus theorem $E = E_1 \oplus E_2$.

Let (y_n) be any normalised Schauder basis of E_1 ; then if $z_{2n-1} = y_n$ and $z_{2n} = x_{2n}$, (z_n) is a normalised Schauder basis of E , and is thus unconditional. Hence (y_n) is unconditional.

Let $u_{2n-1} = z_{2n-1}$, and $u_{2n} = z_{2n} + z_{2n-1}$; then (u_n) is a block perturbation of (z_n) (see Lemma 4.4 of [13]), and is also normalised, as (z_n) is equi-regular and bounded. Similarly, if $v_{2n} = z_{2n}$ and $v_{2n-1} = z_{2n-1} + z_{2n}$, then (v_n) is a normalised Schauder basis of E .

The maps $Q: E \rightarrow E$ and $R: E \rightarrow E$ given by

$$Q \left(\sum_{i=1}^\infty \alpha_i u_i \right) = \sum_{i=1}^\infty \alpha_{2i-1} u_{2i-1} \quad \text{and} \quad R \left(\sum_{i=1}^\infty \alpha_i v_i \right) = \sum_{i=1}^\infty \alpha_{2i} v_{2i}$$

are continuous (the Banach-Steinhaus theorem),

$$\begin{aligned} -Q \left(\sum_{i=1}^\infty \alpha_i v_{2i} \right) &= -Q \left(\sum_{i=1}^\infty \alpha_i z_{2i} \right) = \sum_{i=1}^\infty \alpha_i (u_{2i-1} - u_{2i}) \\ &= \sum_{i=1}^\infty \alpha_i u_{2i-1} = \sum_{i=1}^\infty \alpha_i y_i, \end{aligned}$$



while

$$-R\left(\sum_{i=1}^{\infty} \alpha_i y_i\right) = R\left(\sum_{i=1}^{\infty} \alpha_i (v_{2i} - v_{2i-1})\right) = \sum_{i=1}^{\infty} \alpha_i v_{2i} = \sum_{i=1}^{\infty} \alpha_i w_{2i},$$

so that (w_{2n}) is equivalent to (y_n) ; as E_1 and E_2 are barrelled it follows that $E_1 \cong E_2$, and also that E_1 is a J -space. Further, by the Theorem 3.3, it follows that, as (y_n) is symmetric, (y_n) is equivalent to both (y_{2n-1}) and (y_{2n}) ; for if p is a symmetric norm with respect to (y_n)

$$p\left(\sum_{i=1}^k \alpha_i y_i\right) = p\left(\sum_{i=1}^k \alpha_i y_{2i}\right).$$

Thus $E_1 \cong E_1 \oplus E_1 \cong E_1 \oplus E_2 \cong E$, and so E is a J -space.

THEOREM 4.3. *Let E be a J -space, and let (x_n) be a normalised Schauder basis of E ; then if (y_n) is a normalised block basic sequence with respect to (x_n) , $\lambda_y = \lambda_x$ and $\overline{\text{lin}}(y_n)$ is complemented in E and isomorphic to E .*

Let

$$y_j = \sum_{n_{j-1}+1}^{n_j} \alpha_i x_i,$$

and let $u_j = y_j - \alpha_{n_j} x_{n_j}$. As E is barrelled, (x_n) is a simple (see [11]) Schauder basis, and the set (u_j) is bounded. Thus one can define the block perturbation (z_n) by $z_i = x_i, i \neq n_j$ and $z_i = x_i + u_j$ for $i = n_j$; (z_n) is a normalised Schauder basis of E . Suppose $\beta \in \lambda_x = \lambda_z$; then as (z_i) is equivalent to (x_n) , $\sum_{i=1}^{\infty} \beta_i z_{n_i}$ converges and similarly $\sum_{i=1}^{\infty} \beta_i x_{n_i}$ converges; thus $\sum_{i=1}^{\infty} \beta_i u_i$ converges. As (y_j) is bounded, and (f_n) is equicontinuous, $\sup_j |\alpha_{n_j}| < \infty$, and so $\sum_{i=1}^{\infty} \alpha_{n_j} \beta_i x_{n_i}$ converges; therefore $\sum_{i=1}^{\infty} \beta_i y_i$ converges and so $\lambda_x = \lambda_y$.

As (y_n) is regular, there is a continuous symmetric norm p such that $p(y_j) \geq 1$ for all j . Let h_j be a linear functional on $\overline{\text{lin}}(x_{n_{j-1}+1}, \dots, x_{n_j})$ such that $h_j(y_j) = 1$ and $|h_j(x)| \leq p(x)$; let $g_j \in E'$ be defined by

$$g_j(x) = h_j\left(\sum_{n_{j-1}+1}^{n_j} f_i(x) x_i\right).$$

Then $|g_j(x)| \leq p(x)$, and so the set (g_j) is equicontinuous, and thus strongly bounded in E' ; as (y_j) is bounded (g_j) is regular, and hence is a normalised block basic sequence with respect to (f_n) . As E' is a J -space (Proposition 4.1), $\lambda_g < \lambda_f$; however, $\lambda_g < \lambda_y^g < \lambda_x^g = \lambda_f$ and hence $\lambda_f = \lambda_g$. Therefore $\lambda_x = \lambda_f^g = \lambda_y^g > \lambda_y$ and so $\lambda_x = \lambda_y$.

If $f \in E', f = \sum_{i=1}^{\infty} f(x_i) f_i$ in E' , and as $\lambda_f = \lambda_y, \sum_{i=1}^{\infty} f(x_i) g_i$ converges. For $x \in E, \sum_{i=1}^{\infty} f(x_i) g_i(x)$ converges; as E is reflexive, E is weakly sequentially complete and so $\sum_{i=1}^{\infty} g_i(x) x_i$ converges. As $\lambda_x = \lambda_y, \sum_{i=1}^{\infty} g_i(x) y_i$ converges.

