The following questions may be of a certain interest:

Is the existence of non-increasing null sequences (a_n) with $\delta_n(K) = O(a_n)$ for all compact sets of an F-space E a topological linear property? Does this property characterize the space s in a certain class of F-spaces (of course, such a class should not only contain the B-spaces and the spaces s)? Is this property related to conditions listed in section 1 to characterize the space s?

Concerning the last question it is easy to see that for an F-space E, in which one sequence of finitely-dimensional subspaces approximates slowly, there is a non-increasing null sequence of positive numbers $r_0, r_1, \ldots$ such that

$$\delta_n(B) = O(r_n)$$

for each bounded set $B \subset E$. In this case either each closed bounded set is compact (cf. "(FM)-Räume" in [5], p. 372) or the compact sets cannot be characterized among the closed bounded sets by

$$\lim_{n\to\infty} \delta_n(B) = 0.$$

This characterization is possible e.g. in complete p-normed spaces $(0 < p \leqslant 1)$, where the p-norm differs from a usual norm only by the property $\|\alpha x\| = |\alpha|^p \|x\|$ instead of $= |\alpha|\,\|x\|$ $(\alpha \in C, x \in E)$ (cf. [7], p. 131) or in the space s. May be these facts give hints for an answer to the question at the end of section 1.

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Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

A theorem of Eilenberg-Watts type for tensor products of Banach spaces

by

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Introduction. By the *tensor product of Banach spaces* X and Y we shall mean the projective tensor product $X \hat{\otimes} Y$ defined as the completion of the algebraic tensor product $X \otimes Y$ withrespect to the greatest cross norm

$$\|u\| = \inf\Big\{\sum_{i=1}^{n} \|x_i\|\,\|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i,\ x_i \in X,\ y_i \in Y\Big\}.$$

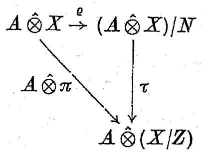
It is well known (cf. Grothendieck [4], Ch. I, § 1, no. 2, Proposition 3 and Théorème 2, Buchwalter [1], p. 33) that for each fixed Banach space A the tensor product $A \hat{\otimes} X$ has the following properties:

(α) If $\varphi: X \to Y$ is a bounded linear operator onto a dense subset of Y, then the induced operator

$$A \hat{\otimes} X \xrightarrow{A\hat{\otimes}\varphi} A \hat{\otimes} Y$$

maps $A \hat{\otimes} X$ onto a dense subset of $A \hat{\otimes} Y$.

(β) If Z is a closed subspace of a Banach space X, then there is a canonical isomorphism τ from $(A \hat{\otimes} X)/N$ onto $A \hat{\otimes}(X/Z)$, where N is the closed subspace of $A \hat{\otimes} X$ generated by the elements of the form $a \otimes z$ with a in A and z in Z; moreover, the corresponding diagram



is commutative; here π and ϱ denote the canonical surjections.

(γ) For any set P the space $A \hat{\otimes} l(P)$ is canonically isomorphic to the space $l(P, A)$ of all indexed families $a = (a_p)_{p \in P}$ with

$$\|a\| = \sum_{p \in P} \|a_p\| < \infty;$$

here $l(P)$ is $l(P, \boldsymbol{F})$, where $\boldsymbol{F}$ is the field of scalars (i.e., $\boldsymbol{F} = \boldsymbol{R}$ or $\boldsymbol{F} = \boldsymbol{C}$).

The purpose of this paper is to show that, roughly speaking, properties (α)-(γ) characterize the tensor product up to isomorphism. This characterization will be formulated as a natural equivalence of functors. An analogous theorem for tensor products of modules is known as the Eilenberg-Watts theorem ([2], [9], [6], p. 157).

