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STUDIA MATHEMATICA, T. XXXVIII. (1970)

Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

Bases in sequentially retractive limit-spaces*

by

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The object of the present paper is to generalize the classical Banach-theorem for the continuity of coefficient functionals of bases in Banach-spaces to some inductive limit-spaces. A suitable class for the treatment is the class of limit-spaces $E = \operatorname{ind}_{n\to} E_n$, for which the convergence of sequences takes place already in some generating space E_n (E_n Fréchet-spaces). These spaces seem to be interesting for other questions too, because they include two of the three most important cases: strict (LF)-spaces and (LS)-spaces (i.e. (LF)-spaces with compact linking mappings), but *not* (LS_w)-spaces (i.e. (LF)-spaces with weakly compact linking mappings; in particular those with generating reflexive normed spaces). The terminology is essentially that of [3].

1. Sequentially retractive sequences.

1.1. A sequence $E_1 \subset E_2 \subset \ldots$ of (F)-spaces E_n is called *sequentially rectractive* if for every sequence (x_i) converging in the (locally convex) inductive limit $E = \operatorname{ind}_{n\to} E_n$ there is an index n such that (x_i) converges in E_n. Furthermore, the limits of (x_i) coincide.

1.2. Sequentially retractive generated (LF)-spaces are separated and sequentially complete.

If an (LF)-space E is twice generated

$$E = \operatorname{ind}_{m\to} E_m = \operatorname{ind}_{n\to} F_n, \qquad E_m, F_n \ (F)\text{-spaces},$$

then by a theorem of *Grothendieck* ([5], I, p. 17) these sequences are mutually cofinal (i.e. for every n there is an m such that $E_n \hookrightarrow F_m$ and $F_n \hookrightarrow E_m$; "$\hookrightarrow$" means continuous embedding). Thus, if one generating sequence of an (LF)-space is sequentially retractive then all are, and one can speak of *sequentially retractive (LF)-spaces*.

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THEOREM. *An (LF)-space E is sequentially retractive if and only if E is regular and fulfills Mackey's condition of convergence.*

(Recall that an (LF)-space $E = \operatorname{ind} E_n$ is *regular* if every bounded set is already bounded in some E_n, and a locally convex space E fulfills *Mackey's condition of convergence* if every convergent sequence converges in the span of a bounded absolutely convex subset with its gauge.)

1.3. The arrangement with the some special spaces gives the following

COROLLARY. (1) *(F)-spaces are sequentially retractive.*

(2) *(LS)-spaces are sequentially retractive. Nuclear, sequentially complete (or reflexive) (DF)-spaces are (LN)-spaces* [8] *and thus (LS).*

(3) *Strictly generated (LF)-spaces* [2] *are sequentially retractive.*

(4) *Sequentially complete (LF)-spaces satisfying Mackey's condition of convergence are sequentially retractive.*

(5) *Bornological sequentially complete (DF)-spaces satisfying Mackey's condition of convergence are sequentially retractive (LB)-spaces.*

(6) *Strong duals of distingué (or even reflexive) quasinormable* ([4], p. 106) *(F)-spaces are sequentially retractive (LB)-spaces.*

(7) *(LS_w)-spaces are in general not sequentially retractive.*

(8) *There is a complete (thus regular) Montel (LB)-space which is not sequentially retractive.*

2. Continuity of coefficient functionals of bases.

2.1. A sequence (x_n) of elements of a locally convex space E forms a *basis*, if every $x \in E$ has a unique expansion

$$x = \sum_{n=1}^{\infty} a_n x_n, \qquad a_n \in \boldsymbol{R} \ (\text{resp.} \in \boldsymbol{C})$$

and a *Schauder-basis*, if all coefficient functionals $x \rightsquigarrow a_n = a_n(x)$ are continuous.

2.2. THEOREM. *Every basis in a sequentially retractive (LF)-space is a Schauder-basis.*

Proof. (a) Let be $E = \operatorname{ind}_{n\to} E_n$, E_n Banach-spaces with norm $|\cdot|_n$ and (y_i) a basis of E with the expansion-operators

$$T_n x = \sum_{i=1}^{n} a_i(x) y_i$$

whose continuity is to be proved.