Let $T(x) = \sum_{i=1}^{\infty} g_i(x) y_i$; then by the Banach-Steinhaus theorem, T is a continuous projection of E onto $\overline{\text{lin}}(y_n)$.

LEMMA 4.4. *Let E be a J -space, and (x_n) be a normalised Schauder basis of E ; then any sequence (v_n) of 2^n -blocks is abnormal.*

Suppose (v_n) is a normal sequence of 2^n -blocks, and suppose (α_i) is a sequence such that $\sum |\alpha_i|^2 = \infty$. Then there is a sequence n_j with $n_0 = 0$ such that

$$\sum_{n_{j-1}+1}^{n_j} |\alpha_i|^2 > 1 \quad \text{for all } j;$$

let $m_0 = 0$ and $m_j = \sum_{i=1}^j 2^{n_j - n_{i-1}}$. Let $E_k = \overline{\text{lin}}(x_{m_{k-1}+1}, \dots, x_{m_k})$, and

let $(y_j)_{m_{k-1}+1}^{m_k}$ and $(z_j)_{n_{k-1}+1}^{n_k}$ be the Haar and Rademacher systems of E_k .

If $x \in E$,

$$\sum_{m_{k-1}+1}^{m_k} f_i(x) x_i = \sum_{m_{k-1}+1}^{m_k} g_i(x) y_i$$

with each g_i continuous on E . If $m_{k-1} + 1 \leq s \leq m_k$, then for any symmetric norm p , by Proposition 3.6,

$$p\left(\sum_{m_{k-1}+1}^s g_i(x) y_i\right) \leq p\left(\sum_{m_{k-1}+1}^{m_k} f_i(x) x_i\right).$$

It follows that (y_i) is a Schauder basis of E .

For each symmetric norm p , and each $j, p(y_j) = p(v_j)$ for some i_j ; for $(f_k(y_i))_{i=1}^{\infty}$ takes only the values ± 1 and zero, and is non-zero on 2^k values of k . As (v_j) is normal, (y_j) is normal; similarly, (z_j) is normal. Let p be a symmetric norm such that $w_j = z_j/p(z_j)$ is normalised; then (w_j) is a normalised block basic sequence with respect to (y_j) , and so $\lambda_w = \lambda_y = \lambda_z$. As p is symmetric,

$$p\left(\sum_{n_{k-1}+1}^{n_k} \alpha_i w_i\right) \geq \frac{1}{8} \quad (\text{Proposition 3.6}).$$

Therefore, $\alpha \notin \lambda_x$ and $\lambda_x = l^2$.

Now let $g_i = (\sum f_i; i \in K_j)$ be a sequence of disjoint 2^j -blocks with respect to (f_n) , and let $u_j = (\sum x_i; i \in K_j)$. Then (u_j) is normal as for any symmetric p , $p(u_j) = p(v_j)$. By Theorem 3.4, for any $x \in E$,

$$\sum_{j=1}^{\infty} \frac{1}{2^j} g_j(x) u_j$$

converges; suppose $(\beta_j u_j)$ is normalised, then it follows that

$$\lim_{j \rightarrow \infty} \frac{1}{2^j \beta_j} g_j(x) = 0$$

so that $((1/2^j \beta_j) g_j)$ is bounded in E' (weakly and strongly).

As

$$\frac{1}{2^j \beta_j} g_j(\beta_j u_j) = 1,$$

it follows that $((1/2^j \beta_j) g_j)$ is regular, and so (g_j) is normal.

By the first part applied to E' , $\lambda_j \in \ell^2$ so that $\ell^2 \subset \lambda_x \subset \ell^2$, i.e. $\lambda_x = \ell^2$. However ℓ^2 is not a J -space, as already observed. This is the required contradiction.

THEOREM 4.5. *A complete barrelled space with a Schauder basis has either an abnormal Schauder basis or a conditional Schauder basis.*

If not E is a J -space (Theorem 4.2); however, by Lemma 4.4 and Theorem 3.4, E has an abnormal Schauder basis.

With a Fréchet space, one can go slightly further, using results established in [2] and [12].

LEMMA 4.6. *Let E be a Fréchet J -space, and let G be a closed non-Montel subspace of E ; then $G \cong G \oplus E$.*

There exists in G a closed bounded set A which is not compact, and so using Theorem 10 of [5], A is not sequentially compact; however, A is weakly sequentially compact, since E is reflexive. Thus there exists in G a sequence y_n tending to zero weakly but not strongly. By Theorem 4.3 of [12], there is a subsequence (z_n) of (y_n) which is a normalised Schauder basic sequence equivalent to a block basic sequence (u_n) of (x_n) , where (x_n) is any normalised basis of E . Furthermore, $\overline{\text{lin}}(u_n)$ is complemented (Theorem 4.3), and, as remarked in [12], $D = \overline{\text{lin}}(z_n)$ is complemented, and as $\lambda_x = \lambda_x$, $D \cong E$.

Therefore

$$G = D \oplus H \cong E \oplus H \cong E \oplus E \oplus H \cong E \oplus G.$$

THEOREM 4.7. *Let E be an F -space with a normalised basis; if E has an infinite-dimensional normed subspace G (i.e. G has a norm topology), then E has a conditional normalised basis.*

Otherwise E is a J -space, and using Lemma 4.6, $\overline{G} \cong E \oplus \overline{G}$, so that E is a Banach space. However, this is impossible, by the results of [16].

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