Literaturnachweis

[1] N. Bourbaki, *Espaces vectoriels topologiques*, Chap. II, Paris 1953.
[2] K. Floret und J. Wloka, *Einführung in die Theorie der lokalkonvexen Räume*, 1968.
[3] A. Pietsch, *Nukleare lokalkonvexe Räume*, Berlin 1965.
[4] — *Quasinukleare Abbildungen in normierten Räumen*, Math. Annalen 165 (1966), S. 76-90.
[5] — und H. Triebel, *Interpolationstheorie für Banachideale von beschränkten linearen Operatoren*, Studia Math. 31 (1968), S. 95-109.
[6] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in L_p-spaces and their applications*, ibidem 29 (1968), S. 275-326.

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Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

The conjugate of the product of operators

by

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Throughout this paper S is a linear operator with domain $D(S)$ dense in Banach space X, kernel $N(S)$ and range $R(S)$ in Banach space Y. T is a linear operator with domain dense in Y and range in Banach space Z. T' and X' denote the conjugate of T and X, respectively.

In [3], Schechter proved the following theorem (Gustafson, Bull. A. M. S. 75, no. 4, proved the theorem for Hilbert space):

1. THEOREM. *If S and T are closed and $R(S)$ has finite codimension, then $(TS)' = S'T'$.*

We show that in a Hilbert space setting, Schechter's theorem is best in the following sense:

2. THEOREM. *Let S be a densely defined closed linear operator with domain and range in Hilbert space H. The adjoint $(TS)^* = S^*T^*$ for all closed densely defined linear operators T with domain and range in H if and only if $R(S)$ has finite codimension.*

In proving theorem 2 the following lemma is used:

3. LEMMA. *Given a closed infinite-dimensional subspace M of Hilbert space H, there exists a closed linear operator with domain a proper dense subspace of H and range contained in M.*

Proof. Choose an infinite orthonormal subset $x_1, x_2, \ldots$ of M with closed linear span denoted by N. Define the map K from N into N by

$$Kz = \sum_{1}^{\infty} 2^{-k}\langle z, x_k\rangle x_k.$$

Then K is compact and 1-1. Since $Kx_j = 2^{-j}x_j$ and K is compact, $R(K)$ is a proper dense subspace of N by [2], III. 1.12. Hence $D = R(K) \oplus \oplus N^{\perp} \cap M \oplus M^{\perp}$ is a proper dense subspace of H and $B: D \to M$, defined by $B(Kz+u+v) = z$, $z \in N$, $u \in N^{\perp} \cap M$ and $v \in M^{\perp}$, is easily seen to be closed.

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Returning to the proof of theorem 2, suppose $(TS)^* = S^*T^*$ for all closed densely defined linear operators T from H to H. Since S^*S is self-adjoint, [1], XII. 7.1, and $N(S^*) = N(SS^*)$, we have $N(S^*)^{\perp} = N(SS^*)^{\perp} = \overline{R(SS^*)}$. Hence $H = \overline{R(SS^*)} \oplus N(S^*)$ and

$$D = \{Sr+n : r \in \overline{R(S^*)} \cap D(S),\ n \in N(S^*)\}$$

is dense in H. Define linear operator T on D by $T(Sr+n) = r$. T is well defined since $Sr = Sr_1$ implies $r - r_1 \in \overline{R(S^*)} \cap N(S) = \overline{R(S^*)} \cap R(S^*)^{\perp} = (0)$. T is closed, for suppose $Sr_k + n_k \to x$ and $r_k \to r$, where $\{r_k\} \subset \overline{R(S^*)} \cap D(S)$ and $\{n_k\} \subset N(S^*) = R(S)^{\perp}$. Then $\{n_k\}$ converges to some $n \in N(S^*)$ and $Sr_k \to x - n$. Since S is closed, r is in $\overline{R(S^*)} \cap D(S)$ and $Sr = x - n$. Hence $x = Sr + n \in D(T)$ and $Tx = r$ which shows that T is closed. In addition, $D\big((TS)^*\big) = H$, for suppose $u \in D(TS)$; i.e., $Su = Sr + n$ for some $r \in \overline{R(S^*)} \cap D(S)$ and $n \in N(S^*)$. Then

$$n \in R(S) \cap N(S^*) = R(S) \cap R(S)^{\perp} = (0)$$

and, therefore, $u - r \in N(S) = R(S^*)^{\perp}$. Hence

$$D(TS) = \{r + z : r \in \overline{R(S^*)} \cap D(S),\ z \in R(S^*)^{\perp}\}.$$