1. We now reformulate properties (α) and (β) in categorical language; unexplained terminology is from Freyd [3].

Let $\mathbf{Ban}_1$ denote the category of Banach spaces (over the field $\boldsymbol{F}$) and linear contractions (i.e., linear maps $\varphi: X \to Y$ satisfying $\|\varphi\| \leqslant 1$). Throughout Sections 1-3 the term "morphism" will refer to this category. The set of all morphisms from X to Y will be denoted by $\langle X, Y \rangle$. A covariant functor

(1) $$T: \mathbf{Ban}_1 \to \mathbf{Ban}_1$$

will be called *linear* if for any Banach spaces X and Y and for any linear maps $\varphi: X \to Y$ and $\psi: X \to Y$ the conditions $\|\varphi\| \leqslant 1$, $\|\psi\| \leqslant 1$, $\|\varphi + \psi\| \leqslant 1$ imply

$$T(\varphi + \psi) = T(\varphi) + T(\psi) \quad \text{and} \quad T(s\varphi) = sT(\varphi)$$

for s in $\boldsymbol{F}$ such that $|s| \leqslant 1$. It is clear that each of the following conditions is necessary and sufficient in order that the functor (1) be linear:

(*) If $\|\varphi\| \leqslant 1$, $\|\psi\| \leqslant 1$ and $|s| + |t| \leqslant 1$, then

$$T(s\varphi + t\psi) = sT(\varphi) + tT(\psi).$$

(**) For each pair (X, Y) of Banach spaces the restriction of T to $\langle X, Y \rangle$ yields a map

$$T_{X,Y}: \langle X, Y \rangle \to \langle T(X), T(Y) \rangle$$

which can be extended to a linear contraction from the space $L(X, Y)$ of all bounded linear maps $X \to Y$ to the space $L(T(X), T(Y))$.

Thus, any linear functor (1) satisfies the condition

(2) $$\|T(\varphi)\| \leqslant \|\varphi\|.$$

Let A be a fixed Banach space. We shall deal with the covariant functors

$$\Omega_A: \mathbf{Ban}_1 \to \mathbf{Ban}_1 \quad \text{and} \quad \Sigma_A: \mathbf{Ban}_1 \to \mathbf{Ban}_1$$

(see [7]); we recall that

$$\Omega_A(X) = L(A, X) \quad \text{and} \quad \Sigma_A(X) = A \hat{\otimes} X;$$

if $\varphi: X \to Y$ is a morphism, then $\Omega_A(\varphi)$ is the corresponding map from $\Omega_A(X)$ to $\Omega_A(Y)$ defined as $\Omega_A(\varphi)(\xi) = \varphi \circ \xi$ for ξ in $\Omega_A(X)$, and $\Sigma_A(\varphi) = A \hat{\otimes} \varphi$. It is well known that Σ_A is a left adjoint of Ω_A ([3], [6], [5], [8], p. 296). In fact, the canonical linear isometrical bijection

$$\langle \Sigma_A X, Y \rangle \to \langle X, \Omega_A Y \rangle$$

is natural in all three variables X, Y, A.

If Z is a closed subspace of X, then X/Z will denote the quotient space with the usual norm $\|x + Z\| = \inf\{\|x + z\|: z \in Z\}$. If $\varphi: X \to Y$ is a morphism, then we define:

$$\operatorname{Ker}\varphi = \{x \in X: \varphi(x) = 0\},$$
$$\operatorname{Im}\varphi = \operatorname{cl}_Y\{\varphi(x): x \in X\},$$
$$\operatorname{Coker}\varphi = Y/\operatorname{Im}\varphi, \quad \operatorname{Coim}\varphi = X/\operatorname{Ker}\varphi;$$

moreover, $\ker\varphi: \operatorname{Ker}\varphi \to X$ and $\operatorname{im}\varphi: \operatorname{Im}\varphi \to Y$ denote the identical injections, while

$$\operatorname{coker}\varphi: Y \to \operatorname{Coker}\varphi \quad \text{and} \quad \operatorname{coim}\varphi: X \to \operatorname{Coim}\varphi$$

denote canonical surjections. It is obvious that

(3) $$\ker(\operatorname{coker}\varphi) = \operatorname{im}\varphi \quad \text{and} \quad \operatorname{coker}(\ker\varphi) = \operatorname{coim}\varphi$$

(cf. [1], p. 8). If $\varphi_1: X \to Y_1$ is also a morphism, then $\varphi =_e \varphi_1$ will mean that there exists a $\mathbf{Ban}_1$-isomorphism (i.e., a linear isometrical bijection) $\eta: Y \to Y_1$ such that $\eta\varphi = \varphi_1$. A morphism φ will be called a *quotient morphism* iff $\varphi =_e \pi$ for some canonical surjection $\pi: X \to X/Z$.

