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STUDIA MATHEMATICA, T. XXXVIII. (1970)
Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

Remarks on a theorem of S. N. Bernstein

by

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As usually $C(0,1)$ denotes the space of all real-valued continuous functions on the closed intervall $[0,1]$. It is well-known that the set of polynomials is dense in $C(0,1)$ with respect to the supremum norm $\|\cdot\|$. If we introduce the minimal deviation

$$d_n(f) = \inf\|f-P_n\|$$

of a function $f \in C(0,1)$ from the linear subspace of polynomials P_n of degree $\leqslant n$, the density of the polynomials can be expressed by

$$\lim_{n\to\infty} d_n(f) = 0 \quad \big(f \in C(0,1)\big).$$

S. N. Bernstein has shown that for each non-increasing null sequence (a_n) of non-negative numbers there is a $g \in C(0,1)$ with

$$d_n(g) = a_n.$$

Shapiro [8] generalized this theorem in the following way. He replaced $C(0,1)$ by an arbitrary B-space $(E, \|\cdot\|)$ and the sequence of n-dimensional subspaces of polynomials with degree $\leqslant n$ by a sequence (M_n) of proper closed linear subspaces in E. In this case for each null sequence (a_n) of non-negative numbers there is a vector $x \in E$ with

$$xM_n := \inf_{u \in M_n}\|x-u\| \neq O(a_n).$$

For two sequences (b_n) and (c_n) of non-negative numbers the formula

$$c_n \neq O(b_n)$$

means that there is no $A > 0$ with $c_n \leqslant Ab_n$ for all n. We shall say briefly that each sequence of proper closed linear subspaces in a B-space *approximates slowly*. In his proof Shapiro used the category argument. Therefore the question seems naturally, whether Shapiro's statement also holds in F-spaces. In this paper F-*space* means a complete metric linear space or, using the terminology of [10], a complete quasinormed space.

In section 1 we answer this question by characterizing those sequences of linear subspaces in an F-space which approximate slowly. The existence of a slowly approximating sequence of finitely dimensional subspaces turns out to be a characteristic property of space s in an extensive class of F-spaces. As usually s denotes the space of all sequences of real or complex numbers with the topology of coordinatewise convergence.

In a second section we deal with the n-th diameter $\delta_n(B)$ of a bounded set B in an F-space $(E, |\cdot|)$. The numbers $\delta_n(B)$ $(n = 1, 2, \ldots)$ are defined by

$$\delta_n(B) = \inf\{\sup_{x\in B} xM \colon \dim M \leqslant n\}, \tag{0.1}$$

where M denotes a linear subspace in E and $xM = \inf_{u\in M}|x-u|$.

1. Let $(E, |\cdot|)$ be any F-space, $M \subset E$ a linear subspace, $x \in E$ and $r > 0$. We write

$$xM \colon = \inf_{u\in M}|x-u|, \qquad \omega_M \colon = \sup_{x\in E} xM$$

and

$$K_r \colon = \{x \in E \colon |x| \leqslant r\}.$$

The following proposition and theorem were proved in [1].

PROPOSITION 1. *It holds the equality*

$$\sup_{x\in K_r} xM = \min\{r, \omega_M\}. \tag{1.1}$$

As this proposition is also valid in incomplete quasinormed spaces, it includes the well-known lemma of F. Riesz on the existence of "nearly orthogonal" elements for proper closed subspaces in normed spaces. We have $\omega_M = \infty$ for any proper closed linear subspace M of a normed space $(E, \|\cdot\|)$ and thus it follows from (1.1)

$$\sup_{\|x\|\leqslant 1} xM = 1.$$

A sequence of linear subspaces $M_n \subset E$ is said to *approximate slowly* if for each null sequence (a_n) of non-negative numbers there is a vector x in E with

$$xM_n \neq O(a_n).$$

THEOREM 2. *A sequence of linear subspaces M_n in $(E, |\cdot|)$ approximates slowly if and only if*

$$\overline{\lim}\, \omega_{M_n} > 0. \tag{1.2}$$

COROLLARY. *For proper closed linear subspaces M_n of a B-space we have $\omega_{M_n} = \infty$ and therefore $\overline{\lim}\, \omega_{M_n} = \infty > 0$. Thus our theorem 2 implies Shapiro's generalization of Bernstein's theorem mentioned above.*