Given $y \in H$ and $u = r + z \in D(TS)$,

$$|\langle TSu, y\rangle| = |\langle TSr, y\rangle| = |\langle r, y\rangle| \leqslant \|r\|\,\|y\| \leqslant \|u\|\,\|y\|$$

which shows that y is in $D\big((TS)^*\big)$. Thus $H = D\big((TS)^*\big) = D(S^*T^*) \subset D(T^*)$ which in turn implies

$$H = D(T^{**}) = D(T) = R(S) \oplus N(S^*). \tag{1}$$

Assert that $N(S^*)$ is finite-dimensional. Suppose this is not the case. Then by lemma 3 there exists a closed linear operator B with $D(B)$ a proper dense subspace of H and range in $N(S^*)$. By hypothesis, $(B^*S)^* = S^*B^{**} = S^*B$. Since $(B^*S)^*$ is closed, $D(B) = N(S^*B)$ is closed which contradicts the property that $D(B)$ is a proper dense subspace of H. Hence $N(S^*)$ is finite-dimensional which, together with (1), proves that $R(S)$ has finite codimension.

By fixing T and letting S "vary" we have

4. THEOREM. *Suppose T is closed. Then $(TS)' = S'T'$ for all densely defined S if and only if T is bounded on Y.*

Proof. It is easy to see that if T is bounded on Y, then $(TS)' = S'T'$ for all densely defined S. Assume that $(TS)' = S'T'$ for all densely defined S but $D(T) \neq Y$. Choose $y \notin D(T)$ and $x' \neq 0$ in X'. Define the bounded operator S on X by $Sx = x'(x)y$. Then $D(TS) = N(x')$ which is not dense in X thereby contradicting the hypothesis that TS is densely defined. Thus T is defined on all of Y and is bounded by the closed graph theorem.

The remaining portion of the paper is concerned with conditions under which $(TS)' = S'T'$.

5. LEMMA. *If S is closed and $\overline{R(S) \cap D(T)} = R(S)$, then TS is densely defined, in which case $(TS)'$ is an extension of $S'T'$.*

Proof. Let $\hat{S}$ be the 1-1 operator induced by S. Since $\hat{S}^{-1}$ is closed and $R(S)$ is complete, $\hat{S}^{-1}$ is continuous on $R(S)$. Thus

$$\begin{aligned} D(S)/N(S) &= \hat{S}^{-1}R(S) = \hat{S}^{-1}\overline{R(S) \cap D(T)} \subset \overline{\hat{S}^{-1}R(S) \cap D(T)} \\ &= \overline{D(TS)/N(S)}. \end{aligned} \tag{1}$$

The assumption that S is densely defined together with (1) imply $\overline{D(TS)} = X$. The last statement in the lemma follows immediately from the definition of a conjugate operator.

The next theorem is a straightforward generalization of theorem 1. We have singled out the essential properties used by Schechter [3].

6. THEOREM. *Suppose S is closed, $Y = R(S) \oplus N$ (direct sum) and T is bounded on the closed subspace $N \subset D(T)$. Then $(TS)' = S'T'$.*

Proof. By [2], IV. 1.12, $R(S)$ is closed. Since

$$D(T) = R(S) \cap D(T) \oplus N$$

is dense in $Y = R(S) \oplus N$, it follows that $\overline{R(S) \cap D(T)} = \overline{R(S)} = R(S)$ and therefore TS is densely defined by lemma 5. Hence $(TS)'$ is an extension of $S'T'$. The rest of the argument is exactly the same as Schechter's [3].

7. COROLLARY. *Suppose S and T are closed and $Y = R(S) \oplus N$, where N is a closed subspace of $D(T)$. Then $(TS)' = S'T'$.*

8. COROLLARY. *Suppose S is closed and $R(S)$ has finite codimension in Y. Then $(TS)' = S'T'$. (T need not be closed.)*

Proof. By [2], IV. 2.8, there exists a finite-dimensional subspace N of $D(T)$ such that $Y = R(S) \oplus N$. Thus T is bounded on N and theorem 6 applies.

9. COROLLARY. *Let $X = Y = Z$ be a Hilbert space. Suppose S and T are closed with the following properties:*

(i) *$N = N(S^*) \cap D(T)$ is closed;*

(ii) *$N^{\perp} \cap N(S^*)$ is finite-dimensional;*

(iii) *$R(S)$ is closed* (e.g., $N(S^*) \subset D(T)$ and $R(S)$ is closed or cod $R(S) < \infty$).

Then $(TS)^* = S^*T^*$.