(b) $G_n = \{x \in E_n \mid T_m x \in E_n$ for all m and $T_m x \underset{E_n}{\to} x\} \subset E_n$ is the space of all $x \in E_n$ whose expansions are all situated in E_n and converge there to x. Obviously $G_n \subset G_{n+1}$ and the assumed sequential retractivity ensures

$$\bigcup_{n=1}^{\infty} G_n = E.$$

(c) Defining

$$\|x\|_n = \sup_{m \in \boldsymbol{N}} |T_m x|_n, \qquad x \in G_n,$$

G_n is a normed space: $\|x\|_n < \infty$ by the boundedness of the (in E_n convergent) sequence $(T_m x)$; the inequality

$$|x|_n = \lim_{m\to\infty} |T_m x|_n \leqslant \sup_{m \in \boldsymbol{N}} |T_m x|_n = \|x\|_n$$

establishes the continuity of the embedding $G_n \hookrightarrow E_n$; in particular, $(G_n, \|\cdot\|_n)$ is separated. Homogeneity and Minkowski's inequality are obvious.

(d) The restricted expansion operators T_l map G_n into G_n and are continuous by

$$\|T_l x\|_n = \sup_m |T_m T_l x|_n = \sup_{m \leqslant l} |T_m x|_n \leqslant \|x\|_n.$$

(e) By the continuous embeddings $G_n \hookrightarrow E_n$, (c) and (b) the identity

$$\varphi\colon \operatorname{ind}_{n\to} G_n \hookrightarrow \operatorname{ind}_{n\to} E_n$$

is continuous and bijective.

(f) To apply a closed-graph-theorem, it is convenient to prove the completeness of the normed spaces G_q. A Cauchy-sequence (x_n) in G_q

$$(+) \qquad \sup_{k,l \geqslant n} \sup_m |T_m x_k - T_m x_l|_q \to 0 \qquad (n \to \infty),$$

is also a Cauchy-sequence in E_q and has an E_q-limit x. By (+), all $(T_n x_k)_k$ form Cauchy-sequences in E_q, such that limits

$$x^n \in [y_1, \ldots, y_n] \cap E_q = F_n$$

($\dim F_n \leqslant n$) exist satisfying

$$(++) \qquad |T_n x_k - x^n|_q \leqslant \sup_{i,j \geqslant k} \|x_i - x_j\|_q \to 0 \qquad (k \to \infty);$$

particularly, $T_n x_k \underset{G_q}{\to} x^n$ $(k \to \infty)$.

The aim is to show $x^n = T_n x$.

Firstly, the inequality

$$|x^n - x|_q \leqslant |x^n - T_n x_l|_q + |T_n x_l - x_l|_q + |x_l - x|_q$$

holds. Choosing $l = l_0$ (by $x_l \underset{E_q}{\to} x$) such that

$$|x_{l_0} - x|_q \leqslant \varepsilon$$

and (by (++))

$$|x^n - T_n x_{l_0}|_q \leqslant \sup_{i,j \geqslant l_0} \|x_i - x_j\|_q \leqslant \varepsilon,$$

the convergence of the expansion of $x_{l_0} \in G_q$ in E_q gives an n_0 satisfying

$$|T_n x_{l_0} - x_{l_0}|_q \leqslant \varepsilon$$

for all $n \geqslant n_0$; thus

$$|x^n - x|_q \leqslant \varepsilon$$

for all $n \geqslant n_0$ is proved and (x^n) converges in E_q to x.

The operator $T_n - T_{n-1}$ is continuous on the finitedimensional space F_m so that $(m \geqslant n+1)$

$$\begin{aligned}[y_n] \ni a_n(x^m) y_n &= (T_n - T_{n-1})(\lim_{k\to\infty} T_m^1 x_k) \\ &= \lim_{k\to\infty} (T_n - T_{n-1})(T_m x_k) = \lim_{k\to\infty} T_n x_k - \lim_{k\to\infty} T_{n-1} x_k \\ &= x^n - x^{n-1}.\end{aligned}$$

But

$$x = \lim_{n\to\infty} x^n = x^1 + (x^2 - x^1) + \ldots + (x^{n+1} - x^n) + \ldots$$

in E_q, all the more in E; therefore the uniqueness of the expansion of x with respect to the basis (y_i) yields

$$x^n - x^{n-1} = a_n(x^m) y_n = a_n(x) y_n,$$

thus $T_n x = x^n$; in particular, the expansion of x converges in E_q: $x \in G_q$.

