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Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

Biprojective tensor products and convolutions of vector-valued measures on a compact group

by

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Introduction. If μ and ν are regular complex-valued Borel measures on a compact Hausdorff group G, then the convolution of μ and ν can be defined by appealing to the Riesz representation theorem and letting $\mu*\nu$ be that unique regular Borel measure on G for which

$$\int_G f(z)\,d\mu*\nu(z) = \int_G \Big\{\int_G f(xy)\,d\mu(x)\Big\}\,d\nu(y)$$

holds for all continuous functions f on G. Moreover, if $\mathscr{M}(G)$ denotes the set of all complex-valued countably additive, regular Borel measures on G, then $\mathscr{M}(G)$ may be made a Banach space if we define linear operations pointwise and the norm as $\|\mu\| = |\mu|(G)$ (total variation of μ). Further $\mathscr{M}(G)$ with convolution multiplication is a Banach algebra (cf. [16], [10], [13]).

In this paper similar questions are dealt with for vector-valued measures. In the Banach space $\mathrm{rca}(\mathscr{B}(G), X)$ of all regular countably additive Borel measures with values in a Banach space X a convolution multiplication is introduced which is a bounded bilinear mapping from $\mathrm{rca}(\mathscr{B}(G), X)\times\mathrm{rca}(\mathscr{B}(G), X)$ into $\mathrm{rca}(\mathscr{B}(G\times G), X\check{\otimes} X)$, where $X\check{\otimes} X$ is the biprojective tensor product of X by X. Some properties of the convolution are given, further results will be given elsewhere.

1. Preliminaries. We need a generalization, for vector-valued measures, of the classical theorem asserting the existence of the product of measures defined on two measurable spaces. This is established in such a way that usual product of two scalars is replaced by the tensor product of two vectors. Namely, let measurable spaces $(S,\mathscr{S})$ and $(T,\mathscr{T})$, complete locally convex topological vector spaces X and Y, and (countably additive) vector-valued measures $\mu\colon\mathscr{S}\to X$ and $\nu\colon\mathscr{T}\to Y$ be given. We denote by $\mathscr{S}\otimes_\sigma\mathscr{T}$ the σ-ring generated by the sets of the form $E\times F$, $E\in\mathscr{S}$,

$F \in \mathcal{T}$. We endow the algebraic tensor product $X \otimes Y$ with the topology determined by the system of seminorms defined as follows:

$$\Big|\sum_{i=1}^{k} x_i \otimes y_i\Big|_{(\alpha,\beta)}^{\vee}$$

$$= \sup\Big\{\Big|\sum_{i=1}^{k} \langle x_i, x'\rangle \langle y_i, y'\rangle\Big| : \|x'\|_\alpha \leqslant 1,\ x' \in X';\ \|y'\|_\beta \leqslant 1,\ y' \in Y'\Big\},$$

$$(\alpha, \beta) \in A \times B,$$

where A, resp. B is the system of continuous seminorms defining a locally convex topology in X, resp. Y; X' and Y' denote dual spaces of X and Y, respectively; for $x' \in X'$ we denote $\|x'\|_\alpha = \sup\{|\langle x, x'\rangle| : \|x\|_\alpha \leqslant 1, x \in X\}$ for every $\alpha \in A$. Similarly for Y. We call this topology (called in [8] the topology of bi-equicontinuous convergence and in [15] the ε-topology) the *biprojective tensor topology* (cf. also [14], where this topology is called inductive). The completion of the space $X \otimes Y$ under this topology we call the *biprojective tensor product* $X \check{\otimes} Y$ of the spaces X and Y.

In [4] the following proposition is established:

PROPOSITION 1. *Let $\mathcal{S}$ and $\mathcal{T}$ be σ-rings. Let $\mu: \mathcal{S} \to X$ and $\nu: \mathcal{T} \to Y$ be vector-valued measures. Then there exists a unique vector-valued measure $\lambda = \mu \check{\otimes} \nu: \mathcal{S} \otimes_\sigma \mathcal{T} \to X \check{\otimes} Y$ for which $\lambda(E \times F) = \mu(E) \otimes \nu(F)$ for $E \in \mathcal{S}$, $F \in \mathcal{T}$.*

Let $\mathcal{B}_0(S)$, $\mathcal{B}_0(T)$, resp. $\mathcal{B}(S)$, $\mathcal{B}(T)$ stand for the σ-ring of Baire, resp. Borel sets in a locally compact Hausdorff spaces S, T (cf. [9] or [2]). In the sequel we need the results proved in [5].