If (1.2) does not hold, only a finite number of terms $\omega_{M_{n_1}}, \ldots, \omega_{M_{n_k}}$ are ∞ and

$$\lim_{j\to\infty} \omega_{M_{n_k+j}} = 0.$$

Then there is a sequence (r_n) such that $r_{n_i} = \infty$ $(i = 1, \ldots, k)$, $\omega_{M_n} < r_n < \infty$ $(n \notin \{n_1, \ldots, n_k\})$ and $\lim_{j\to\infty} r_j = 0$. With $E = \colon K_\infty$ the sequence (K_{r_n}) forms a base of neighbourhoods in E, for which the equalities

$$M_n + K_{r_n} = E \qquad (n = 1, 2., \ldots)$$

hold evidently. It follows that property (1.2) is of a topological linear nature independent of the special quasinorm $|\cdot|$.

We give now some properties of F-spaces, in which slowly approximating sequences of finitely-dimensional subspaces exist.

PROPOSITION 3. *An infinitely-dimensional F-space $(E, |\cdot|)$, in which a sequence of finitely-dimensional subspaces does not approximate slowly, contains a subspace isomorphic to the space s.*

Proof. Without any loss of generality we may assume that for each $x \in E$ the function $t \to |tx|$, $t \geqslant 0$, is non-decreasing and that there is a sequence (M_n) of finitely-dimensional subspaces in E such that (ω_{M_n}) is a null sequence of positive numbers. Let ε be an arbitrary positive number, n a natural number with $\omega_{M_n} < \varepsilon$ and $z \in E \setminus M_n$. Setting

$$L \colon = \{u + \lambda z \colon u \in M_n, \lambda \in C\}$$

(C denotes the set of real or complex numbers) and $H_\varepsilon = K_\varepsilon \cap L$, we have

$$M_n + H_\varepsilon = L.$$

Thus for each natural number m there are vectors $u_m \in M_n$ and $v_m \in H_\varepsilon$ with $v_m = mz - u_m$. As L is finitely-dimensional, there is a norm $\|\cdot\|$ on L equivalent to the restriction of $|\cdot|$ to L. The sequence (v_m) is evidently unbounded. Therefore we may assume

$$1 \leqslant \|v_1\| \leqslant \|v_2\| \leqslant \ldots < \lim_{m\to\infty} \|v_m\| = \infty.$$

In accordance with our assumptions in the beginning of the proof, $(\|v_m\|^{-1} v_m)$ is a bounded sequence in H_ε. These sequence contains a converging subsequence (w_l) with the limit w in H_ε. It is easy to see that

$$\{tw \colon -\infty < t < \infty\} \subset H_\varepsilon.$$

Applying Theorem 9 in [2] we finish the proof.

The following theorem comprehends theorems 5 and 6 in [1] and gives a characterization of the space s in a wider class than the class of locally convex F-spaces (cf. [2], theorem 8):

THEOREM 4. *Let E be an infinitely dimensional F-space, whose topology is given by a sequence (f_j) of semiquasinorms ($f_j(x) = 0$ does not imply $x = o$) with the properties*

(*) *for each $x \in E$ and j is the function $t \to f_j(tx)$, $t \geqslant 0$ non-decreasing and*

(⁑) *for each j there is a positive number r_j such that $\lim_{t\to\infty} f_j(tx)$ is equal to 0 or r_j.*

The space E is isomorphic to the space s if and only if there is a sequence of finitely dimensional subspaces M_n in E, which does not approximate slowly.

Proof. The necessity of the condition is obvious. To prove the sufficiency we remark at first that the sets $\{x \in E: f_j(x) = 0\}$ are closed linear subspaces Z_j in E. The quotient space $E/Z_j = E_j$ can be quasinormed by $q_j(\hat{x}) := f_j(x)$ $(x \in \hat{x} \in E_j)$.