Let us consider the following conditions:

(α′) For every φ, if φ is an epimorphism, then $T(\varphi)$ is an epimorphism.

(β′) For every π, if π is a quotient morphism, then $T(\pi)$ is a quotient morphism and

(4) $$\operatorname{im}T(\ker\pi) = \ker T(\pi).$$

It is obvious that a morphism $\varphi: X \to Y$ is an epimorphism (in $\mathbf{Ban}_1$) iff $\varphi(X)$ is dense in Y, i.e., $\operatorname{Im}\varphi = Y$. Therefore property (α) formulated in the introduction means that the functor $T = \Sigma_A$ satisfies (α′). Moreover, property (β) means that Σ_A satisfies (β′).

It is also clear that a morphism $\xi: Y \to Z$ is a coequalizer (= a difference cokernel, see [3], [6], [8]) of morphisms $\varphi: X \to Y$ and $\psi: X \to Y$

if and only if $\xi =_e \operatorname{coker}\left(\frac{1}{2}(\varphi-\psi)\right)$ (the number $\frac{1}{2}$ guarantees that $\|\frac{1}{2}(\varphi-\psi)\| \leqslant 1$).

PROPOSITION. *Let* (1) *be a linear functor. Then each of the following conditions is equivalent to the conjunction* (α') & (β'):

(i) *T is cokernel-preserving, i.e.,* $T(\operatorname{coker}\varphi) =_e \operatorname{coker} T(\varphi)$ *for every morphism* φ,

(ii) *T is coequalizer-preserving, i.e., if ξ is a coequalizer of φ and ψ, then $T(\xi)$ is a coequalizer of $T(\varphi)$ and $T(\psi)$.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows from the preceding remark.

(i) $\Rightarrow (\alpha')$: Let φ be an epimorphism. Then $\operatorname{coker}\varphi = 0$, and hence

$$\operatorname{coker} T(\varphi) =_e T(\operatorname{coker}\varphi) = T(0) = 0.$$

Thus, $T(\varphi)$ is an epimorphism.

(i) $\Rightarrow (\beta')$: Let π be a quotient morphism; hence $\pi =_e \operatorname{coim}\pi$ and $T(\pi) =_e T(\operatorname{coim}\pi)$. Substituting $\varphi = \ker\pi$ in (i) and applying (3) we get

$$T(\pi) =_e T(\operatorname{coim}\pi) =_e \operatorname{coker} T(\ker\pi).$$

Consequently, $T(\pi)$ is a quotient morphism (as a cokernel of a morphism). Passing to kernels we get

$$\ker T(\pi) = \ker\operatorname{coker} T(\ker\pi) = \operatorname{im} T(\ker\pi).$$

(α') & $(\beta') \Rightarrow$ (i): Let $\varphi\colon X \to Y$ be any morphism. It may be factored as $\varphi = \varepsilon\vartheta$, where $\varepsilon = \operatorname{im}\alpha$ and $\vartheta\colon X \to \operatorname{Im}\varphi$ is the induced epimorphism. It is clear that

$$\operatorname{coker}\varphi = \operatorname{coker}\varepsilon. \tag{5}$$

Acting with T we get $T(\varphi) = T(\varepsilon)T(\vartheta)$. Since ϑ is an epimorphism, (α') implies that $T(\vartheta)$ is an epimorphism. Consequently (cf. [6], p. 15),

$$\operatorname{coker} T(\varphi) = \operatorname{coker} T(\varepsilon) \tag{6}$$

as the range of $T(\varphi)$ is dense in that of $T(\varepsilon)$. Substituting $\pi = \operatorname{coker}\varepsilon$ in (4) and applying (3) we get $\varepsilon = \ker\pi$ and hence

$$\ker\operatorname{coker} T(\varepsilon) = \operatorname{im} T(\ker\pi) = \ker T(\operatorname{coker}\varepsilon).$$