Proof. $M = R(S) \oplus N$ is a closed subspace of X and $M^{\perp} = R(S)^{\perp} \cap N^{\perp} = N(S^*) \cap N^{\perp}$ is finite-dimensional by (ii). Since $D(T)$ is dense in X, there exists, by [2], IV. 2. 8, a finite-dimensional subspace W of $D(T)$ such that $X = M \oplus W = R(S) \oplus N \oplus W$. Then $N_1 = N + W$ is a closed subspace of $D(T)$ and theorem 6 applies.

10. Corollary. *Suppose in corollary 9 that T is also self-adjoint and $R(T) \subset D(S^*)$. Then the adjoint of S^*TS is $S^*\overline{TS}$, where $\overline{TS}$ is the minimal closed linear extension of TS. In particular, if T is self-adjoint,* cod $R(S) < \infty$ *and S is bounded on X, then S^*TS is self-adjoint.*

Proof. Let $C = S^*T = (TS)^*$. Then C is closed and $D(C) = D(T)$ is dense in X. Therefore TS is closable and by corollary 9 applied to C and S, $(S^*TS)^* = (CS)^* = S^*C^* = S^*\overline{TS}$.

In the proofs which follow, use is made of the polar decomposition theorem which appears in [1], XII. 7. 7.

11. Lemma. *Let T and S be closed densely defined linear operators from Hilbert space H into H. Then for $|T| = (T^*T)^{1/2}$, the following statements are equivalent:*

(i) $(TS)^* = S^*T^*$;

(ii) $(|T|S)^* = S^*|T|$;

(iii) $\big((I+|T|)S\big)^*|T| = S^*|T|(I+|T|)$.

Proof. Since $D(T) = D(|T|)$, the assumption that TS, $|T|S$ or $(I+|T|)S$ is densely defined implies that each adjoint operator appearing on the left side of equations (i), (ii) or (iii) exists and is an extension of the corresponding operator appearing on the right side.

(ii) implies (i). T can be expressed in the form $T = V|T|$, where V is a partial isometry with initial set $\overline{R(|T|)}$ and final set $\overline{R(T)}$. Since V is bounded on H,

$$\text{(A)} \qquad (TS)^* = (V|T|S)^* = (|T|S)^*V^* = S^*|T|V^* = S^*T^*.$$

(i) implies (ii). Given $u \in D\big((|T|S)^*\big)$, u is of the form $u = V^*v + n$ for some $n \in R(V^*)^\perp = N(V) = R(|T|)^\perp = N(|T|) \subseteq D(S^*|T|) \subseteq D\big((|T|S)^*\big)$. Hence $v \in D\big((|T|S)^*V^*\big)$ which equals $D(S^*|T|V^*)$ by (A). Thus V^*v and therefore u are in $D(S^*|T|)$.

(ii) implies (iii). Since T is self-adjoint,

$$\langle (I+|T|)Su, |T|v\rangle = \langle |T|Su, (I+|T|)v\rangle, \qquad u \in D(|T|S),\ v \in D(|T|),$$

from which it is easily seen that

$$\big((I+|T|)S\big)^*|T| = (|T|S)^*(I+|T|) = S^*|T|(I+|T|).$$

(iii) implies (ii). Suppose $v \in D\big((|T|S)^*\big)$. Since $R(I+|T|) = H$, there exists a w such that $v = (I+|T|)w$ and

$$\langle u, (|T|S)^*v\rangle = \langle |T|Su, (I+|T|)w\rangle = \langle (I+|T|)Su, |T|w\rangle$$

for all $u \in D(|T|S)$. Hence

$$(|T|S)^*v = \big((I+|T|)S\big)^*|T|w = S^*|T|(I+|T|)w = S^*|T|v.$$

12. Theorem. *Let T and S be closed densely defined linear operators from Hilbert space H into H. If $N\big(S^*(I+|T|)\big)$ and $R(S)$ are closed, then $(TS)^* = S^*T^*$.*

Proof. Since $N\big(S^*(I+|T|)\big)$ is closed,

$$(I+|T|)^{-1}N\big(S^*(I+|T|)\big) = N\big(S^*(I+|T|)^2\big) \equiv N_1$$

is closed in Hilbert space $D(|T|)$ with inner product given by

$$\langle u, v\rangle_{|T|} = \langle (I+|T|)u, (I+|T|)v\rangle, \qquad u, v \in D(|T|).$$