According to (⁑) it holds

$$\lim_{t\to\infty} q_j(t\hat{x}) = r_j$$

for each $\hat{x} \neq \hat{o}$. For any proper finitely-dimensional subspace $M_j \subset E_j$ we have

$$M_j + \{\hat{x} \in E_j: q_j(\hat{x}) \leqslant 2^{-1} r_j\} \neq E_j. \tag{1.3}$$

Provided that there is a subspace Z_{j_0} of infinite codimension in E. Then the dimension of E_{j_0} is infinite and for any finitely-dimensional subspace $M \subset E$ the space $M_{j_0} = M/Z_{j_0}$ is a proper finitely-dimensional subspace of E_{j_0}. Because of (1.3) we have

$$M + \{x \in E: f_{j_0}(x) \leqslant 2^{-1} r_{j_0}\} \neq E.$$

The set $\{x \in E: f_{j_0}(x) \leqslant 2^{-1} r_{j_0}\}$ is a neighbourhood U of o in E. Thus for each finitely-dimensional subspace $M \subset E$ we have $M + U \neq E$. Therefore it follows from the condition of the theorem and from theorem 2 that all Z_j are of finite codimensions in E or, what is the same, that all E_j are of finite dimensions. In this case, however, the quasinorms q_j on E_j can be replaced by equivalent norms $\|\cdot\|_j$. Then the seminorms defined by

$$p_j(x) := \|\hat{x}\|_j \quad (x \in \hat{x} \in E_j)$$

on E are equivalent to the f_j.

We have shown that E is locally convex and that the topology is given by a sequence of seminorms having null spaces of finite codimension. Applying [2], theorems 1 and 8, we finish the proof.

A sequence (r_n) of positive numbers or ∞ tends non-increasingly to zero, if at most a finite number of its terms are ∞ and $r_1 \geqslant \ldots \geqslant r_n > > \lim_{j\to\infty} r_j = 0$.

The linear hull of vectors $x_1, \ldots, x_n$ will be denoted by $[x_1, \ldots, x_n]$. Eventually we remind that $K_\infty = E$.

PROPOSITION 5. *Let E be an F-space with an infinite base $\{u_j\}$. If there is a (non-increasing) null sequence (r_n) of positive numbers or ∞ with*

$$[u_1, \ldots, u_n] + K_{r_n} = E \quad (n = 1, 2, \ldots), \tag{1.4}$$

E is isomorphic to the space s.

Proof. For each $x \in E$ there is a unique representation

$$x = \sum_{j=1}^{\infty} a_j(x) u_j = \sum_{j=1}^{\infty} \xi_j u_j,$$

where a_j are continuous linear functionals on E. Therefore the operator T defined by $Tx = (\xi_j)$ $(x \in E)$ is a continuous 1-1 mapping from E in s. Let (η_j) be any vector in s. Then for each natural number n the vector

$$y_n = \sum_{j=1}^{n} \eta_j u_j$$

belongs to $[u_1, \ldots, u_n]$. According to [6], theorem 1, we may assume that the vector y_n is the only nearest point of each y_m $(m \geqslant n)$ in $[u_1, \ldots, u_n]$. Because of (1.4), (y_n) is a fundamental sequence in E converging to

$$y = \sum_{j=1}^{\infty} \eta_j u_j.$$

We have shown that T maps E onto s. Thus E is isomorphic to s.

There are three properties characterizing the space s in certain classes of F-spaces:

(I) A locally convex F-space or an F-space with a base is isomorphic to s if and only if each infinitely-dimensional subspace is isomorphic to s (cf. [2]);

(II) theorem 4;

and

(III) an F-space E belonging to the class given in theorem 4 or having an infinite base is isomorphic to s if and only if no increasing sequence of finitely-dimensional subspaces, whose union is dense in E, approximate slowly.

The question, whether one of these properties characterizes the space s among all F-spaces or whether some properties are equivalent seem to be not yet answered.

Finally, we remain that the restriction of Shapiro's theorem to the special case of an increasing sequence of finitely-dimensional subspaces in a B-space is a weaker statement than Bernstein's one. As we were concerned with sequences of finitely-dimensional subspaces a great deal the question arises, whether the statement of slow approximation could be formulated more precisely. Indeed, this is possible in many cases and Bernstein's original proof can even be used in a slightly motivated way. We do not want, however, to carry out this work here.