Since the quotient morphisms $\operatorname{coker} T(\varepsilon)$ and $T(\operatorname{coker}\varepsilon)$ have the same kernel, we infer that $\operatorname{coker} T(\varepsilon) =_e T(\operatorname{coker}\varepsilon)$. Thus, by (5) and (6)

$$T(\operatorname{coker}\varphi) = T(\operatorname{coker}\varepsilon) =_e \operatorname{coker} T(\varepsilon) = \operatorname{coker} T(\varphi).$$

Remark. The above proposition may be regarded as a characterization of right exactness of the functor (1).

2. We shall now deal with condition (γ). Given an indexed family $(X_i)_{i\in I}$ of Banach spaces, the *l_1-join* $X = \coprod_{i\in I} X_i$ is the space of all functions $x = (x_i)_{i\in I}$ in the product $\mathsf{P}X_i$ such that

$$\|x\| = \sum_{i\in I}\|x_i\| < \infty.$$

It is well known that X together with the canonical injections $\sigma_i\colon X_i \to X$ is a coproduct (= sum) of $(X_i)_{i\in I}$ in the category $\mathbf{Ban}_1$. We shall say that functor (1) is *coproduct-preserving* iff for any family $(X_i)_{i\in I}$ of Banach spaces the space $T(X)$ together with morphisms $T(\sigma_i)\colon T(X_i) \to T(X)$ is a coproduct of $\left(T(X_i)\right)_{i\in I}$, i.e., there exists a linear isometrical bijection η from $T(X)$ onto the l_1-join $\coprod T(X_i)$ such that for each $i\in I$ the morphism $\eta T(\sigma_i)$ is the canonical injection of $T(X_i)$ into $\coprod T(X_i)$.

The functor Σ_A is coproduct-preserving (as a left adjoint of Ω_A, cf. [3], p. 81, [6], p. 67). This statement is somewhat stronger than saying that $T = \Sigma_A$ satisfies the condition

(γ') If $X_i = \boldsymbol{F}$ for $i\in I$, then there is a $\mathbf{Ban}_1$-isomorphism η making each diagram

$$\begin{array}{ccc}
T\left(\coprod_{i\in I} X_i\right) & \xrightarrow{\eta} & \coprod_{i\in I} T(X_i) \\
{\scriptstyle T(\sigma_j)}\uparrow & \nearrow{\scriptstyle \sigma_j'} & \\
T(X_j) & &
\end{array}$$

commutative (here $\sigma_j\colon X_j \to \coprod_{i\in I} X_i$ and σ_j' are canonical injections, $j\in I$).

The characterization of the tensor product mentioned in the introduction may be formulated as follows:

THEOREM 1. *If a covariant linear functor* (1) *is cokernel-preserving and coproduct-preserving, then it is naturally equivalent to some* Σ_A.

Actually, we shall prove more:

THEOREM 1'. *A linear covariant functor* (1) *is naturally equivalent to some functor* Σ_A *if and only if it satisfies* (α'), (β') *and* (γ').

Proof of Theorem 1'. The "only if" part follows from general theorems on adjoint functors (see [3], p. 81, [6], p. 67).

Now, let (1) be any linear functor. Denote $A = T(\boldsymbol{F})$. If X is a Banach space, let $\beta_x(s) = sx$ for x in X with $\|x\| \leqslant 1$, s in $\boldsymbol{F}$; clearly β_x is a morphism from $\boldsymbol{F}$ to X and $T(\beta_x)$ is a morphism from A to $T(X)$. Moreover, by (2), $\|T(\beta_x)\| \leqslant \|\beta_x\| = \|x\| \leqslant 1$. Letting

$$\xi(a, x) = s(T\beta_{x/s})(a) \quad \text{for } a \text{ in } A,\ x \text{ in } X,\ s > \|x\|,$$

we get a bilinear operator $\xi: A\times X\to T(X)$ with $\|\xi\|\leqslant 1$, which can be factored through a unique linear operator