PROPOSITION 2. *If $\mu_0: \mathcal{B}_0(S) \to X$ is a vector-valued Baire measure on S and $\nu_0: \mathcal{B}_0(T) \to Y$ is a vector-valued Baire measure on T, then $\lambda_0 = \mu_0 \check{\otimes} \nu_0: \mathcal{B}_0(S \times T) \to X \check{\otimes} Y$ is a vector-valued Baire measure on $S \times T$.*

Recall that every vector-valued Baire measure $\mu_0: \mathcal{B}_0(S) \to X$ is regular (cf. [3] and [12]).

PROPOSITION 3. *If $\mu: \mathcal{B}(S) \to X$ and $\nu: \mathcal{B}(T) \to Y$ are regular vector-valued Borel measures on S, T, respectively, then there exists one and only one regular vector-valued Borel measure $\varrho: \mathcal{B}(S \times T) \to X \check{\otimes} Y$ on $S \times T$ which extends $\mu \check{\otimes} \nu$.*

Of course $\varrho = \mu \hat{\otimes} \nu$ if $\mathcal{B}(S) \otimes_\sigma \mathcal{B}(T) = \mathcal{B}(S \times T)$ (for example if all bounded subspaces of either S or T are metrizable, cf. [11]).

A bilinear mapping $U: X \times Y \to Z$, where Z is a locally convex space, is said to be *hypercontinuous* (cf. [14]) if the linear mapping $\overline{U}$ induced by U on $X \otimes Y$ is continuous in the biprojective tensor topology. From Proposition 1 we have

PROPOSITION 4. *Let $U: X \times Y \to Z$ be a hypercontinuous bilinear mapping and let Z be a sequentially complete locally convex space. Let $\mu: \mathcal{S} \to X$ and $\nu: \mathcal{T} \to Y$ be vector-valued measures.*

Then there exists a unique vector-valued measure $\lambda: \mathcal{S} \otimes_\sigma \mathcal{T} \to Z$ for which

$$\lambda(E \times F) = U(\mu(E), \nu(F)), \qquad E \in \mathcal{S}, F \in \mathcal{T}.$$

The measure $\lambda: \mathcal{S} \otimes_\sigma \mathcal{T} \to Z$ is defined by the equation $\lambda = \overline{U}\lambda = \overline{U}(\mu \check{\otimes} \nu)$ (cf. [1], Theorem 2.6.)

In the remainder of the paper, X and Y are Banach spaces, S and T are compact Hausdorff spaces. We denote by $\mathrm{rca}(\mathcal{B}(S), X)$ the family of all regular (countably additive) Borel measures with values in the Banach space X. The set $\mathrm{rca}(\mathcal{B}(S), X)$ is a Banach space if the norm of any $\mu \in \mathrm{rca}(\mathcal{B}(S), X)$ is defined as the semivariation of μ:

$$\|\mu\| = \|\mu\|(S) = \sup \Big\|\sum_{i=1}^{n} \alpha_i \mu(E_i)\Big\|,$$

where the supremum is taken over all finite collections of disjoint sets $E_i \in \mathcal{B}(S)$, $i = 1, 2, \ldots, n$, $\bigcup_{i=1}^{n} E_i = S$, and all finite systems of scalars $\alpha_1, \ldots, \alpha_n$ with $|\alpha_i| \leqslant 1$ (cf. [1] and [6], III. 7 and IV. 10. 4). In paper [7] the symbol $N_s(S, X)$ is used in place of $\mathrm{rca}(B(S), X)$. It follows from paper [1] that $\mathrm{rca}(\mathcal{B}(S), X) = N_s(S, X)$ may be identified with the Banach space of all weakly compact transformations from $C(S)$, the Banach space of all continuous functions on S, into X. This identification is done by means of the formula

$$Uf = \int_S f(s)\, d\mu(s)$$

for $U: C(S) \to X$ weakly compact, $\mu \in \mathrm{rca}(\mathcal{B}(S), X)$ and $f \in C(S)$.

2. The convolution of vector-valued measures. Let $S = T = G$ be a compact Hausdorff group, i.e. G is a compact Hausdorff space with a jointly continuous associative, binary operation $p: G \times G \to G$. We write $p(x, y) = xy$.