We determine the orthogonal complement of N_1 in $D(|T|)$ with respect to $\langle\,,\,\rangle_{|T|}$. Suppose $v \in D(|T|)$ and $\langle v, n\rangle_{|T|} = 0$ for all $n \in N_1$. Then $\langle v, (I+|T|)^2n\rangle = 0$ for all $n \in N_1$ or, equivalently, since $(I+|T|)^2$ is surjective, $\langle v, z\rangle = 0$ for all $z \in (I+|T|)^2N_1 = N(S^*)$. Thus $v \in N(S^*)^\perp = \overline{R(S)} = R(S)$. Hence every $u \in D(|T|)$ can be expressed in the form

$$\text{(a)} \qquad u = Sw + n, \qquad n \in N\big(S^*(I+|T|)^2\big).$$

We now show that $D(|T|S)$ is dense in H. Assume $\langle u, u_0\rangle = 0$ for all $u \in D(|T|S)$. In particular, $u_0 \in N(S)^\perp = R(S^*)$. Since $R((I+|T|)^2) = H$, (a) implies

$$\text{(b)} \qquad R(S^*) = R\big(S^*(I+|T|)^2\big) = R\big(S^*(I+|T|)^2\big)S.$$

Hence $u_0 = S^*(I+|T|)^2Sv_0$ for some v_0 and $0 = \langle u, u_0\rangle = \langle u, S^*(I+|T|)^2Sv_0\rangle = \langle (I+|T|)Su, (I+|T|)Sv_0\rangle$ for all $u \in D(|T|S)$. Taking $u = v_0$, we obtain $(I+|T|)Sv_0 = 0$ which in turn implies $u_0 = 0$. Thus $D(|T|S)$ is dense in H.

In order to complete the proof of the theorem, it suffices to verify (iii) of lemma 11. Assume $v \in D\big(((I+|T|)S)^*|T|\big)$; i.e.,

$$\langle (I+|T|)Su, |T|v\rangle = \langle u, \big((I+|T|)S\big)^*|T|v\rangle \qquad \text{for all } u \in D(|T|S).$$

In particular, since $N(S) \subset D(|T|S)$, we have

$$\big((I+|T|)S\big)^*|T|v \in N(S)^\perp = R(S^*) = R\big(S^*(I+|T|)^2S\big)$$

by (b). Thus $\big((I+|T|)S\big)^*|T|v = S^*(I+|T|)^2Sw$ for some w and since $I+|T|$ is surjective, $|T|v = (I+|T|)d$ for some d. Moreover, by (a), $d = Sz + n$ for some $n \in N(S^*(I+|T|)^2)$. To summarize, $v \in D\big(((I+|T|)S)^*|T|\big)$ implies

$$\langle (I+|T|)Su, (I+|T|)(Sz+n)\rangle = \langle u, \big((I+|T|)S\big)^*|T|v\rangle$$
$$= \langle u, S^*(I+|T|)^2Sw\rangle \qquad \text{for all } u \in D(|T|S).$$

Since $n \in N\big(S^*(I+|T|)^2\big)$, if follows that

$$\langle (I+|T|)Su, (I+|T|)(Sz-Sw)\rangle = 0 \quad \text{for all } u \in D(|T|S).$$

Taking $u = z-w$ and recalling that $I+|T|$ is 1-1, we may conclude that $Sz = Sw$. Therefore

$$\big((I+|T|)S\big)^*|T|v = S^*(I+|T|)^2 Sw = S^*(I+|T|)^2 Sz = S^*(I+|T|)^2 d$$
$$= S^*(I+|T|)|T|v = S^*|T|(I+|T|)v.$$

Thus (iii) of lemma 11 holds.

13. THEOREM. *Let T and S be closed densely defined linear operators from Hilbert space H into H. If $S^*(I+|T|)$ is closed, then $(TS)^* = S^*T^*$.*

Proof. Since $I+|T|$ is self-adjoint and surjective and $S^*(I+|T|)$ is closed, it follows from theorem 1 that

$$S^*(I+|T|) = \big(S^*(I+|T|)\big)^{**} = \big((I+|T|)S\big)^*.$$

In particular, (iii) of lemma 11 holds. Hence $(TS)^* = S^*T^*$.

References

[1] N. Dunford and J. Schwartz, *Linear operators*, part II, New York 1963.
[2] S. Goldberg, *Unbounded linear operators*, New York 1966.
[3] M. Schechter, *The conjugate of a product of closed operators* (to appear).

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Colloquium on
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Linear operations, tensor products, and contractive projections in function spaces*

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Contents

0. INTRODUCTION

One of the main purposes of this paper is to characterize all the subspaces of general Banach function spaces admitting contractive projections onto them, and to extend some of the results when the functions

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