2. Let $(E, |\cdot|)$ be again an F-space and $B \subset E$ a bounded set. Furthermore, let $\delta_n(B)$ $(n = 0, 1, \ldots)$ denote the n^{th} diameter defined in (0,1). We have, obviously, $\lim_{n\to\infty} \delta_n(K) = 0$ for each compact set K in E. We can ask Bernstein's question:

Does for each non-increasing null sequence (a_n) exist a compact set K with $\delta_n(K) = a_n$ $(n = 0, 1, \ldots)$?

For the space s the answer is negative as it is easy to check. For B-spaces, however, we give an affirmative answer by

THEOREM 6([1]). *Let $(E, \|\cdot\|)$ be a B-space and (a_n) a non-increasing null-sequence of non-negative numbers $a_0, a_1, \ldots$ Then there is a compact set $K \subset E$ such that*

$$\delta_n(K) = a_n \quad (n = 0, 1, \ldots).$$

Proof. Let (M_n) be an increasing sequence of linear subspaces in E with $\dim M_n = n$ and let U denote the unit ball in $(E, \|\cdot\|)$.

Setting

$$B_0 := a_0 U,$$

$$B_n := \{x \in B_{n-1}: xM_n \leqslant a_n\} \quad (n = 1, 2, \ldots)$$

and

$$K := \bigcap_{n=0}^{\infty} B_n,$$

we get a non-empty closed bounded set K and sets satisfying the inclusions

$$B_0 \supset B_1 \supset \ldots \supset B_n \supset B_{n-1} \cap M_n \quad (n = 2, 3, \ldots). \tag{2.1}$$

([1]) Meanwhile the author has come to know that theorem 6 is well known. The set constructed in the proof is just a full approximation set in G.G. Lorentz, *Approximation of functions*, New York 1966, p. 139. The theorem was also stated by S.B. Babadžanov, *On geometrical questions in Banach spaces*, Voprosy Kibernet. Vyčisl. Mat. Vyp. 12 (1967), p. 95-97.

If for any natural number p the inclusion $B_{n-1} \cap M_n \subset B_{n+p-1}$ holds, we obtain

$$B_{n-1} \cap M_n = (B_{n-1} \cap M_n) \cap M_{n+p} \subset B_{n+p-1} \cap M_{n+p} \subset B_{n+p}$$

because of (2.1) and $M_n \subset M_{n+p}$. Therefore it follows from (2.1) by induction that $B_{n-1} \cap M_n \subset B_m$ for each fixed n and all m. As a consequence we have

$$B_{n-1} \cap M_n \subset K \quad (n = 1, 2, \ldots). \tag{2.2}$$

Moreover, it is easy to see that we have the inclusions

$$a_n U \cap M_{n+1} \subset B_n \cap M_{n+1} \quad (n = 0, 1, \ldots). \tag{2.3}$$

Applying (2.2) as well as F. Riesz's Lemma we can estimate

$$a_0 = \sup_{X \in B_0} \|x\| \geqslant \sup_{X \in K} \|x\| \geqslant \sup\{\|x\|: x \in B_0 \cap M_1\} = a_0 \tag{2.4}$$

and $a_n = \sup\{xM_n: x \in a_n U \cap M_{n+1}\}$ $(n = 1, 2, \ldots)$.

As the $a_n U \cap M_{n+1}$ are compact sets, there are vectors $x_n \in a_n U \cap M_{n+1}$ for which the suprema are attained. Then we can estimate

$$a_0 \geqslant a_k \geqslant a_n = x_n M_n = \|x_n\| \geqslant x_n M_k \quad (k = 1, \ldots, n;\ n = 1, 2, \ldots).$$

From these inequalities, (2.1), (2.2) and (2.3) we obtain

$$a_n \leqslant \sup\{xM_n: x \in B_n \cap M_{n+1}\} \leqslant \sup_{X \in K} xM_n \leqslant \sup_{X \in B_{n-1}} xM_n \leqslant a_n. \tag{2.5}$$

Because of (2.4) and (2.5) we get

$$\delta_n(K) \leqslant a_n \quad (n = 0, 1, \ldots) \tag{2.6}$$

and the compactness of the set K.