Furthermore, by (++)

$$\begin{aligned}\|x_k - x\|_q &= \sup_m |T_m x_k - T_m x|_q = \sup_m |T_m x_k - x^m|_q \\ &\leqslant \sup_{i,j \geqslant k} \|x_i - x_j\|_q \to 0 \qquad (k \to \infty)\end{aligned}$$

and G_q is complete.

(g) By de Wilde's closed-graph-theorem [7], φ^{-1} is continuous, so that the equality

$$\operatorname{ind}_{n\to} E_n = \operatorname{ind}_{n\to} G_n$$

holds topologically; but by (d) the expansion operators are continuous on $\operatorname{ind} G_n$ so on E.

(h) The restriction that all E_n were Banach- and not Fréchet-spaces was made only for technical reasons: substitute for the norm $|\cdot|_n$ the semi-norms $p_{n,r}$, $r \in N$, of the (F)-space E_n, define the corresponding semi-norms on G_n and prove completeness by the given method; the same closed-graph-theorem is applicable.

2.3. In view of Corollary 1.3, the theorem ensures the continuity of coefficient functionals of bases in (F)-spaces (originally proved by Newns [6]), strict (LF)-spaces (Arsove-Edwards [1]; the spaces considered in theorem 12 of that paper are sequentially retractive, too), in (LS)-spaces, some nuclear spaces (specified in 1.3.(2)), and some classes of (DF)-spaces.

2.4. Another look at the proof yields the

COROLLARY. *The set of y_i with $y_i \in G_n$ forms a basis of G_n.*

For by the convergence of $(T_m x)$ in E_n to x $(x \in G_n)$

$$\begin{aligned}\|T_m x - x\|_n &= \sup_i |T_i T_m x - T_i x|_n \\ &= \sup_{i>m} |T_m x - T_i x|_n \to 0 \qquad (m \to \infty).\end{aligned}$$

2.5. A particular result is that every sequentially retractive (LF)-space with a basis can be represented by an equivalent (i.e. mutually cofinal in the terminology of 1.2.) sequence of (F)-spaces G_n with bases in such a manner, that the basis of G_n grows out of the basis of G_{n-1} by prolongation or (and) enlargement of the associated coefficient-space. An (LS)-Köthe-sequence space $\operatorname{ind}_{n\to} l^p(b^n)$ (with the unit vectors as basis) is an example which enlarges only the coefficient spaces.

2.6. The (weakened) inverse question: "Does an (LF)-space generated by (F)-spaces with bases, have a basis" seems to be incomparably more difficult and is unsolved. Even in the case of nuclear (LN)-spaces (a sequence of Hilbert spaces with nuclear embeddings can be established immediately), this problem, which is equivalent with the existence of bases in nuclear (F)-spaces, is not yet solved.

Added in proof. De Wilde (p. 457) has improved Theorem 2.2.

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Remarks on a theorem of S. N. Bernstein

by

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As usually $C(0,1)$ denotes the space of all real-valued continuous functions on the closed intervall $[0,1]$. It is well-known that the set of polynomials is dense in $C(0,1)$ with respect to the supremum norm $\|\cdot\|$. If we introduce the minimal deviation

$$d_n(f) = \inf\|f-P_n\|$$

of a function $f \in C(0,1)$ from the linear subspace of polynomials P_n of degree $\leqslant n$, the density of the polynomials can be expressed by

$$\lim_{n\to\infty} d_n(f) = 0 \qquad \big(f\in C(0,1)\big).$$

S. N. Bernstein has shown that for each non-increasing null sequence (a_n) of non-negative numbers there is a $g \in C(0,1)$ with

$$d_n(g) = a_n.$$

Shapiro [8] generalized this theorem in the following way. He replaced $C(0,1)$ by an arbitrary B-space $(E, \|\cdot\|)$ and the sequence of n-dimensional subspaces of polynomials with degree $\leqslant n$ by a sequence (M_n) of proper closed linear subspaces in E. In this case for each null sequence (a_n) of non-negative numbers there is a vector $x \in E$ with

$$xM_n := \inf_{u\in M_n}\|x-u\| \neq O(a_n).$$

For two sequences (b_n) and (c_n) of non-negative numbers the formula

$$c_n \neq O(b_n)$$

means that there is no $A>0$ with $c_n \leqslant Ab_n$ for all n. We shall say briefly that each sequence of proper closed linear subspaces in a B-space *approximates slowly*. In his proof Shapiro used the category argument. Therefore the question seems naturally, whether Shapiro's statement also holds in F-spaces. In this paper *F-space* means a complete metric