$$\tau_X\colon \Sigma_A(X)\to T(X) \tag{7}$$

with $\|\tau_X\|\leqslant 1$. Thus, $\tau_X(\sum a_i\otimes x_i) = \sum (T\beta_{x_i})a_i$ if $\|x_i\|\leqslant 1$ for $i=1,\ldots,n$. A routine verification shows that (7) yields a natural transformation

$$\tau\colon \Sigma_A\to T, \tag{8}$$

i.e., for every morphism $\varphi: X\to Y$ the diagram

$$\begin{array}{ccc} \Sigma_A(X) & \xrightarrow{\Sigma_A(\varphi)} & \Sigma_A(Y) \\ {\scriptstyle\tau_X}\downarrow & & \downarrow{\scriptstyle\tau_Y} \\ T(X) & \xrightarrow[T(\varphi)]{} & T(Y) \end{array}$$

is commutative. Moreover, $\tau_F:\Sigma_A(F)\to A$ is an isometrical bijection (in virtue of the canonical isomorphism $A\hat{\otimes}F\cong A$ and $T(s1_F)=s1_A$).

We claim that if T satisfies (α'), (β') and (γ'), then (8) is a natural equivalence, i.e., for every Banach space X the map (7) is an isometrical bijection.

There exist sets P and Q and morphisms

$$l(Q)\xrightarrow{\varrho} l(P)\xrightarrow{\pi} X\to 0 \tag{9}$$

such that π is a quotient morphism and ϱ maps $l(Q)$ onto the kernel of π (one may say that the sequence (9) is exact). Acting with the functors Σ_A and T we get the commutative diagram

$$\begin{array}{ccccc} \Sigma_A(l(Q)) & \xrightarrow{\Sigma_A(\varrho)} & \Sigma_A(l(P)) & \xrightarrow{\Sigma_A(\pi)} & \Sigma_A(X) \\ {\scriptstyle\tau_{l(Q)}}\downarrow & & \downarrow{\scriptstyle\tau_{l(P)}} & & \downarrow{\scriptstyle\tau_X} \\ T(l(Q)) & \xrightarrow[T(\varrho)]{} & T(l(P)) & \xrightarrow[T(\pi)]{} & T(X) \end{array}$$

If T satisfies (γ'), then $\tau_{l(P)}$ and $\tau_{l(Q)}$ are isometrical bijections (since τ_F has this property).

Finally, if T satisfies (α'), (β') and (γ'), then τ_X is also an isometrical bijection. Indeed, in the above diagram we have two $\mathbf{Ban}_1$-isomorphisms $\tau_{l(Q)}$ and $\tau_{l(P)}$ and, by condition (i) of § 1, the set Ker $\Sigma_A(\pi)$ is the closure of the range of $\Sigma_A(\varrho)$, the set Ker $T(\pi)$ is the closure of the range of $T(\varrho)$, and both $\Sigma_A(\pi)$ and $T(\pi)$ are quotient morphisms; therefore τ_X is a $\mathbf{Ban}_1$-isomorphism (note that τ_X is the unique morphism from $\Sigma_A(X)$ to $T(X)$ which makes the diagram commutative).

3. We shall now formulate an analogous characterization of a functor Ω_A.

THEOREM 2. *Let $S:\mathbf{Ban}_1\to\mathbf{Ban}_1$ be a linear covariant functor. S is naturally equivalent to some functor Ω_A if and only if it is product-preserving and kernel-preserving.*

Theorem 2 can be immediately derived from Theorem 1. If S is product-preserving and kernel-preserving, then, by the special adjoint functor theorem of Freyd ([3], p. 89, [6], p. 126), S is a right adjoint of some functor T. The standard argument (cf. [6], p. 120) shows that T is also linear; moreover T is coproduct-preserving and cokernel-preserving. Hence, by Theorem 1, there exists an A such that T is naturally equivalent to Σ_A. This means that S is a right adjoint of Σ_A and, by Kan's uniqueness theorem, S is naturally equivalent to Ω_A.

In an analogous way Theorem 1 can be derived from Theorem 2.