THEOREM 1. *Let μ be in $\mathrm{rca}(\mathcal{B}(G), X)$ and ν be in $\mathrm{rca}(\mathcal{B}(G), Y)$. Define an operation L on $C(G)$ by the formula*

$$L(f) = \int_{G\times G} f(st)\, d\varrho(s, t), \qquad f \in C(G),$$

where $\varrho: \mathcal{B}(G \times G) \to X \otimes Y$ is the vector-valued Borel measure from Proposition 3 ($\varrho = \mu \check{\otimes} \nu$ if $\mathcal{B}(G \times G) = \mathcal{B}(G) \otimes_\sigma \mathcal{B}(G)$; in this case we may write

$$L(f) = \int_{G\times G} f(st)\, d(\mu \check{\otimes} \nu)(s, t).$$

Then L is a weakly compact linear operator from $C(G)$ into $X \check{\otimes} Y$.

Proof. The regular vector-valued Borel measure ϱ is defined on $\mathscr{B}(G\times G)$ and has values in $X\check{\otimes} Y$. According to [1], Theorem 3.2, and [6], VI.7.3, the set

$$\Big\{\int_G g\,d\varrho : \|g\|\leqslant 1,\ g\in C(G\times G)\Big\}$$

is weakly relatively compact in $X\check{\otimes} Y$. It follows that the set

$$\{L(f):\ \|f\|\leqslant 1,\ f\in C(G)\}$$

is also weakly relatively compact in $X\check{\otimes} Y$, i.e. L is a weakly compact linear operator from $C(G)$ into $X\check{\otimes} Y$. The proof is complete.

According to [1], Theorem 3.2, or [6], IV.7.3, it follows from Theorem 1 that there exists a unique regular vector-valued Borel measure $\varrho_*: \mathscr{B}(G)\to X\check{\otimes} Y$ such that

$$L(f) = \int_G f(s)\,d\varrho_*(s), \quad f\in C(S),$$

and $\|L\| = \|\varrho_*\|(G)$.

Definition. Let μ and ν be as in Theorem 1, and let $\mu*\nu$ denote the unique member of $\mathrm{rca}(\mathscr{B}(G), X\check{\otimes} Y)$ such that

$$L(f) = \int_G f\,d\mu*\nu, \quad f\in C(G).$$

Then $\mu*\nu: \mathscr{B}(G)\to X\check{\otimes} Y$ is called the *convolution* of μ and ν.

THEOREM 2. *If μ is in $\mathrm{rca}(\mathscr{B}(G), X)$ and ν is in $\mathrm{rca}(\mathscr{B}(G), Y)$, then for the weakly compact linear operator from $C(G)$ into $X\check{\otimes} Y$ defined by the formula*

$$L(f) = \int_{G\times G} f(st)\,d\varrho(s,t), \quad f\in C(G),$$

there holds

$$\|L\|\leqslant \|\mu\|(G)\,\|\nu\|(G).$$

Proof. We have

$$\|L\| = \sup_{\|f\|\leqslant 1}\|L(f)\| = \sup_{\|f\|\leqslant 1}\Big\|\int_{G\times G} f(st)\,d\varrho(s,t)\Big\| \leqslant \|f\|\ \|\varrho\|(G\times G)\leqslant \|\varrho\|(G\times G).$$

Take an arbitrary $\varepsilon>0$ and consider the number $\|\sum_i a_i\varrho(G_i)\|$ for any scalars a_i with $|a_i|\leqslant 1$ and $G_i\in\mathscr{B}(G\times G)$ mutually disjoint. It follows

([2], p. 221), by the regularity of ϱ, that there exist Baire sets H_i such that

$$\Big\|\sum_i a_i\varrho(G_i)\Big\|\leqslant\Big\|\sum_i a_i\varrho(H_i)\Big\|+\varepsilon = \Big\|\sum_i a_i\mu\check{\otimes}\nu(H_i)\Big\|^{\vee}+\varepsilon.$$

Further, there exist disjoint Baire sets $\bigcup_{j=1}^{n_i} E_j^i\times F_j^i\in\mathscr{B}_0(G\times G)$ such that

$$\begin{aligned}\Big\|\sum_i a_i\mu\check{\otimes}\nu(H_i)\Big\|^{\vee} &\leqslant \Big\|\sum_i a_i\sum_j \mu(E_j^i)\otimes\nu(F_j^i)\Big\|^{\vee}+\varepsilon\\ &= \sup_{\substack{\|x'\|\leqslant 1\\ \|y'\|\leqslant 1}}\Big|\sum_i a_i\sum_j\langle\mu(E_j^i),x'\rangle\cdot\langle\nu(F_j^i),y'\rangle\Big|+\varepsilon\\ &\leqslant \sup_{\substack{\|x'\|\leqslant 1\\ \|y'\|\leqslant 1}}|\langle\mu(\ ),x'\rangle\times\langle\nu(\ ),y'\rangle|(G\times G)+\varepsilon\\ &= \sup_{\|x'\|\leqslant 1}|\langle\mu(\ ),x'\rangle|(G)\cdot\sup_{\|y'\|\leqslant 1}|\langle\nu(\ ),y'\rangle|(G)+\varepsilon\\ &= \|\mu\|(G)\,\|\nu\|(G)+\varepsilon.\end{aligned}$$