The rest of the proof consists in showing that the inequalities $\delta_n(K) \geqslant a_n$ $(n = 0, 1, \ldots)$ are also true. This will be done by means of a result of M. A. Krasnoselskij, M. G. Krein and P. D. Milman (cf. [4] or [3] or [9], p. 250), which is a refinement of F. Riesz's Lemma. It states that for any linear subspace $F \subset E$ with $\dim F \leqslant n$ there is a vector $y_0 \neq o$ in M_{n+1} such that $\|y_0\| = y_0 F$. The vector $y_n := a_n \|y_0\|^{-1} y_0$ belongs to $a_n U \cap M_{n+1}$ and has o as a best approximation in F. Together with (2.2) and (2.3) we obtain

$$a_n = \|y_n\| = y_n F \leqslant \sup\{xF: x \in a_n U \cap M_{n+1}\} \leqslant \sup_{x \in K} xF \quad (n = 1, 2, \ldots).$$

These estimates, the inclusion $a_0 U \cap M_1 \subset K$ and (2.6) show that $\delta_n(K) = a_n$ holds. Theorem 6 is proved.

Remark. For the special case $(E, \|\cdot\|) = l^1$ the same result is given in [7], p. 130, by other means.

The following questions may be of a certain interest:

Is the existence of non-increasing null sequences (a_n) with $\delta_n(K) = O(a_n)$ for all compact sets of an F-space E a topological linear property? Does this property characterize the space s in a certain class of F-spaces (of course, such a class should not only contain the B-spaces and the spaces s)? Is this property related to conditions listed in section 1 to characterize the space s?

Concerning the last question it is easy to see that for an F-space E, in which one sequence of finitely-dimensional subspaces approximates slowly, there is a non-increasing null sequence of positive numbers $r_0, r_1, \ldots$ such that

$$\delta_n(B) = O(r_n)$$

for each bounded set $B \subset E$. In this case either each closed bounded set is compact (cf. "(FM)-Räume" in [5], p. 372) or the compact sets cannot be characterized among the closed bounded sets by

$$\lim_{n\to\infty} \delta_n(B) = 0.$$

This characterization is possible e.g. in complete p-normed spaces $(0 < p \leqslant 1)$, where the p-norm differs from a usual norm only by the property $\|ax\| = |a|^p \|x\|$ instead of $= |a|\,\|x\|$ $(a \in C, x \in E)$ (cf. [7], p. 131) or in the space s. May be these facts give hints for an answer to the question at the end of section 1.

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Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

A theorem of Eilenberg-Watts type for tensor products of Banach spaces

by

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Introduction. By the *tensor product of Banach spaces* X and Y we shall mean the projective tensor product $X \hat{\otimes} Y$ defined as the completion of the algebraic tensor product $X \otimes Y$ withrespect to the greatest cross norm

$$\|u\| = \inf\Big\{\sum_{i=1}^{n} \|x_i\|\,\|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i,\ x_i \in X,\ y_i \in Y\Big\}.$$

It is well known (cf. Grothendieck [4], Ch. I, § 1, no. 2, Proposition 3 and Théorème 2, Buchwalter [1], p. 33) that for each fixed Banach space A the tensor product $A \hat{\otimes} X$ has the following properties:

(α) If $\varphi: X \to Y$ is a bounded linear operator onto a dense subset of Y, then the induced operator

$$A \hat{\otimes} X \xrightarrow{A \hat{\otimes} \varphi} A \hat{\otimes} Y$$

maps $A \hat{\otimes} X$ onto a dense subset of $A \hat{\otimes} Y$.

(β) If Z is a closed subspace of a Banach space X, then there is a canonical isomorphism τ from $(A \hat{\otimes} X)/N$ onto $A \hat{\otimes} (X/Z)$, where N is the closed subspace of $A \hat{\otimes} X$ generated by the elements of the form $a \otimes z$ with a in A and z in Z; moreover, the corresponding diagram

$$\begin{array}{ccc} A \hat{\otimes} X & \xrightarrow{\varrho} & (A \hat{\otimes} X)/N \\ & {\scriptstyle A \hat{\otimes} \pi} \searrow & \downarrow {\scriptstyle \tau} \\ & & A \hat{\otimes} (X/Z) \end{array}$$

is commutative; here π and ϱ denote the canonical surjections.