4. We shall now outline another proof of Theorem 2 (and hence, by the preceding remark, another proof of Theorem 1 as well). The argument is valid in any autonomous category $\mathfrak{A}$ (in the sense of Linton [5]) satisfying the following conditions:

(a) If the underlying map of a morphism α in $\mathfrak{A}$ is a bijection, then α is an isomorphism in $\mathfrak{A}$.

(b) The identity functor $\mathfrak{A}\to\mathfrak{A}$ is strongly representable, i.e., there is an object E such that $\mathbf{Hom}\,(E, ?)$ is naturally equivalent to the identity.

(c) $\mathfrak{A}$ satisfies the assumptions of the special adjoint functor theorem of Freyd.

Under these assumptions we have

WATT'S THEOREM. *Every covariant left-root-preserving strong functor $S:\mathfrak{A}\to\mathfrak{A}$ is naturally equivalent to some functor $\Omega_A=\mathbf{Hom}(A, ?)$.*

The argument is as follows: By the special adjoint functor theorem S has a left adjoint $T:\mathfrak{A}\to\mathfrak{A}$. Since S is strong, by (a), T is also strong and S is a strong right adjoint of T. Consequently, by (b),

$$\mathbf{Hom}(T(E), ?)\cong\mathbf{Hom}(E, S(?))\cong S.$$

Thus, S is naturally equivalent to $\Omega_{T(E)}$.

The category $\mathbf{Ban}_1$ becomes an autonomous category if the closed unit ball is regarded as the underlying set and $\mathbf{Hom}\,(X, Y) = L(X, Y)$; thus, the underlying set of the object $\mathbf{Hom}(X, Y)$ is the set of all linear contractions from X to Y. Conditions (a)-(c) are obviously satisfied: (a) means that a one-to-one linear operator mapping the unit ball onto the unit ball is an isometrical bijection; (b) is satisfied if $E = F$.

Let us note that typical categories of topological vector spaces do not satisfy (a); moreover, the proof of Theorem 1 presented in § 2 has no obvious extension to that case.

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Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

Unconditional and normalised bases

by

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1. Introduction. A Schauder basis (x_n) of a locally convex space E is unconditional if, whenever $\sum_{i=1}^{\infty} a_i x_i$ converges, the convergence is unconditional. In [16], Pełczyński and Singer proved that every Banach space with a basis possesses a conditional (i.e. not unconditional) basis. In this paper I shall generalise this theorem using the concept of normalisation introduced in [12].

A sequence (x_n) is *regular* if there is a neighbourhood V of zero with $x_n \notin V$ for all n; a regular bounded sequence is said to be *normalised*. If there exists a scalar sequence (a_n) with $(a_n x_n)$ normalised, then (x_n) is said to be *normal*; otherwise (x_n) is abnormal.

If (x_n) is a Schauder basis of E, then (f_n) will always denote its dual sequence in E'; if $(f_n)_{n=1}^{\infty}$ is equicontinuous, then (x_n) is *equi-regular*, and hence regular; if E is barrelled, then any regular basis is equi-regular. The sequence space of all a such that $\sum_{i=1}^{\infty} a_i x_i$ converges will be denoted by λ_x, and μ_x is the sequence space $\{(f(x_n))_{n=1}^{\infty};\, f \in E'\}$. If E is sequentially complete, then (x_n) is unconditional if and only if λ_x is solid (see [4]), that is if $a \in \lambda_x$ and $|\theta_n| \leqslant 1$ for all n, then $(\theta_n a_n) \in \lambda_x$. If E is also barrelled, it can be shown that the topology on E may be given by a collection of solid semi-norms p such that

$$p(x) = \sup_{|\theta_i| \leqslant 1} p\Big(\sum_{i=1}^{\infty} \theta_i f_i(x) x_i\Big).$$

A sequentially complete barrelled space with a Schauder basis is complete (see [10]); in this paper I shall restrict attention almost exclusively to complete barrelled spaces.

2. Reflexivity and unconditional bases. A Schauder basis (x_n) is γ-complete or boundedly-complete if whenever $\big(\sum_{i=1}^{n} a_i x_i;\ n = 1, 2\ldots\big)$ is