It follows that $\|\varrho\|(G\times G)\leqslant\|\mu\|(G)\,\|\nu\|(G)$.

COROLLARY. *If μ is in $\mathrm{rca}(\mathscr{B}(G), X)$ and ν is in $\mathrm{rca}(\mathscr{B}(G), Y)$, then*

$$\|\mu*\nu\|\leqslant\|\mu\|\ \|\nu\|.$$

In fact, if μ and ν are in $\mathrm{rca}(\mathscr{B}(G), X)$, $\mathrm{rca}(\mathscr{B}(G), Y)$, respectively, then

$$\begin{aligned}\|\mu*\nu\| &= \sup_{\substack{|a_i|\leqslant 1\\ G_i\subset G}}\Big\|\sum_i a_i\mu*\nu(G_i)\Big\| = \sup_{\substack{\|f\|\leqslant 1\\ f\in C(G)}}\Big\|\int f(s)\,d\mu*\nu(s)\Big\|\\ &= \sup_{\substack{\|f\|\leqslant 1\\ f\in C(G)}}\Big\|\int f(st)\,d\varrho(s,t)\Big\| = \|L\|\leqslant\|\mu\|\ \|\nu\|.\end{aligned}$$

It is easy to see that $\mu*\nu$ is bounded bilinear mapping from the product $\mathrm{rca}(\mathscr{B}(G), X)\times\mathrm{rca}(\mathscr{B}(G), Y)$ into $\mathrm{rca}(\mathscr{B}(G), X\check{\otimes} Y)$.

If X is a complete locally convex algebra such that multiplication is hypercontinuous, then in virtue of Propositions 3 and 4, for any $\mu,\nu\in\mathrm{rca}(\mathscr{B}(G), X)$ there exists a unique regular Borel vector-valued measure $\varrho: \mathscr{B}(G\times G)\to X$ for which $\varrho(E\times F) = \mu(E)\nu(F)$ for $E, F\in\mathscr{B}(G)$.

In particular, it may be proved that if in a Banach algebra X the multiplication is hypercontinuous (for example if X is finite-dimensional), then $\mathrm{rca}(\mathscr{B}(G), X)$ with convolution multiplication is a Banach algebra which is commutative if G is. Further results in this direction will be published later.

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On a theorem of L. Schwartz and its applications to absolutely summing operators

by

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1. Introduction. In a recently published paper [12], L. Schwartz has presented a theory of p-radonifying operators. He has found numerous applications of its main theorem (the "duality theorem for radonifying operators"; cf. [13]). It has been also observed by him that p-radonifying operators and p-absolutely summing are "almost identical". This creates the possibility of applying the "duality theorem" to the theory of p-absolutely summing operators. The aim of this paper is to show how this theorem may be used to obtain in a simple way already known results as well as new ones in the theory of p-absolutely summing operators. To make this paper self-contained § 2 restates some of the results of L. Schwartz. We use here neither the theory of cylindrical measures nor of radonifying operators. All the theorems are formulated in the language of absolutely summing operators. The "duality theorem" is essentially the same as Theorem 1 of this paper.

Let us recall that if E, F are Banach spaces and $0 < p < +\infty$, then an operator $u\colon E \to F$ is said to be *p-absolutely summing* (we shall write $u \in \pi_p(E, F)$) if there exists a constant C such that for each $x_1, \ldots, x_n \in E$

$$\sum_{i=1}^{n} \|u(x_i)\|^p \leqslant C \sup_{x' \in E', \|x'\| \leqslant 1} \sum_{i=1}^{n} |\langle x_i, x'\rangle|^p.$$

u is said to be *0-absolutely summing* ($u \in \pi_0(E, F)$) if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x_1, x_2, \ldots, x_n \in E$ and

$$\sup_{x' \in E', \|x'\| \leqslant 1} \sum_{i=1}^{n} \frac{1}{n} \min\{1, |\langle x_i, x'\rangle|\} < \delta,$$